IG_0606
Additional Maths

Quadratic Functions
and
Simultaneous Equations
Notes

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Quadratic Function

§ Quadratic Function:

ax^2+bx+c $\quad a\neq 0,\ a,b,c\in R$

§ Factorising a quadratic function:

ax^2+bx+c

When $a=1$ $\quad\quad$ $a\neq 1$

x^2+bx+c

Case I Take two factors of c, whose sum is b.
when c is $+$

Example 1. $x^2+7x+12$ $\left\{ \begin{array}{l} 12=4\times 3 \\ 4+3=7 \end{array}\right.$

$= x^2+4x+3x+12$

$= x(x+4)+3(x+4)$

$= (x+3)(x+4)$ ✓

Example 2.

$x^2-13x+42$

$= x^2-7x-6x+42$ $\left\{ \begin{array}{l} 7\times 6=42 \\ \text{and} \\ 7+6=13 \end{array}\right.$

$= x(x-7)-6(x-7)$

$= (x-6)(x-7)$ ✓

Case II when c is $-$ve, take two factors of c whose numerical diffence is b.

Example 3. x^2-x-42 $\left\{ \begin{array}{l} 7\times 6=42 \\ 7-6=1 \end{array}\right.$

$= x^2-7x+6x-42$

$= x(x-7)+6(x-7)$

$= (x+6)(x-7)$

Example 4. $x^2+4x-12$

$= x^2+6x-2x-12$ $\left\{ \begin{array}{l} 6\times 2=12 \\ 6-2=4 \end{array}\right.$

$= x(x+6)-2(x+6)$

$= (x-2)(x+6)$

ax^2+bx+c

Case III consider the product 'ac' >0
Now take two factors of 'ac' whose sum is b

Example 1'. $2x^2+7x+6$ $\left\{ \begin{array}{l} 2\times 6=12 \\ \text{Now} \\ 4\times 3=12 \\ \text{and } 4+3=7 \end{array}\right.$

$= 2x^2+4x+3x+6$

$= 2x(x+2)+3(x+2)$

$= (x+2)(2x+3)$

Example 2'.

$3x^2-13x+14$ $\left\{ \begin{array}{l} \text{Consider} \\ 3\times 14=42 \\ \text{Now} \\ 7\times 6=42 \\ 7+6=13 \end{array}\right.$

$= 3x^2-7x-6x+14$

$= x(3x-7)-2(3x-7)$

$= (3x-7)(x-2)$

Case IV Consider the product $ac<0$
Now take two such factors of 'ac' whose numerical difference is b.

Example 3'. $12x^2+4x-1$ $\left\{ \begin{array}{l} \text{Consider} \\ 12\times 1=12 \\ \text{Now } 6\times 2=12 \\ 6-2=4 \end{array}\right.$

$= 12x^2+6x-2x-1$

$= 6x(2x+1)-1(2x+1)$

$= (6x-1)(2x+1)$

Example 4'. $7x^2-x-6$ $\left\{ \begin{array}{l} 7\times 6=42 \\ 7-6=1 \end{array}\right.$

$= 7x^2-7x+6x-6$

$= 7x(x-1)+6(x-1)$

$= (x-1)(7x+6)$

Quadratic Equations

§ Solving Quadratic Equations by Factorisation:

Example 5. Solve by factorisation.

$$3x^2 - 7x - 20 = 0$$
$$\Rightarrow 3x^2 - 12x + 5x - 20 = 0$$
$$\Rightarrow 3x(x-4) + 5(x-4) = 0$$
$$(x-4)(3x+5) = 0$$

either $x - 4 = 0$ or $3x + 5 = 0$

$$\Rightarrow x = 4 \text{ or } x = -5/3$$

Consider $3 \times -20 = -60$
Now $-12 \times 5 = -60$
and $-12 + 5 = -7$

Example 6. Solve $3x^2 + 2x - 8 = 0$

or $3x^2 + 6x - 4x - 8 = 0$

$$3x(x+2) - 4(x+2) = 0$$
$$(x+2)(3x-4) = 0$$
$$\therefore x = -2 \text{ or } x = 4/3$$
$$\therefore x = -2 \text{ or } 1\tfrac{1}{3}$$

$3 \times (-8) = -24$
Now $6 \times (-4) = -24$
and $6 + (-4) = 2$

Example 7. Solve. $x^2 - 40x + 300 = 0$

$$\Rightarrow x^2 - 30x - 10x + 300 = 0$$
$$\Rightarrow x(x-30) - 10(x-30) = 0$$
$$\Rightarrow (x-30)(x-10) = 0$$
$$\Rightarrow x - 30 = 0 \text{ or } x - 10 = 0$$
$$\Rightarrow x = 30 \text{ or } x = 10$$

$300 = 30 \times 10$
and $30 + 10 = 40$

Quadratic Equations

§ Solving Quadratic Equation using Quadratic Formula.

Given a quadratic equation:

$$ax^2+bx+c=0 \qquad a\neq 0, \quad a,b,c\in R$$

Then $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Here (b^2-4ac) is called discriminant.

denoted by, or $D=b^2-4ac$

§ Nature of roots:

Case I If $D\geq 0$ the quadratic equation has two real roots.

In particular:

(I) If $D>0$ two real and distinct roots.

(II) If $D=0$ two equal roots (or only one root)

Case II If $D<0$ then no real roots (or No Solution).

Example 6'.

Solve the equation:

$$3x^2+2x-8=0$$

$$\begin{cases} a=3,\ b=2,\ c=-8 \\ b^2-4ac=2^2-4\times 3\times(-8) \\ \qquad =4+96 \\ \qquad =100>0 \end{cases}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-2\pm\sqrt{100}}{2\times 3}$$

$$=\frac{-2\pm 10}{6}$$

$$=\frac{-2+10}{6} \text{ or } \frac{-2-10}{6}$$

$$\therefore x=8/6 \text{ or } -2$$

$$\text{or } x=4/3 \text{ or } -2 \checkmark$$

Quadratic Equations

Example 8. Show that the roots of $px^2+(p-q)x-q=0$ are real for all real values of p and q. [S-17/22/Q6] --[4]

Solution: Given a quad. equation $px^2+(p-q)x-q=0$; $p, q \in R$

$\because$ for $ax^2+bx+c=0$: $a=p$, $b=p-q$, $c=-q$

$$D = b^2-4ac = (p-q)^2 - 4 \cdot p \times (-q)$$
$$= p^2+q^2-2pq+4pq$$
$$= p^2+q^2+2pq$$
$$D = (p+q)^2 \geq 0$$ Square of a real no. ≥ 0

$\Rightarrow D \geq 0$

$\therefore$ The given quadratic equation has real roots.

Example 9. Find the value of k, for which the curve $y = 2x^2-3x+k$
(i) Passes through the point $(4,-7)$ --[1]
(ii) Meets the x-axis at one point only. [S-16/11/Q1] --[2]

Solution (i) Given equation of curve $y = 2x^2-3x+k$ —— ①
passes through the point $(4,-7)$
satisfy ① $\Rightarrow -7 = 2\times 4^2 - 3\times 4 + k$
$-7 = 32-12+k \Rightarrow k = -27$ ✓

(ii) The curve $y = 2x^2-3x+k$
meet x-axis at only one point,
$\therefore$ The quad equation $2x^2-3x+k=0$
has only one solution $\Rightarrow D = b^2-4ac = 0$; $a=2$, $b=-3$, $c=k$
$\Rightarrow D = (-3)^2 - 4\times 2\times k = 0$
$9-8k=0$
$\Rightarrow k = \frac{9}{8}$ ✓

§ Solving a Quadratic Equation by Completing the Square:

Note. $x^2 \pm 2ax + a^2 = (x \pm a)^2$

$\left\{\begin{array}{l} \text{here coefficient of } x^2 = 1 \\ \text{and the constant } a^2 = \left(\frac{\text{Coefficient of } x}{2}\right)^2 \end{array}\right.$

Example 10. Solve by completing the square.

$$x^2 + 6x - 16 = 0$$
$$\Rightarrow x^2 + 6x = 16$$
$$\Rightarrow x^2 + 6x + 3^2 = 16 + 3^2$$
$$\Rightarrow (x+3)^2 = 25$$
$$\Rightarrow (x+3) = \pm\sqrt{25}$$
$$x = -3 \pm 5$$
$$\therefore x = 2 \text{ or } x = -8$$

$\left\{\begin{array}{l} \text{Coeff of } x = 6 \\ \therefore \text{constant term} = \left(\frac{\text{Coeff of } x}{2}\right)^2 \\ = \left(\frac{6}{2}\right)^2 = 3^2 \end{array}\right.$

Example 11 Solve by completing the square.

$$3x^2 + 2x - 8 = 0$$

Divide each term by 3 (To make, coeff of $x^2 = 1$)

$$x^2 + \frac{2}{3}x - \frac{8}{3} = 0$$
$$\Rightarrow x^2 + \frac{2}{3}x + \left(\frac{1}{3}\right)^2 = \frac{8}{3} + \left(\frac{1}{3}\right)^2$$
$$\Rightarrow \left(x + \frac{1}{3}\right)^2 = \frac{25}{9}$$
$$\text{or } \left(x + \frac{1}{3}\right)^2 = \left(\frac{5}{3}\right)^2$$
$$\therefore x + \frac{1}{3} = \pm \frac{5}{3}$$
$$\therefore x = \frac{-1}{3} \pm \frac{5}{3}$$
$$x = \frac{-1}{3} + \frac{5}{3} \text{ or } x = \frac{-1}{3} - \frac{5}{3}$$
$$x = \frac{4}{3} \text{ or } x = -2$$

$\left\{\begin{array}{l} \text{Coeff. of } x^2 = \frac{2}{3} \\ \therefore \text{Constant term} = \left(\frac{\text{Coeff of } x}{2}\right)^2 \\ = \left(\frac{1}{2} \times \frac{2}{3}\right)^2 = \left(\frac{1}{3}\right)^2 \end{array}\right.$

Quadratic Function

Example 12(a) Express $12x^2-6x+5$ in the form $p(x-q)+r$, where where p, q and r are constants to be found. ---[3]

(b) Hence find the greatest value of $(12x^2-6x+5)^{-1}$ and state the value of x at which this occur. SP-20|01|Q3 ---[2]

Solution: $12x^2-6x+5$

$= 12\left[x^2 - \frac{6}{12}x\right] + 5$

$= 12\left[x^2 - \frac{1}{2}x + --\right] + 5$

$= 12\left[x^2 - \frac{1}{2}x + \left(\frac{1}{4}\right)^2 - \frac{1}{16}\right] + 5$

$= 12\left[x^2 - \frac{1}{2}x + \left(\frac{1}{4}\right)^2\right] - 12\times\frac{1}{16} + 5$

$= 12\left(x - \frac{1}{4}\right)^2 - \frac{3}{4} + 5$

$= 12\left(x - \frac{1}{4}\right)^2 + \frac{17}{4}$ ✓

Coeff of $x^2 = 1$
Coeff of $x = \frac{1}{2}$
$\therefore$ Constant term $= \left(\frac{1}{2}\times\text{Coeff } x\right)^2$
$= \left(\frac{1}{2}\times\frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2$

Comparing with the form $p(x-q)^2 + r$

$\therefore$ Min. value of

$12x^2 - 6x + 5 = 12\left(x - \frac{1}{4}\right)^2 + \frac{17}{4}$

$= 0 + \frac{17}{4}$

$= \frac{17}{4}$

here $\left(x - \frac{1}{4}\right)^2 \geq 0$
Min Value of $\left(x - \frac{1}{4}\right)^2 = 0$
for $x - \frac{1}{4} = 0$
$x = \frac{1}{4}$ —— (1)

$\therefore$ Greatest value of $\frac{1}{(12x^2-6x+5)} = \frac{1}{\frac{17}{4}} = \frac{4}{17}$ ✓

or Greatest Value of $(12x^2-6x+5)^{-1} = \frac{4}{17}$ ✓
from (1) it occurs at $x = \frac{1}{4}$ ✓

Quadratic Function

Example 13 (i) Show that $y = 3x^2 - 6x + 5$ can be written in the form, $y = a(x-b)^2 + c$, where a, b and c the constants to be found; --- [3]

(ii) Hence or otherwise, find the coordinates of the station points on the curve $y = 3x^2 - 6x + 5$. [S-14/13/Q1] --- [1]

Solution: $y = 3x^2 - 6x + 5$

$= 3(x^2 - 2x + --) + 5$

$= 3[x^2 - 2x + 1^2 - 1] + 5$

$= 3(x-1)^2 - 3 + 5$

$= 3(x-1)^2 + 2$ ✓

Coeff of $x^2 = 1$
Coeff of $x = 2$
Constant term $= \left(\frac{\text{Coeff of } x}{2}\right)^2$
$= \left(\frac{2}{2}\right)^2 = 1^2$

∴ Stationary point is $(1, 2)$ ✓

is in the required form $a(x-b)^2 + c$
Here the stationary point is (b, c)
and is (i) Min if $a > 0$
(ii) Max if $a < 0$

§ Solution of Quadratic Inequalities:

(i) $x^2 + bx + c > 0$

(ii) $x^2 + bx + c < 0$

Solution:

(i) Let $x^2 + bx + c = (x-p)(x-q)$ where p and q are the solution of $x^2 + bx + c = 0$ and let $p > q$

Then the solution of $x^2 + bx + c > 0$
$(x-p)(x-q) > 0$
is $x > p$ and $x < q$

$x < q$ $\qquad$ $x > p$

p

(ii) Solution of, $x^2 + bx + c < 0$
$(x-p)(x-q) < 0$
is $q < x < p$

$q < x < p$

here $p > q$

Example 14. Determine the set of values of k for which the equation, $(3-2k)x^2+(2k-3)x+1=0$, has no real roots. [M-18/22/Q2] --- [5]

Solution: Given $(3-2k)x^2+(2k-3)x+1=0$

{ quad eqn $ax^2+bx+c=0$ has no real roots if $b^2-4ac<0$

$$a=(3-2k),\ b=(2k-3),\ c=1$$

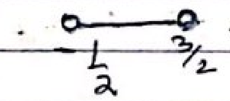
$$b^2-4ac=(2k-3)^2-4\times1(3-2k)<0$$

For no real root or $4k^2-4k-3<0$

$$\Rightarrow (2k-3)(2k+1)<0$$

for $(2k-3)(2k+1)=0$

$k=\frac{3}{2},-\frac{1}{2}$



$$\Rightarrow -\frac{1}{2}<x<\frac{3}{2}$$

or $-0.5<x<1.5$ ✓

Example 15. Find the range of values of k for which the equation, $kx^2+k=8x-2xk$ has two real distinct roots. --- [4] [W-15/11/Q1]

Solution: Given $kx^2+k=8x-2xk$

or $kx^2+(2k-8)x+k=0$

Quad. eqn $ax^2+bx+c=0$ has two real distinct roots, for $b^2-4ac>0$

$$b^2-4ac=(2k-8)^2-4\times k\times k>0$$

$$\Rightarrow 64-32k>0$$

$$\Rightarrow 32k<64$$

$$\Rightarrow k<2$$ ✓

Example 16. Find the set of values of k for which the curve, $y=(k+1)x^2-3x+(k+1)$ lies below x-axis. [W-13/11/Q2] --- [4]

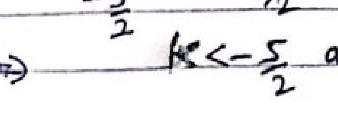
Solution: $y=(k+1)x^2-3x+(k+1)$ lies below x-axis, / $y=ax^2+bx+c$

for (i) Curve does not intersect x-axis $\Rightarrow b^2-4ac<0$; below x-axis $\Rightarrow a<0$

Now $b^2-4ac=(-3)^2-4(k+1)(k+1)<0$

$$\Rightarrow 4k^2+8k-5>0$$

$(2k+5)(2k-1)>0$ (for eqn) $k=\frac{1}{2},-\frac{5}{2}$



$\Rightarrow k<-\frac{5}{2}$ and $k>\frac{1}{2}$ — ①

for below x-axis, $a<0 \Rightarrow k+1<0$

$k<-1$ — ②

for ① and ②

$$k<-\frac{5}{2}$$ ✓

§ Graph of Quadratic Function. $y = ax^2 + bx + c$ $a \neq 0$

(is a parabola)

when $a > 0$ Parabola opening upwards.

§ **Case I** when $b^2 - 4ac > 0$, the curve intersects x-axis at two distinct points and the vertex is the minimum at $x = \frac{\alpha+\beta}{2}$ where α, β are zero's of the polynomial.

Example 17. Draw the graph of $y = x^2 + 4x - 12$ —— (i)

Consider $x^2 + 4x - 12 = 0$ $[b^2 - 4ac = 4^2 - 4 \times 1 \times (-12)]$ $= 64 > 0$

$x^2 + 6x - 2x - 12 = 0$

$x(x+6) - 2(x+6) = 0$

$(x+6)(x-2) = 0$

$x = -6$ and $x = 2$

∴ Curve intersects x-axis at $(-6, 0)$ and $(2, 0)$

(zero's or $\alpha = -6$, $\beta = 2$)

∴ Vertex at $x = \frac{-6+2}{2} = -2$

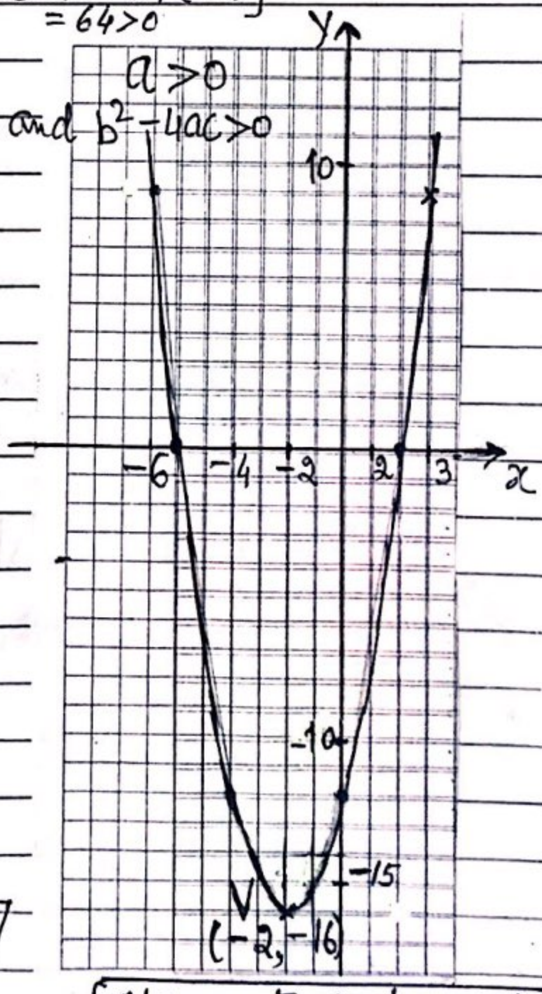
for (i) $y = (-2)^2 + 4 \times (-2) - 12$

$= 4 - 8 - 12$

$= -16$

Vertex at $(-2, -16)$

x	-7	-6	-4	-2	0	2	3
y	9	0	-12	-16	-12	0	9



[Also vertex is at $x = -\frac{b}{2a}$]

$x = \frac{-4}{2 \times 1} = -2$ ✓

Alternate Method: Completing the Square:

$ax^2 + bx + c = a(x-h)^2 + k$; $V(h, k)$

here. $x^2 + 4x - 12$

$= (x^2 + 4x + 2^2) - 4 - 12$

$= (x+2)^2 - 16$

$= [x - (-2)]^2 + (-16)$

∴ Vertex at $(-2, -16)$ ✓

Quadratic Function

§ **Graph of Quadratic Function**: $y = ax^2 + bx + c$ $a \neq 0$

$a > 0$, Graph is a parabola opening upwards.

§ **Case II** $b^2 - 4ac = 0$, the curve meets x-axis at exactly one point.

Example 18. Draw the graph of $y = x^2 - 2x + 1$ —— ①

Consider the equation $x^2 - 2x + 1 = 0$

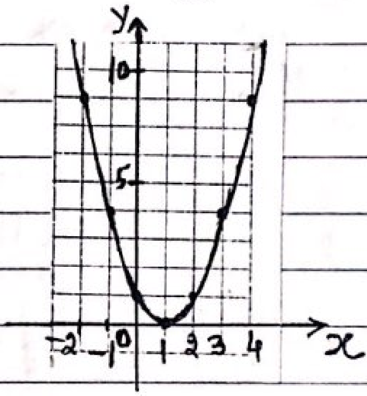
or $(x-1)^2 = 0$

$x = 1, 1$

Two coincident root.

Vertex at $x = 1$ and from ① $y = 0$

Vertex $(1, 0)$



x	-2	-1	0	1	2	3	4
y	9	4	1	0	1	4	9

$a > 0$

§ **Case III** $b^2 - 4ac < 0$, $\boxed{\text{The curve does not intersect x-axis, and the curve is above x-axis}}$

Example 19. Draw the graph of $y = 4x^2 - 12x + 11$

$b^2 - 4ac = (-12)^2 - 4 \times 4 \times 11$
$= 144 - 176 = -32 < 0$
(No real roots)

Consider $y = 4x^2 - 12x + 11$ (using completing the square method)

$4[x^2 - 3x\ ] + 11$

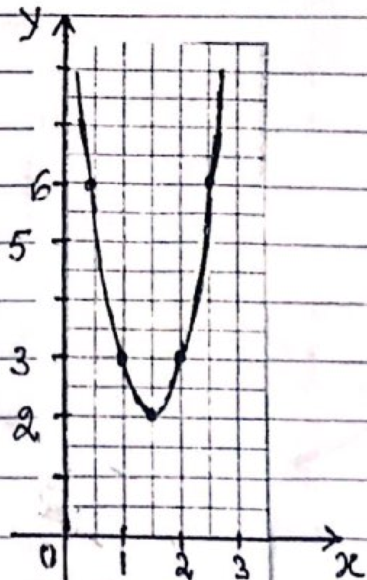
$4\left[x^2 - 3x + \left(\frac{3}{2}\right)^2 - \frac{9}{4}\right] + 11$

$4\left(x - \frac{3}{2}\right)^2 - 9 + 11$

$y = 4(x - 3/2)^2 + 2$ $[a(x-h)^2 + k]$

Vertex at (h, k)

$V\left(\frac{3}{2}, 2\right)$



x	0.5	1	$3/2$	2	2.5
y	6	3	2	3	6

$\boxed{\text{Alternatively, Vertex } x = -\frac{b}{2a} = \frac{+12}{8} = \frac{3}{2}}$

Quadratic Functions

§ Graph of quadratic Function: $ax^2+bx+c=0$, $a\neq 0$

$\boxed{a<0,\ \text{Graph a parabola opening down wards}}$

§ Case I $b^2-4ac>0$, the curve intersects x-axis at two distinct points.

Example 20. Draw the graph of $y=-x^2+5x-6$ —— (1)

Consider the equation,

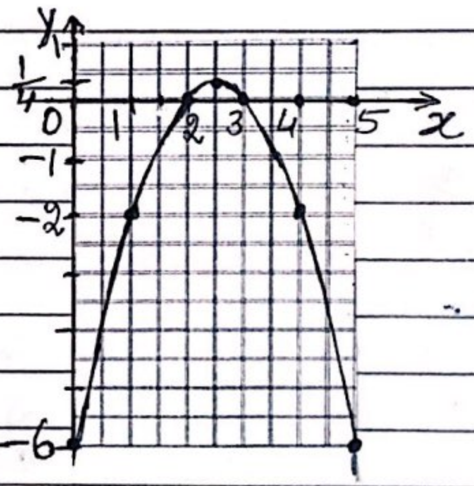
$$-x^2+5x-6=0$$

or $$x^2-5x+6=0$$

$$(x-2)(x-3)=0$$

$x=2$ & $x=3$ are the roots.

Vertex at $x=\frac{2+3}{2}=\frac{5}{2}$



from (1) Vertex $\left(\frac{5}{2}, \frac{1}{4}\right)$

x	0	1	2	$5/2$ ✓	3	4	5
y	-6	-2	0	$1/4$	0	-2	-6

Alternate Method 1: Vertex at

$$x=-\frac{b}{2a}=\frac{-5}{-2}=\frac{5}{2}\checkmark$$

Alternate 2. Completing the square:

$$y=-x^2+5x-6$$
$$=-[x^2-5x+--]-6$$
$$=-\left[x^2-5x+\left(\frac{5}{2}\right)^2-\frac{25}{4}\right]-6$$
$$=-\left[\left(x-\frac{5}{2}\right)^2\right]+\frac{25}{4}-6$$
$$=-\left(x-\frac{5}{2}\right)^2+\frac{1}{4}$$

(Comparing with $a(x-h)^2+k$)

$$V\left(\frac{5}{2}, \frac{1}{4}\right)$$

Quadratic functions

§ Graph of Quadratic Function: $y = ax^2 + bx + c$; $a \neq 0$

$a < 0$, Graph is a parabola opening downwards.

§ Case II when $b^2 - 4ac = 0$ the curve touches the x-axis.

Example (21) $y = -x^2 + 4x - 4$ —— ①

Consider $-x^2 + 4x - 4 = 0$

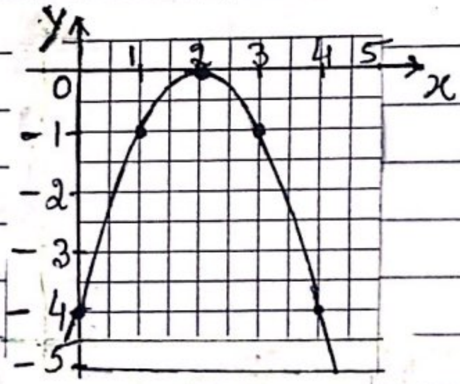
or $x^2 - 4x + 4 = 0$

or $(x-2)^2 = 0$

$\therefore$ Solution is $x = 2$ & 2

Vertex is at $(2, 0)$

x	0	1	2	3	4
y	−4	−1	0	−1	−4



The graph touches the x-axis at $(2,0)$ and opening downwards.

$a < 0$

§ Case III. $b^2 - 4ac < 0$ The curve does not intersect the x-axis and curve lies below x-axis

Example 22. Draw the graph of $y = -4x^2 + 12x - 11$

the corresponding eqn in this case when $b^2 - 4ac < 0$ has no solution.

$\left\{ b^2 - 4ac = 12^2 - 4(-4)(-11) = 144 - 176 = -32 < 0 \right.$

Using completing the square method.

$$y = -4x^2 + 12x - 11$$

$$= -4\left[x^2 - 3x\right] - 11$$

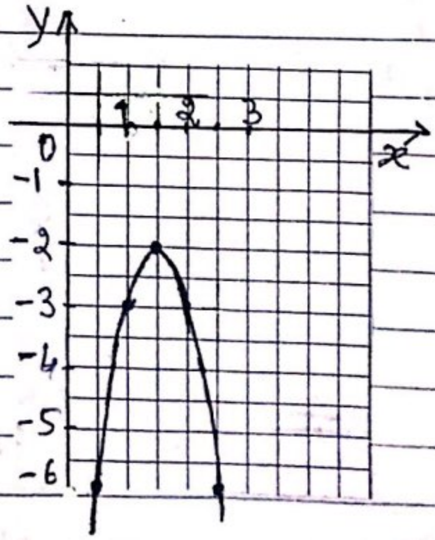
$$= -4\left[x^2 - 3x + \left(\frac{3}{2}\right)^2 - \frac{9}{4}\right] - 11$$

$$= -4\left[\left(x - \frac{3}{2}\right)^2\right] + 9 - 11$$

$$= -4\left(x - \frac{3}{2}\right)^2 - 2 \qquad \left[a(x-h)^2 + k\right] \text{ Vertex}(h, k)$$

$\therefore$ Vertex $\left(\frac{3}{2}, -2\right)$

x	0.5	1	3/2	2	2.5
y	−6	−3	−2	−3	−6



Alternatively: Vertex $x = \frac{-b}{2a}$

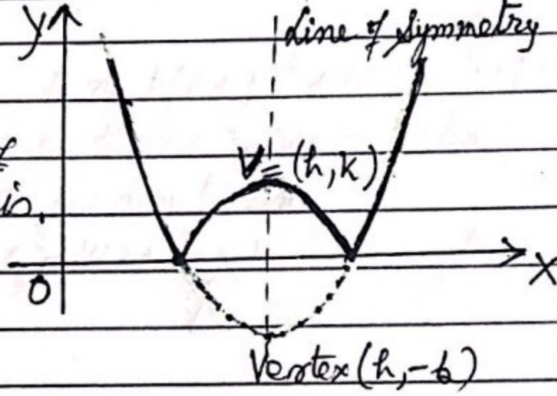
or $x = \frac{-12}{-8} = \frac{3}{2}$ ✓

Quadratic Functions

§ Graph of modulus of a Quad. Function: $y=|ax^2+bx+c|$

Draw the graph of $y=ax^2+bx+c$

Now the graph of $y=|ax^2+bx+c|$ is obtained by taking the reflection of the graph below x-axis → above x-axis.



Example 23 (i) Express $4x^2+8x-5$ in the form $p(x+q)^2+r$, where p, q, r are constants to be found. ---[3]

(ii) State the coordinates of the vertex of $y=|4x^2+8x-5|$ ---[2]

(iii) Sketch the graph of $y=|4x^2+8x-5|$, showing the coordinates of the points where the curve meets the axes. $\boxed{S-16/21/Q6}$ ---[3]

Solution (i) Consider $4x^2+8x-5 = 4[x^2+2x\ \ ]-5$

$=4[x^2+2x+1^2-1]-5$

$=4[(x+1)^2-1]-5$

$=4(x+1)^2-4-5$

$=4(x+1)^2-9$ — 1 ✓

Vertex of $y=4x^2+8x-5$ is $(-1,-9)$

(ii) Vertex of $y=|4x^2+8x-5|$ is the reflection of $(-1,-9)$ in x-axis.

$=V(-1,9)$ ✓

(iii) Consider $y=|4x^2+8x-5|$

x	-3	-2.5	-1	0	.5	1
y	7	0	9	5	0	7

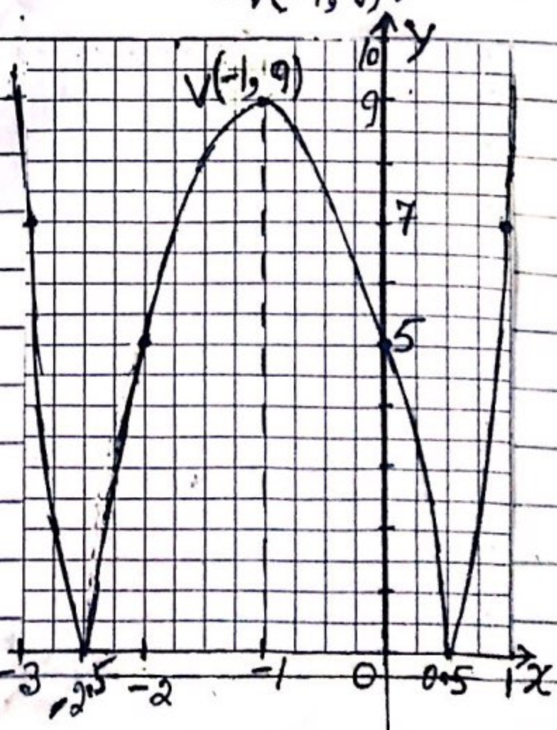
Consider $4x^2+8x-5=0$

$4x^2+10x-2x-5=0$

$2x(2x+5)-1(2x+5)=0$

$(2x+5)(2x-1)=0$

roots $x=\frac{1}{2}, -\frac{5}{2}$



Graph of Quadratic function: $y = |ax^2+bx+c|$

Example 24 (i) Express $5x^2-14x-3$ in the form $p(x+q)^2+r$. ---[3]
(ii) Sketch the graph of $y=|5x^2-14x+3|$ on the axes, showing clearly any points where your graph meets the coordinates axes. ---[4]
(iii) State the values of k for which $|5x^2-14x+3| = k$ has exactly four solutions. $\boxed{S\text{-}18/21/Q9}$ ---[2]

Solution (i) $5x^2-14x-3 = 5\left(x^2-\frac{14}{5}x\right)-3$ $\qquad\left(\frac{1}{2}\times\frac{14}{5}=\frac{7}{5}\right)$

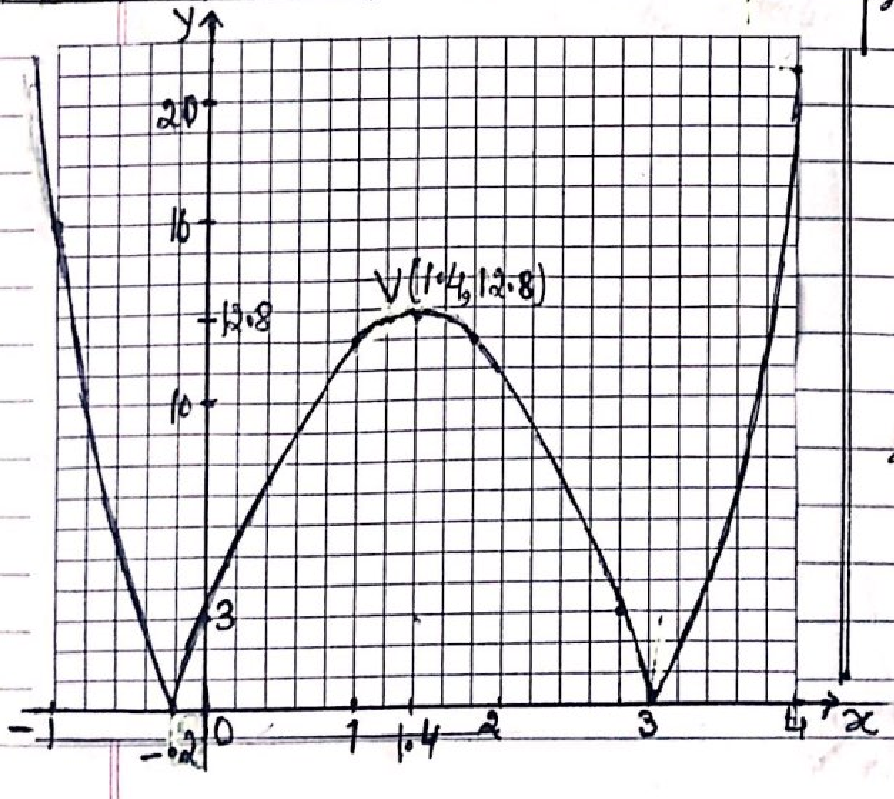
$$= 5\left[x^2-\frac{14}{5}x+\left(\frac{7}{5}\right)^2-\left(\frac{7}{5}\right)^2\right]-3$$
$$= 5\left[\left(x-\frac{7}{5}\right)^2-\frac{49}{25}\right]-3$$
$$= 5\left(x-\frac{7}{5}\right)^2-\frac{49}{5}-3$$
$$= 5\left(x-\frac{7}{5}\right)^2-\frac{64}{5} \quad \text{— (1)} \rightarrow \left(\frac{7}{5}, -12.8\right)$$

Vertex of $y = 5x^2-14x-3 \longrightarrow \left(\frac{7}{5}, -\frac{64}{5}\right)$ for (1)

(ii) $\therefore$ Vertex of $y=|5x^2-14x-3|$ is $V\left(\frac{7}{5}, 12.8\right) = (1.4, 12.8)$ ✓

x	-1	-0.2	0	1.4	2	3	4
y	16	0	3	12.8	+11	0	+21

Consider $5x^2-14x-3=0$
$\Rightarrow 5x^2-15x+x-3=0$
$\Rightarrow (x-3)(5x+1)=0$
$x=-0.2$ and $x=3$ ✓ are point of Int with X-axis



(iii) $|5x^2-14x+3| = k$ has exactly four solutions for,

$$0 < k < \frac{64}{5} \checkmark$$

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Quadratic functions

Example 25 (a) Find the set of values of x for which $4x^2+19x-5 \leq 0$ ---[3]

(b) (i) Express x^2+8x-9 in the form $(x+a)^2+b$, where a & b are integers. --[2]

(ii) Use your answer to part (i) to find the greatest value of $9-8x-x^2$ and the value of x at which this occurs. --[2]

(iii) Sketch the graph of $y=9-8x-x^2$, indicating the coordinates of any point of intersection with the coordinates axes. [S-15/21/Q9] --[2]

Solution (a) Given $4x^2+19x-5 \leq 0$

or $(4x-1)(x+5) \leq 0$

or $4\left(x-\frac{1}{4}\right)(x+5) \leq 0$

zeros of the corresponding eqn are $\frac{1}{4}$ & -5

$\therefore$ Solution of $4x^2+19x-5 \leq 0$ is $-5 \leq x \leq \frac{1}{4}$ ✓

(b)(i) x^2+8x-9

$= x^2+8x+4^2-16-9$

$= (x+4)^2-25$ ✓ ——(1)

(ii) $9-8x-x^2$

$= -[x^2+8x-9]$

$= -[(x+4)^2-25]$ fm (1)

$= 25-(x+4)^2$ ——(2)

$\therefore$ Greatest Value $= 25$ at $x=-4$.

(iii) $y=9-8x-x^2$

or $y = 25-(x+4)^2$ — fm (2)

Vertex of the Parabola $V(-4,25)$ ✓

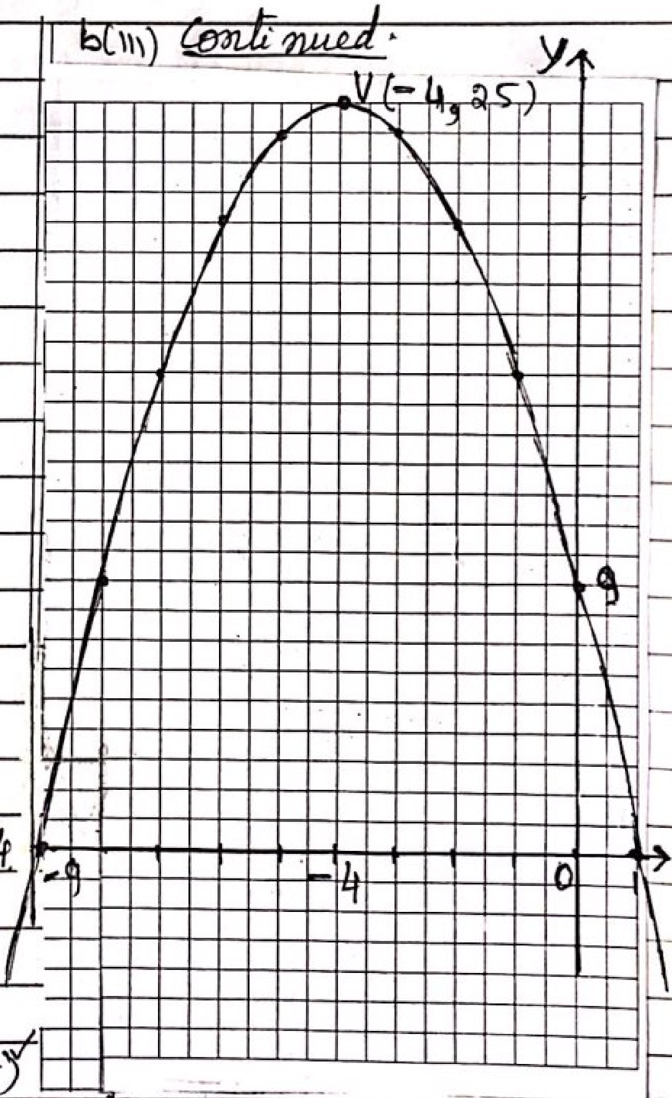
is opening down wards as $a<0$ (Coeff of x^2)

also $y = 25-(x+4)^2=0$

$\Rightarrow (x+4) = \pm 5$

$x=-9, 1$ are the values of x where the curve meet x-axis ✓

b(iii) Continued.



x	−10	−9	−4	0	1
y	−11	0	25	9	0

Curve meet the y-axis at $(0,9)$ ✓

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Simultaneous equations

(One linear and one quad.)

Example 26. Find the set of values of k for which the line $y=3x+k$ and the curve $y=2x^2-3x+4$ do not intersect. [M-17/22/Q4] --[4]

Solution: Given $y=3x+k$ —— ①

$y=2x^2-3x+4$ —— ②

Eliminating y from ① and ②

$3x+k=2x^2-3x+4$

or $2x^2-6x+(4-k)=0$ —— ③

line don't intersect the curve for the eqn ③ has no solution.

$\Rightarrow b^2-4ac<0$

$\Rightarrow (-6)^2-4\times2\times(4-k)<0$

$\Rightarrow 36-32+8k<0$

$\Rightarrow 8k<-4$

$\Rightarrow k<-\frac{1}{2}$ ✓

$\left\{\begin{array}{l} a=2 \\ b=-6 \\ c=(4-k) \end{array}\right.$

Example 27. Find the coordinates of the points where the line $2y-3x=6$ intersects the curve $\frac{x^2}{4}+\frac{y^2}{9}=5$ [S-18/22/Q4] --[5]

Solution: Give $2y-3x=6$

or $y=\left(\frac{6+3x}{2}\right)$ —— ①

and $\frac{x^2}{4}+\frac{y^2}{9}=5$ —— ②

from ① & ② $\frac{x^2}{4}+\frac{1}{9}\left(\frac{6+3x}{2}\right)^2=5$

$\Rightarrow x^2+2x-8=0$

$(x+4)(x-2)=0$

$\Rightarrow x=2$ or $x=-4$

from ① $x=2, y=6$

and $x=-4, y=-3$

∴ Points of Intersection are $(2,6)$ and $(-4,-3)$

Example 28. The line $y = kx+3$, where k is a positive constant, is a tangent to the curve $x^2 - 2x + y^2 = 8$, at point P.

(i) Find the value of k. --[4]

(ii) Find the coordinates of P --[3]

(iii) Find the equation of normal to the curve at P. [W-17/21/Q11] --[2]

Solution: $y = kx+3$ —— ①

(i) $x^2 - 2x + y^2 = 8$ —— ②

fm ① & ② $x^2 - 2x + (kx+3)^2 = 8$

or $(1+k^2)x^2 + (6k-2)x + 1 = 0$ —— ③

As the line ① is a tangent to the curve ②

they will intersect at one point only.

$\therefore$ fm ③ $b^2 - 4ac = 0$

$\Rightarrow (6k-2)^2 - 4(1+k^2)\times 1 = 0$

$\Rightarrow 32k^2 - 24k = 0$

$\Rightarrow 8k(4k-3) = 0$

$\Rightarrow k = \frac{3}{4}$ ✓ or $k = 0$ ✗ (given k is +)

(ii) for ③ has only one solution, for $k = 3/4$

$x = -\frac{b}{2a} = \frac{-(6k-2)}{2(1+k^2)} = \frac{-(6\times\frac{3}{4} - 2)}{2(1+(\frac{3}{4})^2)} = \frac{-2.5}{2\times 1.5625} = -0.8$ ✓

fm ① $y = \frac{3}{4}x + 3$ —— ④

$= 0.75\times(-0.8) + 3 = 2.4$ $\therefore P(-0.8, 2.4)$ ✓

(iii) fm ① gradient of tangent $= \frac{3}{4}$

$\therefore$ grad of Normal $= -\frac{4}{3}$

$\therefore$ Eqn of normal at $P(-0.8, 2.4)$ is $y - 2.4 = -\frac{4}{3}(x + 0.8)$

$\Rightarrow 3y = 4 - 4x$

or $4x + 3y = 4$ ✓

Simultaneous equations

Example 29. Find the values of a for which the line $y = ax + 9$, intersects the curve $y = -2x^2 + 3x + 1$ at two distinct points. --- [4]

[M-16/12/Q1]

Solution: Given line $y = ax + 9$ — ①

and curve $y = -2x^2 + 3x + 1$ — ②

Solving ① & ②

$$-2x^2 + 3x + 1 = ax + 9$$

$$\Rightarrow 2x^2 + (a-3)x + 8 = 0 \text{ — ③}$$

as the line ① intersects curve ② at two distinct points,

hence for ③ $b^2 - 4ac > 0$

$$\Rightarrow (a-3)^2 - 4 \times 2 \times 8 > 0$$

$\left\{ \text{fm ③ } a = 2,\ b = a - 3,\ c = 8 \right.$

$$\Rightarrow (a-3)^2 > 64$$

$$\Rightarrow (a-3)^2 > 8^2$$

$\left[\because x^2 > a^2 \Rightarrow x > a \text{ or } x < -a \right.$

$$\Rightarrow a - 3 > 8 \text{ or } a - 3 < -8$$

$$\Rightarrow a > 11,\ a < -5 \checkmark$$

Example 30. Solve $xy = 3$ and $x^4 y^5 = 486$ [SP-20/02/Q1] --- [3]

Solution. $xy = 3 \Rightarrow y = \frac{3}{x}$ — ①

and $x^4 y^5 = 486$ — ②

from ① and ② $x^4 \cdot \left(\frac{3}{x}\right)^5 = 486$

$$\Rightarrow \frac{243x^4}{x^5} = 486$$

$$\Rightarrow x = \frac{1}{2} \checkmark$$

fm ① $y = \frac{3}{x} = \frac{3}{\frac{1}{2}} = 6 \checkmark$

$$\therefore x = \frac{1}{2},\ y = 6.$$

Example 31. Solve the simultaneous equations:

$$\log_2(x+4) = 2\log_2 y$$

and $\log_2(7y-x) = 4$ ---[5]

[W-17/22/Q4]

Solution. Given $\log_2(x+4) = 2\log_2 y$

$$\Rightarrow \log_2(x+4) = \log_2 y^2$$

$$\Rightarrow x+4 = y^2 \Rightarrow x = y^2-4 \quad \text{—— (1)}$$

Also $\log_2(7y-x) = 4$

$$\Rightarrow (7y-x) = 2^4$$

$$\Rightarrow 7y-x = 16 \quad \text{—— (2)}$$

fr (1) in (2) $\quad 7y-(y^2-4) = 16$

$$\Rightarrow y^2-7y+12 = 0$$

or $(y-4)(y-3) = 0$

$$y = 3, 4$$

fr (1) $\quad y=3 \Rightarrow x = 3^2-4 = 5 \quad \Rightarrow x=5 \;\&\; y=3$ ✓

and $\quad y=4 \Rightarrow x = 4^2-4 = 12 \quad \Rightarrow$ and $x=12 \;\&\; y=4$ ✓

Example 32. Solve the following simultaneous equations:

$\log_2(x+3) = 2+\log_2 y$ and $\log_2(x+y) = 3$ [W-14/21/Q3] --[5]

Solution: Given $\log_2(x+3) = 2+\log_2 y$

$$\Rightarrow \log_2(x+3) - \log_2 y = 2$$

$$\Rightarrow \log_2\left(\frac{x+3}{y}\right) = 2$$

$$\Rightarrow \frac{x+3}{y} = 2^2$$

$$\Rightarrow x+3 = 4y \quad \text{—— (1)}$$

Also $\log_2(x+y) = 3$

$$\Rightarrow x+y = 2^3$$

$$\Rightarrow x+y = 8 \quad \text{—— (2)}$$

Now fr (2) $y = 8-x$ put in (1)

$$x+3 = 4(8-x)$$

$$5x = 29$$

$$x = \frac{29}{5} = 5.8 \checkmark$$

fr (2) $\quad 5.8+y = 8$

$$y = 2.2 \checkmark$$

$\therefore \quad x = 5.8, \; y = 2.2$

Simultaneous Equations

Example 33. Solve the simultaneous equations.

$$\frac{4^x}{256^y} = 1024 \text{ and } 3^{2x} \times 9^y = 243 \qquad \boxed{W-13/21/Q5} \quad \text{---[5]}$$

Solution: Given $\frac{4^x}{256^y} = 1024$

$$\Rightarrow \frac{4^x}{(4^4)^y} = 4^5 \Rightarrow 4^{x-4y} = 4^5$$

$$\Rightarrow x - 4y = 5 \text{ --- (1)}$$

Also given $3^{2x} \cdot 9^y = 243$

$$\Rightarrow 3^{2x} \cdot (3^2)^y = 3^5$$

$$\Rightarrow 3^{(2x+2y)} = 3^5$$

$$\Rightarrow 2x + 2y = 5 \text{ --- (2)}$$

Solving (1) and (2) $\left.\begin{array}{l} x = 3 \\ y = -0.5 \end{array}\right\} \checkmark$