IG 0606
Additional Maths

Series-2
A.P. and G.P.
Notes

Reviewed by
Mr Madan Saini
(Maths Deptt),
A.W.S.

Suresh Goel (Director)
Alliance World School.
Noida, Delhi-NCR. India.

§ Sequences (or Progressions).

Consider the following collection of numbers:

(i) 1, 4, 9, 16, 25, --- n^2 → here n is a positive integer.

(ii) 2, 5, 8, 11, 14, ---$(3n-1)$ → the difference between any two consecutive terms is 3.

(iii) 3, 6, 12, 24, 48, --- $3\times 2^{n-1}$ → the ratio two consecutive terms is 2.

(iv) $\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, --- \frac{n}{n+2}$.

In each of the above collection, numbers are arranged in a definite order with some rule.
These are called **sequences** (or progressions).
The terms of a sequence are denoted by; $t_1, t_2, t_3 --- t_n, t_{n+1}$ --
(or $a_1, a_2, a_3, --- a_n, a_{n+1}, --$)

First term $= t_1$

Second term $= t_2$

General term or n_{th} term $= t_n$

next to it $(n+1)_{th}$ term $= t_{n+1}$.

---

§ Arithmetic Sequence (or Arithmetic Progression – A.P)

Consider a sequence 7, 10, 13, 16, 19, ----
The difference $10-7 = 13-10 = 16-13 = $ -- 3 is constant
is defined as common difference

General Form of A.P: $a, (a+d), (a+2d), (a+3d)$
Here the first term $= a$
common difference $= d$

$$\boxed{n_{th} \text{ term of A.P.} = a + (n-1)d}$$

Scanned with CamScanner

Arithmetic Progression.

§ Arithmetic Series (or Sum of first 'n' terms of a Arithmetic seq.) denoted by S_n.

Given an arithmetic series.

$$S_n = a + (a+d) + (a+2d) - - - n \text{ terms}.$$

$$\boxed{S_n = \frac{n}{2}[2a+(n-1)d] \text{ or } \frac{n}{2}[a+l] \quad \therefore (l = n_{th} \text{ term} = a+(n-1)d)}$$

Proof:

$$S_n = a + (a+d) + (a+2d) + - - + (l-2d) + (l-d) + l \quad \text{--- (1)}$$

or $$S_n = l + (l-d) + (l-2d) + - - - + (a+2d) + (a+d) + a \quad \text{--- (2)}$$

add (1) & (2)

$$2S_n = (a+l) + (a+l) - - - - \; n \text{ terms}$$

$$2 S_n = n(a+l)$$

$$S_n = \frac{n}{2}(a+l) \text{ ———— ✓}$$

or $$S_n = \frac{n}{2}\left[a + \left(a+(n-1)d\right)\right] \quad \left\{ \begin{array}{l} \text{as } l = n_{th} \text{ term} \\ = a+(n-1)d \end{array}\right.$$

or $$S_n = \frac{n}{2}[2a + (n-1)d] ✓$$

---

§ $$\Sigma n = 1+2+3+ - - n \quad \left\{ \begin{array}{l} a=1 \\ d=1 \\ n_{th} \text{ term} = n \end{array}\right.$$

$$= \frac{n}{2}[1+n]$$

$$\therefore \Sigma n = \frac{n(n+1)}{2} ✓$$

Example 1. Given an Arithmetic Progression, 7, 10, 13, 16, - - -

(i) Find 8_{th} term

(ii) Sum of first 8 terms.

Solution (i) $a = 7$, $d = 10-7 = 3$

$\therefore 8_{th}$ term $= a + (n-1)d$

$= 7 + (8-1)\times 3$

$= 7 + 7\times 3$

$= 28.$ ✓

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_8 = \frac{8}{2}[2\times 7 + (8-1)\times 3]$

$= 4[14+21]$

$= 4\times 35 = 140$ ✓

Example 2. An arithmetic progression has a first term of 5 and a common difference of -3, Find the number of terms, such that the sum of n terms is first less than -200. --[4]

SP-20/01/Q10(a)

Solution: Given first term $a = 5$

Common difference $d = -3$

number of terms $n = ?$

Given Sum of n_{th} terms $\frac{n}{2}[2a + (n-1)d] < -200$

or $\frac{n}{2}[2\times 5 + (n-1)\cdot(-3)] < -200$

or $n[10 - 3n + 3] < -400$

$\Rightarrow -3n^2 + 13n < -400$ $\left[\begin{array}{l} 3n^2 - 13n - 400 = 0 \\ n = -9.6, 13.9 \end{array} \right.$

$\Rightarrow 3n^2 - 13n - 400 > 0$

-9.6 13.9

$n = 13.9$

$\therefore$ nearest positive integral value of $n = 14$ ✓

Example 3: Given is an arithmetic progression, 32, 25, 18, 11, ---

Which term of the sequence is -332.

Solution: Given first term $a = 32$

and common difference $d = 25 - 32 = -7$

.... Let n_{th} term $= -332$ $\left[\begin{array}{l} n_{th} \text{ term} \\ = a + (n-1)d \end{array} \right.$

or

$32 + (n-1)(-7) = -332$

or $32 - 7n + 7 = -332$

or $-7n = -332 - 39$

$n = \frac{+371}{+7} = 53$ ✓

$\therefore$ Required term $= \underline{53rd}$

Geometric Progression

§ Geometric Sequence:

Consider a sequence of numbers.

2, 6, 18, 54, ----

observe $\frac{6}{2} = \frac{18}{6} = \frac{54}{18} = 3$ Constant ratio denoted by 'r' in General.

General form of a Geometric sequence:

$a, ar, ar^2, ar^3, ---$

First term $= a$

Second term $= ar$

third term $= ar^2$

$$\boxed{n_{th} \text{ term of geometric sequence, } t_n = a \cdot r^{n-1}}$$

Example 4. (i) Find the next two terms and the n_{th} term of the sequence, 2, 6, 18, 54. (ii) which term of the sequence is 4374

Solution: we not $\frac{6}{2} = \frac{18}{6} = \frac{54}{18} = 3$ constant

$\therefore$ Common ratio $r = 3$ ✓

Given first term $a = 2$

$\therefore$ 5_{th} term $= 54 \times 3 = 162$ ✓

and 6_{th} term $= 162 \times 3 = 486$ ✓

and the n_{th} term $= ar^{n-1}$

$= 2 \cdot 3^{n-1}$ ✓ ——— (1)

(ii) Let n_{th} term is 4374.

from (1) $2 \cdot 3^{n-1} = 4374$

or $3^{n-1} = \frac{4374}{2} = 2187 = 3^7$

$\therefore n - 1 = 7$

$\Rightarrow n = 8$ ✓

Geometric Series

§ Sum of first n terms of a geometric sequence, denoted by S_n

$S_n = a + ar + ar^2 + \text{---} + n$ terms.

$$S_n = \frac{a(r^n - 1)}{(r-1)} \quad \text{if } r > 1$$

$$\text{or} = \frac{a(1-r^n)}{(1-r)} \quad \text{if } r < 1$$

Proof: $S_n = a + \cancel{ar} + \cancel{ar^2} + \text{----} + ar^{n-1}$ —— ①

$-r \cdot S_n = \cancel{-ar} \pm \cancel{ar^2} + \text{---} \pm ar^{n-1} \pm ar^n$ —— ②

Subtract ② fr ①, $S_n(1-r) = a - ar^n$

or $S_n = \dfrac{a(1-r^n)}{(1-r)}$ ✓ ③ if $r < 1$

§ Sum of Infinite geometric series:

$S_\infty = a + ar + ar^2 + \text{---} \infty$ when $|r| < 1$

$S_\infty = \dfrac{a}{(1-r)}$ ✓ fr ③ [∵ If $|r| < 1$ or $-1 < r < 1$, $r^n \to 0$ when $n \to \infty$

Example 5. In a G.P, the 3rd term is 24 and the 6th term is 192. Find 10th term.

Solution: 3rd term of G.P, $ar^2 = 24$ —①

and 6th term $ar^5 = 192$ —②

② ÷ ① $\dfrac{ar^5}{ar^2} = \dfrac{192}{24} \Rightarrow r^3 = 8$

$\Rightarrow r = 2$ ✓, fr ① $a = 6$

Now 10th term $= ar^9 = 6 \times 2^9$

$= \underline{3072}$ ✓

Example 6. Find the sum of first 5 terms of the term of G.P.
$1 + \frac{1}{3} + \frac{1}{9} + ----$

Solution: Given Geometric Series $1 + \frac{1}{3} + \frac{1}{9} + --$

here first term $a = 1$

and common ratio $r = \frac{1}{3}/1 = \frac{1}{9}/\frac{1}{3} = \frac{1}{3}$ ✓

$\therefore$ Sum of first 5 terms, $n = 5$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\therefore S_5 = \frac{1\left[1-\left(\frac{1}{3}\right)^5\right]}{1-\frac{1}{3}} = \frac{1-\frac{1}{243}}{\frac{2}{3}}$$

$$= \frac{3}{2} \times \frac{242}{243} = \frac{121}{81} ✓$$

Example 7. A Geometric progression is such that its 3rd term is $81/64$ and 5_{th} term is equal to $\frac{729}{1029}$.

(i) Find first term of this progression and the positive common ratio of this progression. --- [5]

(ii) Hence find the sum to infinity of this progression. --- [1] $\boxed{SP\text{-}20|01|Q10(b)}$

Solution (i) Third term $ar^2 = \frac{81}{64}$ — ①

and 5_{th} term $ar^4 = \cancel{729}\frac{729}{1024}$ — ②

② ÷ ① $r^2 = \frac{729}{1024} \times \frac{64}{81} = \frac{9}{16}$

$\therefore r = \frac{3}{4}$ ✓

fr ① $a = \frac{9}{4}$ ✓

(ii) $S_\infty = \frac{a}{1-r} = \frac{\frac{9}{4}}{1-\frac{3}{4}} = \frac{\frac{9}{4}}{\frac{1}{4}} = \frac{9}{4} \times \frac{4}{1} = 9$

$\therefore S_\infty = 9$ ✓

§ **Three terms in G.P:**

Given three terms a, b, c are in G.P.

$\therefore$ common ratio. $\frac{b}{a} = \frac{c}{b} \Rightarrow \boxed{b^2 = ac}$

Example 8. The first, second and the third term of G.P. are $2k+6$, $2k$ and $k+2$ respectively, where k is a positive constant.

(i) Find the value of k ---[3]

(ii) Find the 6th term of G.P. --[3]

Solution: Given $(2k+6)$, $2k$ and $(k+2)$ are G.P $\left\{ \begin{array}{l} a, b, c \text{ are in G.P} \\ \frac{b}{a} = \frac{c}{b} \end{array} \right.$

$\Rightarrow \frac{2k}{2k+6} = \frac{k+2}{2k}$

$\Rightarrow 4k^2 = (k+2)(2k+6)$

$\Rightarrow k^2 - 5k - 6 = 0$

$\Rightarrow k = 6$ ✓ or $k = -1$ ✗ as k is positive.

(ii) The first and second terms

$2k+6$ and $2k$

$= 18$ and 12 $\quad (\because k = 6$

$\therefore$ Common ratio of G.P $= \frac{t_2}{t_1} = \frac{12}{18} = \frac{2}{3}$ ✓ $\Rightarrow r = \frac{2}{3}$ ✓

$\therefore$ 6th term of G.P $= ar^5$

$= 18 \times \left(\frac{2}{3}\right)^5 \quad [a = 18]$

$= \frac{64}{27}$ ✓

Example 9. How many terms of the G.P. $3, 3^2, 3^3, ---$ are needed to give the sum 120.

Solution $a = 3$, $r = 3$; $n = ?$

$S_n = \frac{a(r^n - 1)}{r-1} = 120$ given

or $\frac{3(3^n - 1)}{3-1} = 120 \Rightarrow 3^n - 1 = 80$

$\Rightarrow 3^n = 81$

$\Rightarrow n = 4$ ✓

§ Three terms in A.P.

Given three numbers a, b, c in A.P

$$\Rightarrow b-a=c-b \quad \text{—(i)}$$

$$\text{or} \quad \underline{2b=a+c} \quad \text{—(ii)}$$

Example 10. The first term of an arithmetic progression is $4x$ and the second term is x^2, with a common difference of 12. Find the possible values of x and the third term.

Solution: The three terms in A.P.

$$4x,\ x^2 \text{ and } x^2+12 \qquad (\text{as } d=12)$$

$$\Rightarrow x^2-4x=(x^2+12)-x^2$$

$$\Rightarrow x^2-4x-12=0$$

$$\Rightarrow x=-2 \text{ or } 6 \checkmark$$

$$\therefore \text{third term } x^2+12 = 16 \text{ and } 48 \checkmark \quad [\text{for } x=-2 \,\&\, 6]$$

Example 11. In an A.P the sum, S_n, of the first n terms is given by, $S_n = 2n^2+8n$. Find the first term and the common difference of the progression.

Solution: Given $S_n = 2n^2+8n$ —①

First term $a = S_1 = 2\times1^2+8\times1 = 10 \checkmark$

Second term $= S_2-S_1 = (2\times2^2+8\times2)-10$

$= 24-10 = 14 \checkmark$

$$\left\{\begin{array}{l} S_1 = a \\ S_2 = a+(a+d) \end{array}\right.$$

now $d = t_2-t_1$

$= 14-10$

Common difference. $\underline{d = 4} \checkmark$

Example 12. An A.P has first term 'a' and common difference 'd', It is given that the sum of first 200 terms is 4 times the sum of first 100 terms.
(i) Find 'd' in term of a
(ii) Find the 100th term in terms of a.

Solution (i) $S_n = \frac{n}{2}[2a+(n-1)d]$

$\therefore \quad S_{200} = \frac{200}{2}[2a+199d] = 100(2a+199d)$ —— ①

and $\quad S_{100} = \frac{100}{2}[2a+99d] = 50(2a+99d)$ —— ②

Given $S_{200} = 4 \times S_{100}$

$\Rightarrow 100[2a+199d] = 4\times 50[2a+99d]$ fm ① & ②

$\Rightarrow d = 2a$ ✓ —— ③

(ii) nth term of an A.P $= a+(n-1)d$

$\therefore$ 100th term $= a+99d \qquad d=2a$

$= a+99\times 2a \quad$ fm ③

$= 199a$ ✓

Example 13. The 12th term of an A.P. is 17 and the sum of first 31 terms is 1023. Find the sum of 31st term.

Solution: nth term of an A.P $= a+(n-1)d$

$\therefore$ 12th term $= a+11d = 17$ —— ①

$S_n = \frac{n}{2}[2a+(n-1)d]$

$\therefore S_{31} = \frac{31}{2}[2a+30d] = 1023$

or $2a+30d = 66$

or $a+15d = 33$ —— ②

Solving ① & ②, $d = 4, a = -27$ ✓

$\therefore$ 31st term $= a+(31-1)d = a+30d$

$= -27+30\times 4 = 93$ ✓

Example 14. The first, second and third terms of a G.P. are 1st, 9th and 21st terms respectively of an A.P. The first term of each progression is 8 and the common ratio of G.P is r, where $r \neq 1$. Find,

(i) the value of r,

(ii) the fourth term of each progression.

Solution: First three terms of G.P. $8, 8r$ and $8r^2$

and terms in AP $8, 8+8d$ and $8+20d$

Given $8+8d = 8r \Rightarrow d = (r-1)$ — ①

and $8+20d = 8r^2$

$\Rightarrow 8+20(r-1) = 8r^2$ [from ① $d = (r-1)$]

$\Rightarrow 2r^2 - 5r + 3 = 0$

$(2r-3)(r-1) = 0$

$r = 3/2$ ✓ and $r = 1$ ✗ as $r \neq 1$

$\therefore$ from ① $d = r - 1 = \frac{3}{2} - 1 = \frac{1}{2}$ ✓

(ii) 4th term of G.P $= ar^3$

$= 8 \times \left(\frac{3}{2}\right)^3 = \underline{27}$ ✓

and 4th term of A.P $= a + 3d$

$= 8 + 3 \times \frac{1}{2} = \underline{9.5}$ ✓

Example 15. The third and the fourth terms of a G.P. are $\frac{1}{3}$ and $\frac{2}{9}$ respectively. Find the sum to infinity of the progression.

Solution: Third term of G.P $ar^2 = \frac{1}{3}$ — ①

and the fourth term of GP $ar^3 = \frac{2}{9}$ — ②

② ÷ ① $r = \frac{2}{3}$ ✓

from ① $a = \frac{3}{4}$

$\therefore$ Sum to infinity $S_\infty = \frac{a}{1-r} = \frac{\frac{3}{4}}{1-\frac{2}{3}} = \frac{3}{4} \times \frac{3}{1} = \underline{\frac{9}{4}}$ ✓

Example 16. The first term of a G.P. is $4x$ and the second term is x^2; and the sum to infinity is 8, find the third term.

Solution: 1st term of G.P $a = 4x$ — ①
2nd term G.P $ar = x^2$ — ②
fr ① & ② $r = \frac{x^2}{4x} = \frac{x}{4}$ — ③

and $S_\infty = \frac{a}{1-r} = 8$ give

$\Rightarrow \frac{4x}{1-\frac{x}{4}} = 8 \Rightarrow \frac{16x}{4-x} = 8 \Rightarrow 16x = 32 - 8x$
$\Rightarrow x = 4/3$

fr ① $a = 4 \times \frac{4}{3} = \frac{16}{3}$ ✓

and fr ② $ar = x^2$
$\Rightarrow \frac{16}{3} \times r = \left(\frac{4}{3}\right)^2$
$\Rightarrow r = \frac{1}{3}$

$\therefore$ the third term of G.P $= ar^2$
$= \frac{16}{3} \times \left(\frac{1}{3}\right)^2 = \frac{16}{27}$ (or 0.593) ✓

Example 17. The sum of the 1st and 2nd term of a G.P. is 50 and the sum of 2nd and 3rd terms is 30. Find the sum to infinity.

Solution. Given $a + ar = 50 \Rightarrow a(1+r) = 50$ — ①
and $ar + ar^2 = 30 \Rightarrow ar(1+r) = 30$ — ②
② ÷ ① $\Rightarrow r = 3/5$ and fr ① $a = \frac{125}{4}$

$\therefore S_\infty = \frac{a}{1-r} = \frac{\frac{125}{4}}{1-3/5} = \frac{625}{8}$ ✓