IG_0606
Additional Maths

Series_1
Binomial Theorem
Notes

Reviewed By.
Mr Madan Saini,
(Maths Deptt.)
A.W.S.

Suresh Goel
(Director)
Alliance World School,
Noida, NCR-Delhi-India.

Binomial Theorem

§ Consider the following binomials:

(i) $(x+a)^1 = \underset{^1C_0}{x} + \underset{^1C_1}{a}$ → degree of each term 1

(ii) $(x+a)^2 = \underset{^2C_0}{x^2} + \underset{^2C_1}{2xa} + \underset{^2C_2}{a^2}$ → '2'

(iii) $(x+a)^3 = \underset{^3C_0}{x^3} + \underset{^3C_1}{3x^2a} + \underset{^3C_2}{3xa^2} + \underset{^3C_3}{a^3}$ degree of each term is '3'.

(iv) $(x+a)^4 = \underset{^4C_0}{x^4} + \underset{^4C_1}{4x^3a} + \underset{^4C_2}{6x^2a^2} + \underset{^4C_3}{4xa^3} + \underset{^4C_4}{a^4}$ (deg of each term $=4$)

Note:
Degree of $5x = 1$
degree of $5x^2a = 3$
degree of $2a = 1$
degree of $12x^3a^2 = 5$

Note:
(i) $^nC_r = \dfrac{n!}{r!(n-r)!}$
(ii) $^nC_r = {}^nC_{n-r}$

§ Binomial Theorem:

$$(x+a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} a^2 + \cdots + {}^nC_r x^{n-r} a^r + \cdots + {}^nC_n a^n$$

(degree of each term $= n$)

No. of terms $= (n+1)$.

General term $= {}^nC_r x^{n-r} a^r$, $r = 0, 1, 2, \ldots n$.

Some Particular cases:

§

(i) $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_r x^r + \cdots + {}^nC_n x^n$

$(r+1)_{th}$ term $=$ General term $= {}^nC_r x^r$.

Coefficient of $x^r = {}^nC_r$

(ii) $(1+ax)^n = {}^nC_0 + {}^nC_1(ax) + {}^nC_2(ax)^2 + \cdots$

Example 1 The first three terms in the expansion of $(2+ax)^n$ are equal to $1024-1280x+bx^2$, where a, b and n are constants.

(i) Find the value of n, a and b. ---[5]

(ii) Hence find the term independent of x in the expansion of $(2+ax)^n\left(x-\frac{1}{x}\right)^2$ $\boxed{M\text{-}18/12/Q5}$ --[3]

Solution: $(2+ax)^n = {}^nC_0\, 2^n + {}^nC_1\, 2^{n-1}\cdot ax + {}^nC_2\, 2^{n-2}\cdot (ax)^2 + \cdots$

$= 1024 - 1280x + bx^2 \cdots$

Comparing the first term, ${}^nC_0\, 2^n = 1024$

$\Rightarrow 1\times 2^n = 2^{10} \Rightarrow \underline{n=10}$ ✓ —— (i)

Second term, ${}^nC_1\, 2^{n-1}\cdot a\cancel{x} = -1280\cancel{x}$

$\Rightarrow n\cdot 2^{n-1}\cdot a = -1280$ (from (i) $n=10$)

$\Rightarrow 10\times 2^9\cdot a = -1280$

$a = \frac{-1280}{10\times 512} \Rightarrow \underline{a = -\frac{1}{4}}$ ✓ —— (ii)

third term, ${}^nC_2\cdot 2^{n-2}\cdot (ax)^2 = bx^2$

$\Rightarrow {}^{10}C_2\cdot 2^8\cdot a^2\, \cancel{x^2} = b\cancel{x^2}$

$\Rightarrow 45\times 256\times\left(-\frac{1}{4}\right)^2 = b \Rightarrow \underline{b = 720}$ ✓

(ii) Given $(2+ax)^n\cdot\left(x-\frac{1}{x}\right)^2 = \left(2-\frac{x}{4}\right)^{10}\cdot\left(x-\frac{1}{x}\right)^2$

$= \left(x^2+\frac{1}{x^2}-2\right)\left[1024-1280x+720x^2 \cdots\right]$

$\therefore$ Term Independent of $x = \frac{1}{x^2}\times 720x^2 - 2\times 1024$

$= 720 - 2048$

$= \underline{-1328}$ ✓

Example 2. In the expansion of $(1+2x)^n$, the coefficient of x^4 is ten times the coefficient of x^2, find the value of the positive integer n, [SP-20/01/Q9] ---[6]

Solution. We know $(1+ax)^n = {}^nC_0 + {}^nC_1(ax) + {}^nC_2(ax)^2 + \dots {}^nC_r(ax)^r - - -$

and the coefficient of $(ax)^2 = {}^nC_2$ ——— (1)

$\therefore$ In the expansion of $(1+2x)^n$.

Term containing $x^2 = {}^nC_2(2x)^2 = {}^nC_2 \times 4x^2$ ——— (2)

Term containing $x^4 = {}^nC_4(2x)^4 = {}^nC_4 \times 16x^4$ ——— (3)

Given Coeff of $x^4 = 10 \times$ Coeff. of x^2

$\Rightarrow 16 \cdot {}^nC_4 = 4 \cdot {}^nC_2 \times 10$ fr (2) & (3)

$\Rightarrow \frac{16 \cdot n(n-1)(n-2)(n-3)}{4 \times 3 \times 2 \times 1} = \frac{4 \times n(n-1)}{2 \times 1} \times 10$

$\Rightarrow \frac{2(n-2)(n-3)}{3} = 20$

$\Rightarrow n^2 - 5n + 6 = 30$

$\Rightarrow n^2 - 5n - 24 = 0$

$(n-8)(n+3) = 0$

$n = 8$ or $n = -3^{\times}$ as n is +

$\therefore n = 8$ ✓

Binomial Theorem

Example 3 (i) Find the first 3 term in the expansion of $\left(2x-\frac{1}{16x}\right)^8$ in the descending powers of x. --- [3]

(ii) Hence find the coefficient of x^4 in the expansion of,
$$\left(2x-\frac{1}{16x}\right)^8 \cdot \left(\frac{1}{x^2}+1\right)^2$$ --- [3]

$\boxed{S-18/11/Q9}$

Solution: $\left(2x-\frac{1}{16x}\right)^8 = (2x)^8 + {}^8C_1(2x)^7\cdot\left(-\frac{1}{16x}\right) + {}^8C_2(2x)^6\cdot\left(\frac{-1}{16x}\right)^2 + \cdots$

(i)
$$= 256x^8 - 8\times 128x^7\cdot\frac{1}{16x} + 28\times 64x^6\cdot\frac{1}{256x^2} + \cdots$$
$$= 256x^8 - 64x^6 + 7x^4. \checkmark \longleftarrow ①$$

(ii) $\left(2x-\frac{1}{16x}\right)^8\cdot\left(\frac{1}{x^2}+1\right)^2$

$$= \left(256x^8 - 64x^6 + 7x^4 \cdots\right)\left(\frac{1}{x^4}+\frac{2}{x^2}+1\right) \quad \text{fr ①}$$

terms involving $x^4 = 256x^8\times\frac{1}{x^4} - 64x^6\times\frac{2}{x^2} + 7x^4\times 1$

$$= (256\times 1 - 64\times 2 + 7\times 1)x^4$$

$\therefore$ Coeff. of $x^4 = 256 - 128 + 7$
$= 135 \checkmark$

Example 4: Find the coefficient of x^5 in the expansion of $(3-2x)^8$ -- [2]

$\boxed{S-14/21/Q6(a)}$

Solution: General Term in the expansion of $(3-2x)^8$
$$= {}^8C_r\cdot 3^{8-r}\cdot(-2x)^r$$

General Term in $(x+a)^n = {}^nC_r\, x^{n-r}\cdot a^r$

for term involving x^5, $r=5$
$$= {}^8C_5\cdot 3^{8-5}\cdot(-2x)^5$$
$$= {}^8C_5\times 3^3\times(-2)^5\cdot x^5$$

${}^8C_5 = {}^8C_{8-5} = {}^8C_3$
$= \frac{8\times 7\times 6}{3\times 2\times 1}$
$= 56 \checkmark$

$\therefore$ Coeff. of $x^5 = {}^8C_3\times 27\times(-32)$
$= -56\times 27\times 32 = -48384 \checkmark$

Binomial Theorem.

Example 5 (a) Given that the first 4 terms in the expansion of $(2+kx)^8$ are $256+256x+px^2+qx^3$, find the value of k, p and q. ---[3]

(b) Find the term that is independent of x, in the expansion of $\left(x-\frac{2}{x^2}\right)^9$ --[3]

W-15/13/Q8

Solution: Consider,

(a) $(2+kx)^8 = 2^8 + {}^8C_1 \cdot 2^7 (kx)^1 + {}^8C_2\, 2^6 \cdot (kx)^2 + {}^8C_3\, 2^5 (kx)^3 + \cdots$

$= 256 + 8\times 128 \times kx + 28\times 64 \times k^2x^2 + 56\times 32 \times k^3x^3$

$= 256 + 1024kx + 1792k^2x^2 + 1792k^3x^3$

$= 256 + 256x + px^2 + qx^3$

Comparing the coeff. of x, x^2 and x^3

Coeff of x, $1024k = 256 \Rightarrow \underline{k = \frac{1}{4}}$ ✓

Coeff of x^2, $p = 1792k^2 = 1792\times\left(\frac{1}{4}\right)^2 = 112 \Rightarrow \underline{p = 112}$ ✓

Coeff of x^3, $q = 1792k^3 = 1792\times\left(\frac{1}{4}\right)^3 = 28 \Rightarrow \underline{q = 28}$ ✓

(b) To find the term independent of x in $\left(x-\frac{2}{x^2}\right)^9$

General term in the expansion of $\left(x-\frac{2}{x^2}\right)^9 = {}^9C_r\, x^{9-r} \cdot \left(\frac{-2}{x^2}\right)^r$

$= {}^9C_r \cdot x^{9-r} \cdot (-2)^r \cdot x^{-2r}$

$= {}^9C_r\, (-2)^r \cdot x^{9-3r}$ —— (1)

for term independent of x, Exponent of $x = 0$

$\Rightarrow 9-3r=0$

$\Rightarrow r = 3$ ✓

$\therefore$ from (1) the term independent of x

$= {}^9C_3 (-2)^3 \cdot x^{9-3\times 3}$

$= -84\times 8 \times x^0$ $(x^0 = 1)$

$= \underline{-672}$ ✓

Binomial Theorem.

Example 6(i) Given that n is a positive integer, find the first 3 terms in the expansion of $(1+\frac{1}{2}x)^n$ in ascending powers of x. ---[2]

(ii) Given that the coefficient of x^2 in the expansion of $(1-x)(1+\frac{1}{2}x)^n$ is $\frac{25}{4}$, find the value of n. $\boxed{S-13/12/Q9}$ ---[5]

Solution (i) $(1+\frac{1}{2}x)^n = 1+n(\frac{1}{2}x)+\frac{n(n-1)}{2!}(\frac{1}{2}x)^2+\cdots$

$$= 1+\frac{n}{2}x+\frac{n(n-1)}{8}x^2+\cdots \quad (1)$$

(ii) Now Consider.

$$(1-x)(1+\tfrac{1}{2}x)^n = (1-x)\left[1+\frac{n}{2}x+\frac{n(n-1)}{8}x^2+\cdots\right] \text{ from (1)}$$

terms in $x^2 = 1\times\frac{n(n-1)}{8}x^2 - x\times\frac{n}{2}x$

$$= \left(\frac{n(n-1)}{8}-\frac{n}{2}\right)\cdot x^2$$

$\therefore$ Coeff of x^2, $\frac{n(n-1)}{8}-\frac{n}{2}=\frac{25}{4}$

$$\Rightarrow \quad n^2-5n-50=0$$

$$(n-10)(n+5)=0$$

$$n=10 \checkmark \text{ or } n=-5 \times$$

Example 7. Find the value of the term that is independent of x in the expansion of $\left(2x+\frac{1}{4x^3}\right)^{12}$. $\boxed{W-14/13/Q9(b)}$ ---[3]

In the expansion of $(x+a)^n$

Gen. Term $= {}^nC_r\, x^{n-r}\cdot a^r$

Solution: The General term in the expansion of

$$\left(2x+\frac{1}{4x^3}\right)^{12} = {}^{12}C_r\,(2x)^{12-r}\cdot\left(\frac{1}{4x^3}\right)^r$$

$$= {}^{12}C_r\cdot 2^{12-r}\cdot x^{12-r}\cdot\left(\frac{1}{4}\right)^r\cdot x^{-3r}$$

$$= {}^{12}C_r\cdot 2^{12-r}\cdot 4^{-r}\cdot x^{12-4r} \quad (1)$$

for the term independent of x,

exponent of $x=0$

from (1) $\quad 12-4r=0$

$$r=3$$

$\therefore$ from (1) Independ of $x = {}^{12}C_3\cdot 2^{12-3}\cdot 4^{-3}\cdot x^0$

$$= {}^{12}C_3\cdot 2^9\cdot\frac{1}{4^3}$$

$$= 220\times 8$$

$$= 1760 \checkmark$$

Example 8 (i) Given that a is a constant, expand $(2+ax)^4$, in ascending powers of x, simplify each term of your expansion. --[2]

Given also that the coefficient of x^2 is equal to the coefficient of x^3.

(ii) Show that $a=3$ --[1]

(iii) Use your expansion to show that the value of 1.97^4 is 15.1 to 1 decimal place. $\boxed{S\text{-}17/21/Q5}$ --[2]

Solution: (i) $(2+ax)^4 = 2^4 + {}^4C_1\, 2^3 \cdot (ax) + {}^4C_2\, 2^2 (ax)^2 + {}^4C_3\, 2^1 \cdot (ax)^3 + {}^4C_4\, (ax)^4$

or $(2+ax)^4 = 16 + 32ax + 24a^2x^2 + 8a^3x^3 + a^4x^4$ ✓ (1)

(ii) Given Coeff of x^2 = Coeff of x^3.

$\therefore \quad 24a^2 = 8a^3$ for (1)

$\Rightarrow \underline{a = 3}$ ✓

(iii) Now $(2+ax)^4 = (2+3x)^4$ for $a=3$

$= [2+3\times(-0.01)]^4$ for $x = -0.01$

$= (2-0.03)^4$

$= (1.97)^4$

$\therefore$ To get the value of $(1.97)^4$ Put $a=3$ and $x=-0.01$ in (1)

we get

$(1.97)^4 = 16 + 32\times3\times(-0.01) + 24\times3^2\times(-0.01)^2 + 8\times3^3\times(-0.01)^3 +$

$= 16 - 0.96 + .0216 + - + -$

$= 15.0616$

$= 15.1$ (1 decimal place)

$= \underline{15.1}$ ✓