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IG_0606 Additional Maths

Functions Notes

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Functions

§ Functions

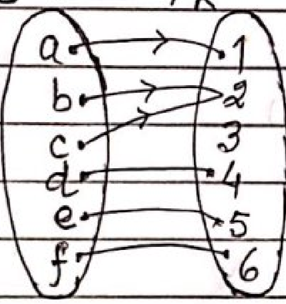
Let consider a situation,
Some students who got admission in a college and are seeking hostel accomodation,

Set of Students $S = \{a, b, c, d, e, f\}$

Set of hostel rooms $R = \{1, 2, 3, 4, 5, 6\}$

$f_1 : S \longrightarrow R$

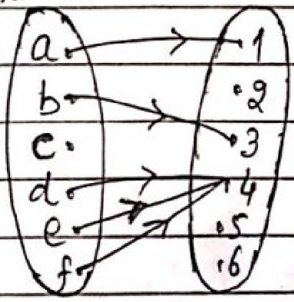
Case I



(i) each student is alloted a unique room.
(ii) Student b & c are sharing room no. 2,
(iii) room no 3 is lying vacant.

It is a many-one function. ✓

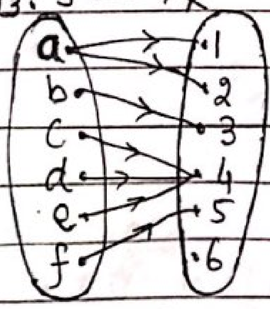
Case II $f_2 : S \longrightarrow R$



Student 'c' has not been alloted any room.
Each student should be alloted a room.

$\therefore$ f_2 is not a function ×

Case III $f_3 : S \longrightarrow R$



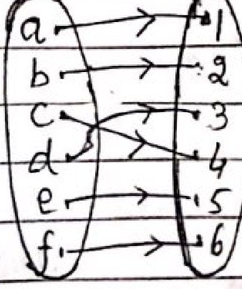
Student 'a' has occupied two room.

$\boxed{\text{One to Many is not allowed}}$

$\therefore$ f_3 is not a function ×

$f_4 : S \longrightarrow R$

Case (iv)



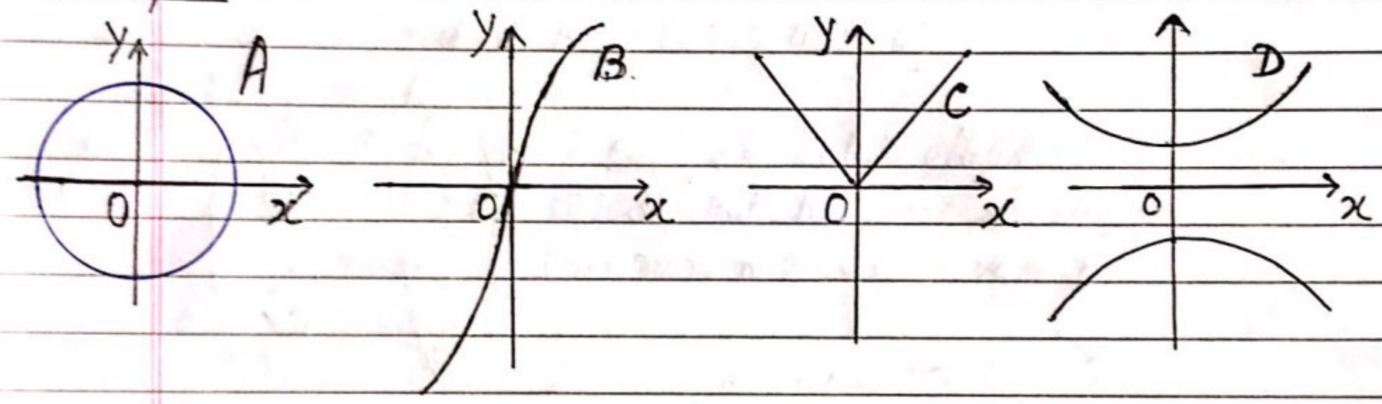
Each student has been alloted a single room, and conversely.

f_4 is a One-One function. ✓

Functions

(Definition) **Function:** A function $f: A \to B$ is a mapping or association such that **each** element of set A is associated to a **Unique** element (one and only one) of set B.

Example 1.



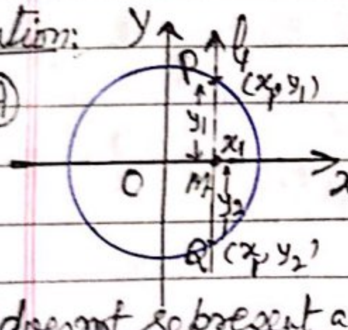
The four graphs above are labelled A, B, C and D.

(i) Write down the letter of each graph that represents a function, giving the reasons for your choice. ---[2]

(ii) Write down the letter of each graph the represents a one-one function, giving a reason for your choice. [S-17/23/Q2] ---[2]

Solution:

✗ (A)



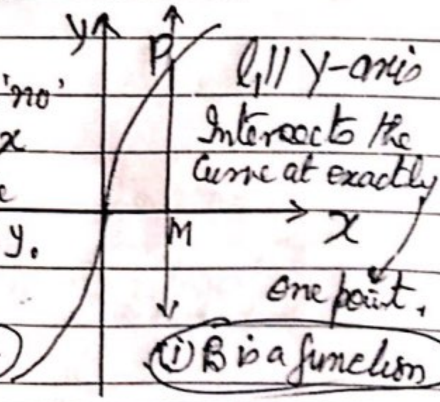
Draw a line l parallel to y-axis intersects the curve at two different points.

∴ (A) does not represent a function, as it is a (One to Many) $x_1 \begin{smallmatrix} \nearrow y_1 \\ \searrow y_2 \end{smallmatrix}$

(B) ✓✓

(ii) there are 'no' two values of x which has the same value of y.

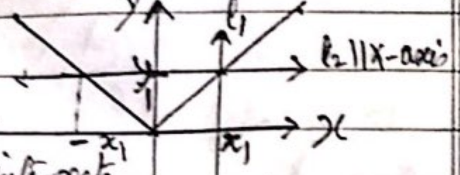
∴ (ii) One - One



$l_1 \parallel$ y-axis intersects the curve at exactly one point.

(i) B is a function

(C) ✓



(i) is a function.

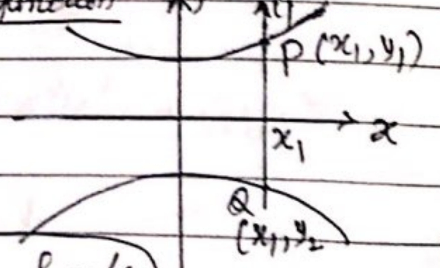
as $l_1 \parallel$ y-axis intersects the curve at exactly one point,

(ii) But Not a One-One function

as x_1 & $-x_1$ both have the same value of y_1 of the

(D) ✗ not a function

$x_1 \begin{smallmatrix} \nearrow y_1 \\ \searrow y_2 \end{smallmatrix}$



(One to Many function)

hence not a function.

Functions

§ Function Notation:
(1) Domain
(2) Range.

Example 2(i) $f: \theta \mapsto \sin\theta ; \theta \in$ Real or $f(\theta) = \sin\theta$

$f(30^\circ) = \sin 30^\circ = \frac{1}{2}$ $\begin{cases} -1 \leq \sin\theta \leq 1 \\ \text{for } \theta \text{ is any real number} \end{cases}$

Given a function $y = f(x)$.

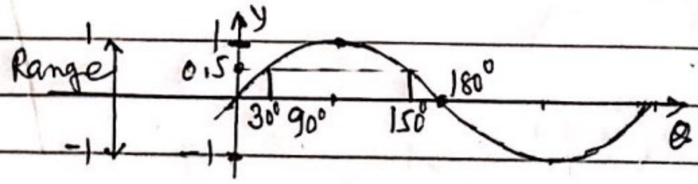
1 Domain of a Function is the set of values 'x' for which the function is defined (corresponds to some value $f(x)$)

2. Range of a Function is the set of values of $f(x)$ which corresponds to the values of x in the domain of function.

Again in Example 2(i) $y = \sin\theta$ is a Many to one function $\begin{cases} \sin 30^\circ = \frac{1}{2} \\ \sin 150^\circ = \frac{1}{2} \end{cases}$

Domain is the set of all real numbers. $\theta \in R$

and Range is $-1 \leq \sin\theta \leq 1$



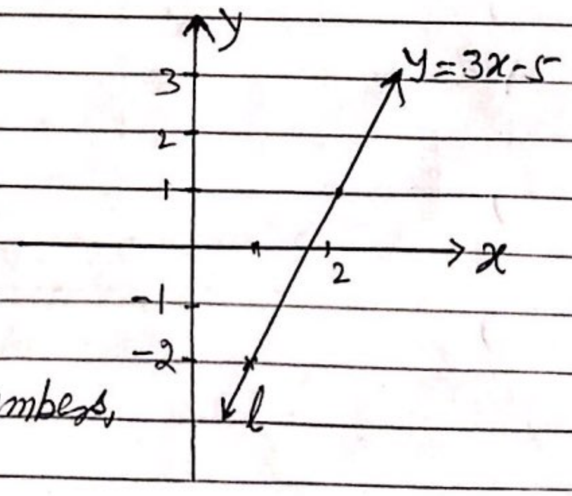
Example 2(ii) $f: x \longmapsto 3x - 5$

or $f(x) = 3x - 5$

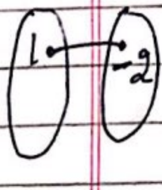
is a One-One function.

Domain is set of real numbers.

Range is also the set of real numbers,



$f: x \mapsto 3x - 5$



$f(1) = -2$ we say that f-image of 1 is 2,

and pre-image of -2 is 1

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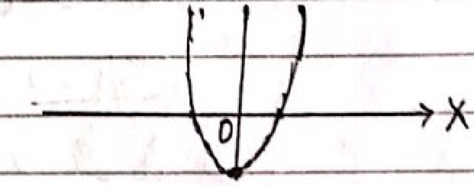
Functions

Example 3. A function f is such that $f(x) = 3x^2 - 1$ for $-10 \leq x \leq 8$.

(i) Find the range of f. [W-13/11/Q12(a)] ---[3]

(ii) Write down a suitable domain for f for which it is one-one. --[1]



Solution: Given $f(x) = 3x^2 - 1$

(i) Given domain of $f = -10 \leq x \leq 8$.

for Range of $f \rightarrow$ $f(x) = 3x^2 - 1$ is minimum at $x = 0$

$f(0) = -1$

and $f(-10) = 300 - 1 = 299$

$\therefore$ Range of $f = -1 \leq f(x) \leq 299$

(ii) f is one-one when $x \geq 0$ (since $f(x)$ is increasing for $x \geq 0$) (or $x \leq 0$)

Example 4. A function f is such that $f(\theta) = \sin 2\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$

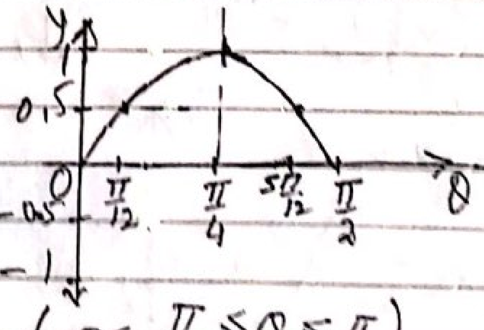
(i) Write down the range of f. ---[1]

(ii) Write down a suitable restricted domain for f to be one-one. [M-15/12/Q8(a)] ---[1]

Solution: $f(\theta) = \sin 2\theta \quad 0 \leq \theta \leq \frac{\pi}{2} \Rightarrow 0 \leq 2\theta \leq \pi$

(i) Range of f (when $0 \leq \theta \leq \frac{\pi}{2}$)

range. $0 \leq f(\theta) \leq 1$



(ii) function one-one when. $0 \leq \theta \leq \frac{\pi}{4}$ (or $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$)

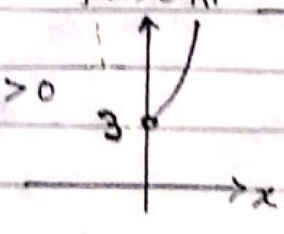
Example 4. Function f and g are defined, for $x > 0$ by $f(x) = \ln x$.
and $g(x) = 2x^2 + 3$. [W-17/12/Q10]

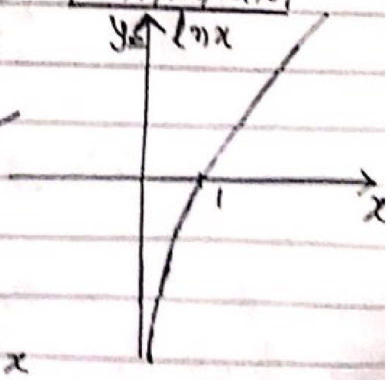
(i) Write down the range of f. --[1]

(ii) Write down the range of g. --[1]

Solution (i) Range of f is $-\infty < f(x) < \infty$ or $f(x) \in R$.

(ii) Range of g is $g(x) > 3$ for $x > 0$





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Functions

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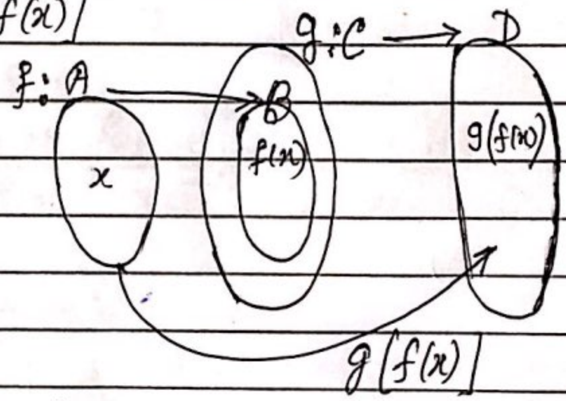
§ Composite Function:

Given two functions $f: A \to B$
and $g: C \to D$

(i) for the composit function (gf) to be defined →
the range of 'f' should be a subset of the domain of 'g'
or $B \subseteq C$

Then $(gf)(x) = g[f(x)]$



(ii) for the composite function $(fg)(x) = f[g(x)]$

here range of 'g' lies in the domain of 'f'

Example 5. Functions f and g are such that, for $x \in R$,
$f(x) = x^2 + 3$ and $g(x) = 4x - 1$

(i) State the range of f. ---[1]

(ii) Solve $fg(x) = 4$ | W-17/11/Q6(a) | ---[3]

Solution (i) $f(x) = x^2 + 3$, $x \in R$; Range of f. is $f(x) \geq 3$ [∵ $x^2 \geq 0$, $x \in R$; $x^2 + 3 \geq 3$]

(ii) Solve $fg(x) = 4$

$f[4x-1] = 4$

$\Rightarrow (4x-1)^2 + 3 = 4$

$\Rightarrow (4x-1)^2 = 1$

$\Rightarrow 4x - 1 = \pm 1$

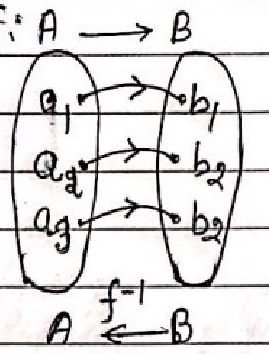
$\Rightarrow x = 0, \; x = \frac{1}{2}$ ✓

Functions

§ **Inverse of a function.** $f: A \rightarrow B$

Such that $f(a_1) = b_1$

Then $f^{-1}(b_1) = a_1$



f^{-1} exist if "f is a One-One function"

$f: x \rightarrow y$

Note. (i) $ff^{-1}(y) = f(x) = y$

(ii) $f^{-1}f(x) = f^{-1}(y) = x$

(iv) Domain of f^{-1} is same as the range of f.

(iii) The Graphs of $f(x)$ and $f^{-1}(x)$ are Reflection of each other in line $y = x$ (Mirror Line)

Example 6 (a) It is given that $f(x) = 3e^{-4x} + 5$ for $x \in R$

(i) State the range of f. ---[1]

(ii) Find f^{-1} and state its domain. ---[4]

(b) It is given that $g(x) = x^2 + 5$ and $h(x) = \ln x$ for $x > 0$, solve $hg(x) = 2$. $\boxed{S\text{-}17/11/Q4}$ ---[3]

Solution: $f(x) = 3e^{-4x} + 5$ for $x \in R$

(a) (i) Range of f, $f(x) > 5$ ✓

$\left\{ \begin{array}{l} e^{-4x} \rightarrow 0 \text{ as } x \rightarrow \infty \\ \therefore 3e^{-4x} + 5 \rightarrow 5^+ \end{array} \right.$

(ii) To find f^{-1}

Let $f(x) = 3e^{-4x} + 5 = y$

$\Rightarrow e^{-4x} = \frac{y-5}{3}$

$\Rightarrow -4x = \ln\left(\frac{y-5}{3}\right)$

$\Rightarrow x = -\frac{1}{4}\ln\left(\frac{y-5}{3}\right)$

Interchange x & y

$y = -\frac{1}{4}\ln\left(\frac{x-5}{3}\right)$

or $f^{-1}(x) = -\frac{1}{4}\ln\left(\frac{x-5}{3}\right)$

$= \frac{1}{4}\ln\left(\frac{3}{x-5}\right)$

or $(x) = \frac{1}{4}[\ln 3 - \ln(x-5)]$ ✓

and Domain of f^{-1} = Range of $f \rightarrow x > 5$ ✓

(b) $g(x) = x^2 + 5$

and $h(x) = \ln x$ for $x > 0$

To Solve

Given $h[g(x)] = 2$

$\Rightarrow h[x^2 + 5] = 2$

$\Rightarrow \ln(x^2 + 5) = 2$

$\Rightarrow x^2 + 5 = e^2$

$\Rightarrow x^2 = e^2 - 5 = 2.389$

$x = \sqrt{2.389} = 1.55$

$\therefore x = 1.55$ ✓

Functions

Example 7. Functions g and h are such that,

$g(x) = 2 + 4\ln x$ for $x > 0$

$h(x) = x^2 + 4$ for $x > 0$

(a) Find g^{-1}, stating its domain and its range. ---[4]

(b) Solve, $gh(x) = 10$ ---[3]

(c) Solve $g'(x) = h'(x)$ $\boxed{SP\text{-}20/02/Q3}$ --[3]

Solution (a) Given $g(x) = 2 + 4\ln x = y$ (let)

$$\Rightarrow \ln x = \frac{y-2}{4}$$

$$\Rightarrow x = e^{\frac{(y-2)}{4}}$$

Interchanging x and y

$$y = e^{\frac{(x-2)}{4}}$$

or $g^{-1}(x) = e^{\left(\frac{x-2}{4}\right)}$ ✓

Domain $x \in \mathbb{R}$ ✓

and Range $y > 0$ ✓

(b) Solve $gh(x) = 10$

$$\Rightarrow g[h(x)] = 10$$

$$g[x^2+4] = 10$$

or $2 + 4\ln(x^2+4) = 10$

$$\Rightarrow \ln(x^2+4) = 2$$

$$x^2 + 4 = e^2$$

$$x^2 = e^2 - 4 = 3.39$$

$$x = \underline{1.84}$$ ✓

(c) Given $g'(x) = h'(x)$

$$\Rightarrow \frac{4}{x} = 2x \Rightarrow 2x^2 = 4$$

$$\Rightarrow x^2 = 2$$

$$\Rightarrow x = \underline{\sqrt{2}}$$ ✓ as $x > 0$

Example 8 (a) The function f is defined by $f(x)=\sqrt{1+x^2}$, for all values of x. The graph of $y=f(x)$ is given below.

(i) Explain, with reference to the graph, why f does not have an inverse. ---[1]

(ii) Find $f^2(x)$ ---[2]

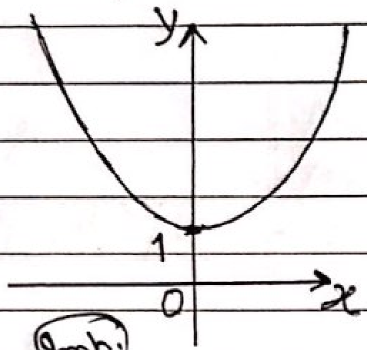
(b) The function $g(x)=\sqrt{1+x^2}$ for $x>k$ and g has an inverse.

(i) Write down a possible value for k. ---[1]

(ii) Find $g^{-1}(x)$ ---[2]

(c) The function h is defined, for all real values of x, by $h(x)=4e^x+2$.

Sketch the graph of $y=h(x)$. Hence on the same axes sketch the graph of $y=h^{-1}(x)$. Give the coordinates of any points where your graph meets the coordinate axis. ---[4]



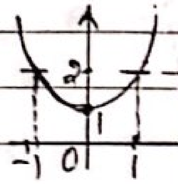
Imp.

The graph of $h^{-1}(x)$ is the reflection of the graph of $h(x)$ in line $y=x$

M-18/22/Q10

Solution:

(a)(i) $f(1)=2$
$f(-1)=2$



$\therefore f(x)$ is not a one-one function.

$\therefore f$ does not have an inverse.

(ii) $f^2(x)=f[f(x)]=f[\sqrt{1+x^2}]$

$=\sqrt{1+(\sqrt{1+x^2})^2}$

$\therefore f^2(x)=\sqrt{2+x^2}$ ✓

(b) (i) for $g(x)=\sqrt{1+x^2}$; $x>k$

for $g(x)$ to have an inverse

$k=0 \Rightarrow x\geq 0$

(ii) $g(x)=\sqrt{1+x^2}=y$ (let)

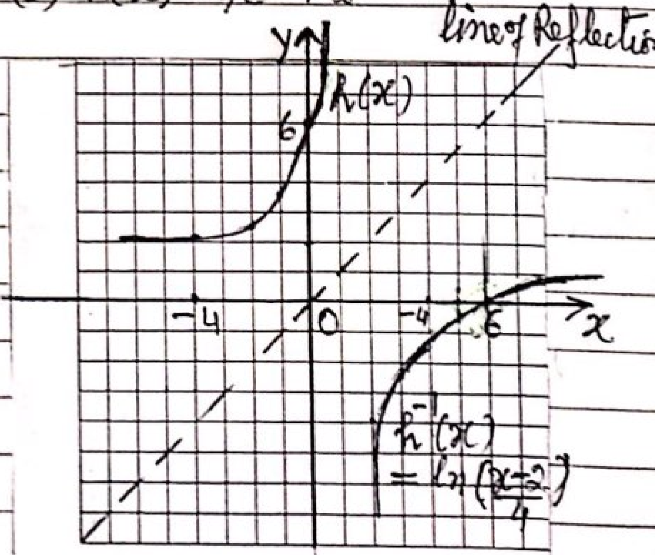
$\Rightarrow y^2=1+x^2$

$\Rightarrow x=\sqrt{y^2-1}$

Interchange x & y $\Rightarrow$ $y=\sqrt{x^2-1}$

or $g^{-1}(x)=\sqrt{x^2-1}$ ✓

(c) $h(x)=4e^x+2$



$h(x)=4e^x+2=y$ (let)

$\Rightarrow \frac{y-2}{4}=e^x$

or $x=\ln\left(\frac{y-2}{4}\right)$

Interchange x & y $y=\ln\left(\frac{x-2}{4}\right)$

or $h^{-1}(x)=\ln\left(\frac{x-2}{4}\right)$ ✓

Functions

Example 9. Functions f and g are such that,

$$f(x) = \frac{x^2-2}{x} \quad \text{for } x \geq 2; \quad g(x) = \frac{x^2-1}{2} \quad \text{for } x \geq 0$$

(i) State the range of g. ---[1]

(ii) Explain why fg(1) does not exist. ---[2]

(iii) Show that $gf(x) = ax^2 + b + \frac{c}{x^2}$, where a, b and c, to be found. --[3]

(iv) State the domain of gf. --[1]

(v) Show that $f^{-1}(x) = \frac{x+\sqrt{x^2+8}}{2}$ -- [4]

[M-17/22/Q11]

Solution (i) Given $g(x) = \frac{x^2-1}{2}$ for $x \geq 0 \Rightarrow$ Range of g; $g(x) \geq -\frac{1}{2}$ ✓

(ii) $fg(1) = f[g(1)] = f(0)$ is not defined ✓ as '0' does not lie in the domain of $f(x)$, $x \geq 2$. $\therefore$ $fg(1)$ is not defined.

(iii) $gf(x) = g[f(x)] = g\left[\frac{x^2-2}{x}\right] = \frac{\left(\frac{x^2-2}{x}\right)^2 - 1}{2} = \frac{1}{2}x^2 - \frac{5}{2} + \frac{2}{x^2}$ ✓

(iv) Domain of gf is $x \geq 2$

(v) Let $f(x) = y = \frac{x^2-2}{x}$

$$\Rightarrow x^2 - xy - 2 = 0$$

$$\Rightarrow x = \frac{y \pm \sqrt{y^2+8}}{2}$$

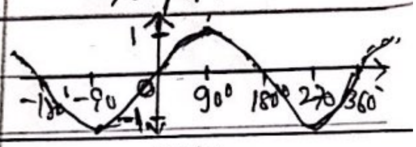
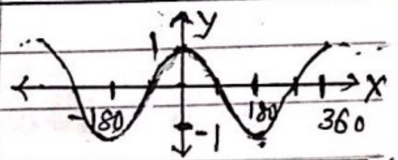
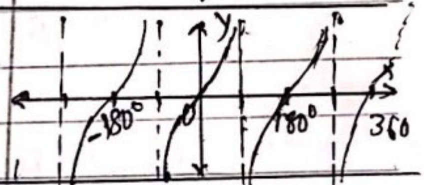
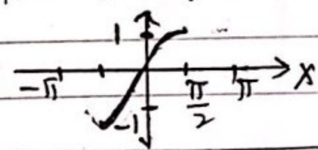
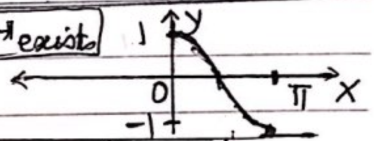
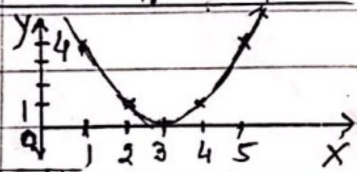
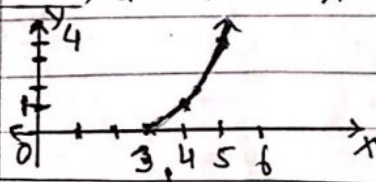
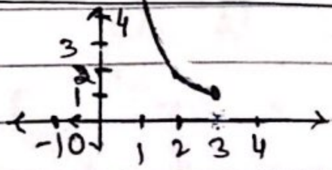
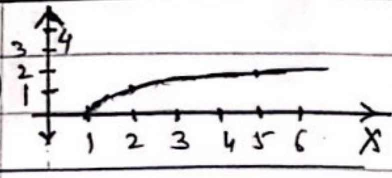
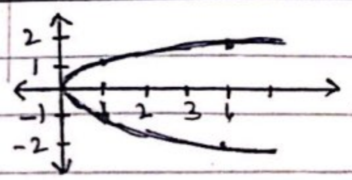
[−ve sign to be discarded as $x \geq 2$]

Interchange x & y

$$y = \frac{x + \sqrt{x^2+8}}{2}$$

$$\text{or } f^{-1}(x) = \frac{x+\sqrt{x^2+8}}{2} \checkmark$$

Functions and Graph:

	Function	Domain	Range.	One-One / Many-One	Graph.
1.	$f(x) = \sin x^0$	$x \in R$	$-1 \le f(x) \le 1$	Many one $f(30^0) \rightarrow \frac{1}{2}$, $f(150^0) \nearrow$	
2.	$f(x) = \cos x^0$	$x \in R$	$-1 \le f(x) \le 1$	Many-one $f(30) = \frac{\sqrt{3}}{2}$, $\cos(330^0) \nearrow$	
3.	$f(x) = \tan x^0$	$R - \{180n \pm 90^0\}$, $n \in I$	$-\infty < f(x) < \infty$	Many one	
4.	$f(x) = \sin x$	$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$	$-1 \le f(x) \le 1$	One-One $\boxed{f^{-1} \text{ exist}}$ (Increasing function)	
5.	$f(x) = \cos x$	$0 \le x \le \pi$	$-1 \le f(x) \le 1$	One-One $\boxed{f^{-1} \text{ exists}}$ (decreasing function)	
6.	$f(x) = \tan x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$	$-\infty < f(x) < \infty$	One-One ($\boxed{f^{-1} \text{ exist}}$ Increasing funct!)	
7(i)	$f(x) = (x-3)^2$	$x \in R$	$f(x) \ge 0$	Many-one	
7(ii)	$f(x) = (x-3)^2$	$x \ge 3$	$f(x) \ge 0$	One-one (Increasing funct.) $\boxed{f^{-1} \text{ exist}}$	
7(iii)	$f(x) = (x-3)^2 + 1$	$x \le 3$	$f(x) \ge 1$	One-One (decreasing fn!) $\boxed{f^{-1} \text{ exist}}$	
8(i)	$f(x) = \sqrt{x-1}$	$x \ge 1$	$f(x) \ge 0$	One-One (Increasing fn) $\boxed{f^{-1} \text{ exists}}$	
(ii)	$y^2 = x$, $y = \pm\sqrt{x}$	$x \ge 0$	$f(x) \in R$		

$\boxed{\text{Not a function}}$

as $f(4) \rightarrow 2, -2$

$\boxed{\text{One-to Many}}$