(FP1)

Matrices - Transformations Notes

- **Cartesian coordinates, Position vector (column vector) and Matrices:**

  - Given A and B two points on x-axis and y axis respectively at unit distances.

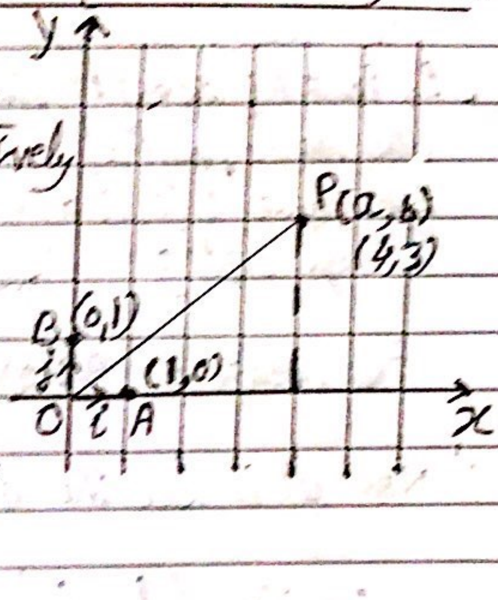
  Then $A(1,0)$, $\vec{OA} = i$

  may also be $\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{2\times1}$ ✓

  Also $B(0,1)$, $\vec{OB} = j$

  or as a column vector $= \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{2\times1}$ ✓

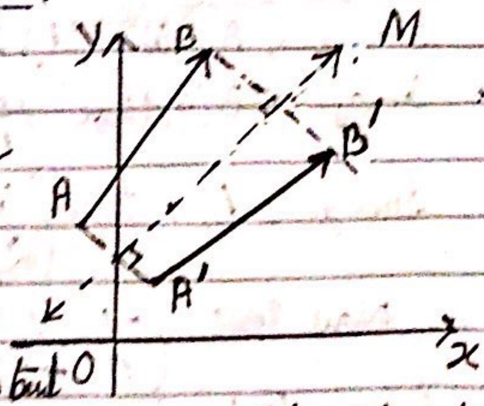
  or as a matrix.



- Similary $P(4,3)$, $\vec{OP} = 4i + 3j = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

- **Transformation:**

  With the application of a transformation a shape changes its position or size or both. The original shape is called object and the new shape is called its image.



- 1. **Reflection transformation:**

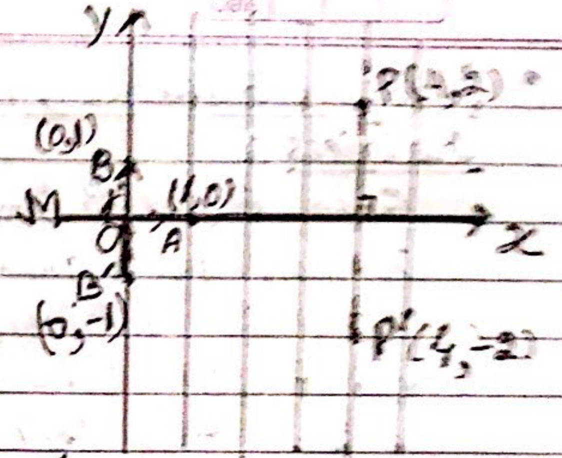
  Reflection is the flip of shape over the line of reflection, called mirror line, M. Any point of the object and the corresponding point on the image are equidistant from the mirror line, along a line perpendicular to the mirror line.

  A'B' is the image of AB over the mirror line M.

(continued →)

• 1 (i) Reflection in x-axis:

x-axis is the mirror line M.
for Point $A(1,0)$ or $\begin{pmatrix}1\\0\end{pmatrix}$, image is the same point A.
for Point $B(0,1)$ or $\begin{pmatrix}0\\1\end{pmatrix}$ the image is $B'\begin{pmatrix}0\\-1\end{pmatrix}$ and



The image of $P(4,2)$ or $\begin{pmatrix}4\\2\end{pmatrix}$ is $P'(4,-2)$ or $\begin{pmatrix}4\\-2\end{pmatrix}$ — (i)

Now let us see the effect of $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ matrix.

Consider for A, $\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ A ✓ same imge.

for B $\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} = B'$ $\boxed{P\begin{pmatrix}x\\y\end{pmatrix} \to P'\begin{pmatrix}x\\-y\end{pmatrix}}$

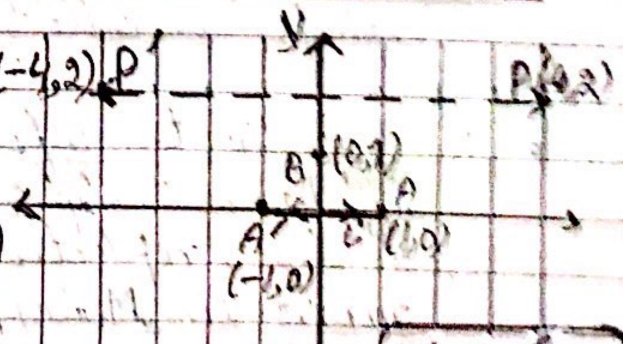
and P $\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}4\\-2\end{pmatrix} = P'$ image of P.

∴ Reflection transformation in X-axis $= \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ ✓

• 1 (ii) Reflection in Y-axis:

Image of $A(1,0)$ or $\begin{pmatrix}1\\0\end{pmatrix}$ is $A'\begin{pmatrix}-1\\0\end{pmatrix}$
Image of $B(0,1)$ or $\begin{pmatrix}0\\1\end{pmatrix}$ is B only

Image of $P(4,2)$ or $\begin{pmatrix}4\\2\end{pmatrix}$ is $P'\begin{pmatrix}-4\\2\end{pmatrix}$



Now Test $\begin{pmatrix}-1&0\\0&1\end{pmatrix}$

$\boxed{P\begin{pmatrix}x\\y\end{pmatrix} \to P'\begin{pmatrix}-x\\y\end{pmatrix}}$

for A, $\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$ A' ✓

for B $\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$ B same

and for P $\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}-4\\2\end{pmatrix}$ P'

Hence Reflection in Y-axis Transformation is:

matrix $\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ ✓

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• 1(iii) Reflection in Origin:

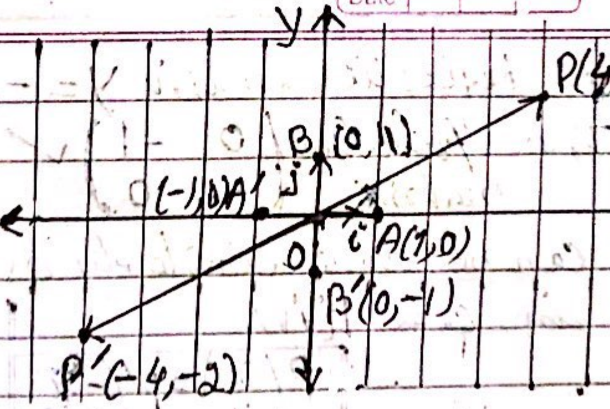


Image of $A(1,0)$ or $\begin{pmatrix}1\\0\end{pmatrix}$ in Origin $= \begin{pmatrix}-1\\0\end{pmatrix} A'$

Image of $B(0,1)$ or $\begin{pmatrix}0\\1\end{pmatrix}$ in origin is $\begin{pmatrix}0\\-1\end{pmatrix} B'$.

and the image of $P(4,2)$ in Origin $= (-4,-2)\ P'$

Consider a matrix $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$.

for A, $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} A'$ ✓

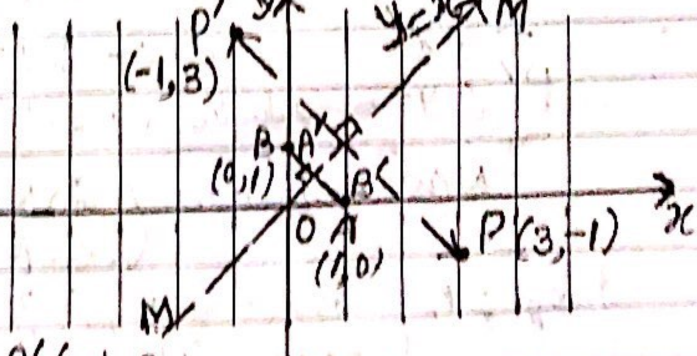
for B $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} B'$ ✓

$$\boxed{P\begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}-x\\-y\end{pmatrix} P'}$$

and the image of P, $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}-4\\-2\end{pmatrix} P'$ ✓

∴ The Reflection in Origin transformation matrix is: $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$ ✓

• 1(iv) Reflection in line $y = x$



In line $y = x$ the reflection of $A\begin{pmatrix}1\\0\end{pmatrix} \to B\begin{pmatrix}0\\1\end{pmatrix}$

The image of $B\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix} A$

and the image of $P(3,-1)$ in mirror line $y = x$ is $P'(-1,3)$.

Now consider a matrix $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

In General:

$$\boxed{P\begin{pmatrix}x\\y\end{pmatrix} \to P'\begin{pmatrix}y\\x\end{pmatrix}}$$

$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}^{A} \to \begin{pmatrix}0\\1\end{pmatrix} B$

$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} B \longrightarrow \begin{pmatrix}1\\0\end{pmatrix} A$

and $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}3\\-1\end{pmatrix} P \longrightarrow \begin{pmatrix}-1\\3\end{pmatrix} P'$

Reflection transformation (in line $y = x$) Matrix

$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ ✓

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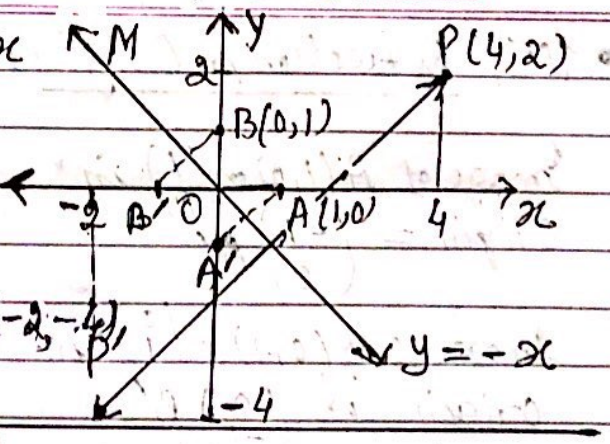
- Reflection in line $y = \tan\theta \cdot x$
  Transformation matrix $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

- **1(V)** Reflection in line $y = -x$

Transformation matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ ✓

Can be done similarly as in case 1(iv).

$$\boxed{P\begin{pmatrix} x \\ y \end{pmatrix} = P'\begin{pmatrix} -y \\ -x \end{pmatrix}}$$



- **2(i)** <u>Rotation transformation about origin, in the anticlockwise direction by θ°:</u>

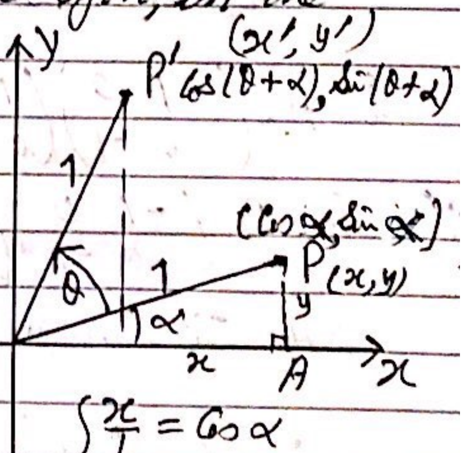
Given a point $P(x, y)$, $OP = 1$

Let angle $POA = \alpha$

$P(x, y) \equiv (\cos\alpha, \sin\alpha)$ —— (1)

Now OP is rotated about origin in anticlockwise direction by θ°.

Let image of P after rotation is $P'(x', y')$: $\begin{cases} x' = \cos(\theta + \alpha) \\ y' = \sin(\theta + \alpha) \end{cases}$



$\begin{cases} \dfrac{x}{1} = \cos\alpha \\ \dfrac{y}{1} = \sin\theta \end{cases}$

$\Rightarrow P' \begin{cases} x' = \cos\theta\cos\alpha - \sin\theta\sin\alpha = x\cos\theta - y\sin\theta \\ y' = \sin\theta\cos\alpha + \cos\theta\sin\alpha = x\sin\theta + y\cos\theta \end{cases}$ from (1) —— (2)

Now consider $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ ✓

Image of P. $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix} P'$ from (2)

$\therefore$ Rotation Transformation through origin in anticlock direction by θ° $= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ ✓

- In particular <u>Rotation through 90° about origin in anticlockwise direct</u> $= \begin{pmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{pmatrix}$

$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ✓

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Matrices Transformations Notes

• 2(ii) Rotation transformation about origin in the clockwise-direction by $\theta°$ is

(as in 2(i) Replace θ by $-\theta$) Matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ ✓

Note: Rotation transformation about origin in the clockwise-direction by $90°$ is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ✓

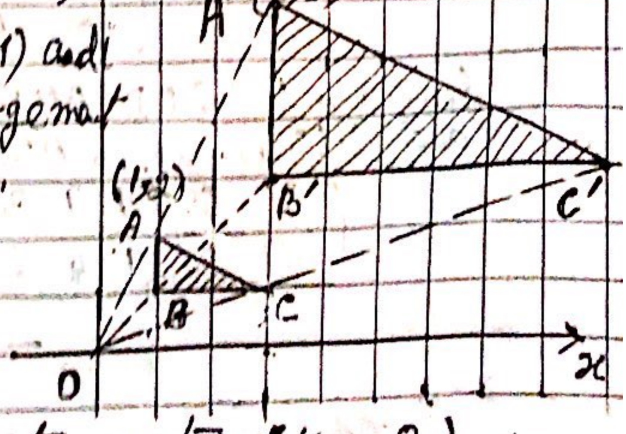
• 2(iii). Rotation transformation about origin through $180°$ in anticlockwise (or clockwise) direction:

Matrix. $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(Note: It is same as reflection in origin).

• 3(i) Enlargement transformation:

In the enlargement transformation the object is enlarged, the image is geometrically similar to the object (may be larger or smaller in size).
For "Enlargement transformation" we are given the position of the centre of enlargement and the scale factor.
Given the centre of enlargement origin (0,0) and the scale factor $k>0$, then the object and image are on the same side of the origin.



Given a triangle ABC, $A(1,2)$, $B(1,1)$ and $C(3,1)$; now centre of enlargement is origin (0,0), scale factor 3.

Consider a matrix $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

for A, $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} A'$ ✓

Similarly for B and C,

$\therefore$ Required transformation matrix $\begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$ ✓

here $K>0$ is the scale factor.

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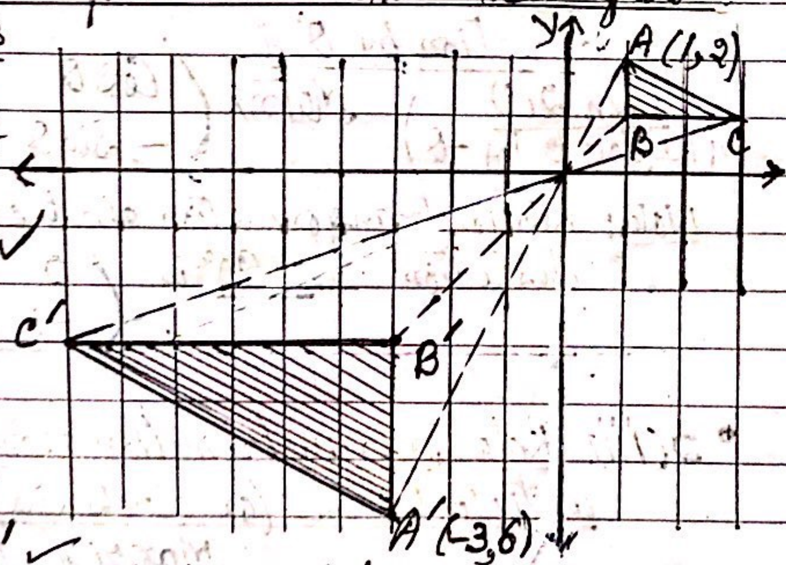
• 3(ii) Enlargement Transformation when the scalar factor $K<0$: let $K=-3$



Consider the enlargement Transformation $\begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$ ✓

$K=-3$

Image of A

$$\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} A$$

$$= \begin{pmatrix} -3 \\ -6 \end{pmatrix} A' ✓$$

The image $A'B'C'$ is on the opposite side of Origin. Note: $M = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$ Transformation matrix

$\det M = (-3)(-3) = 9.$

area of object $\Delta ABC = \frac{1}{2} \times 2 \times 1 = 1$ sq unit

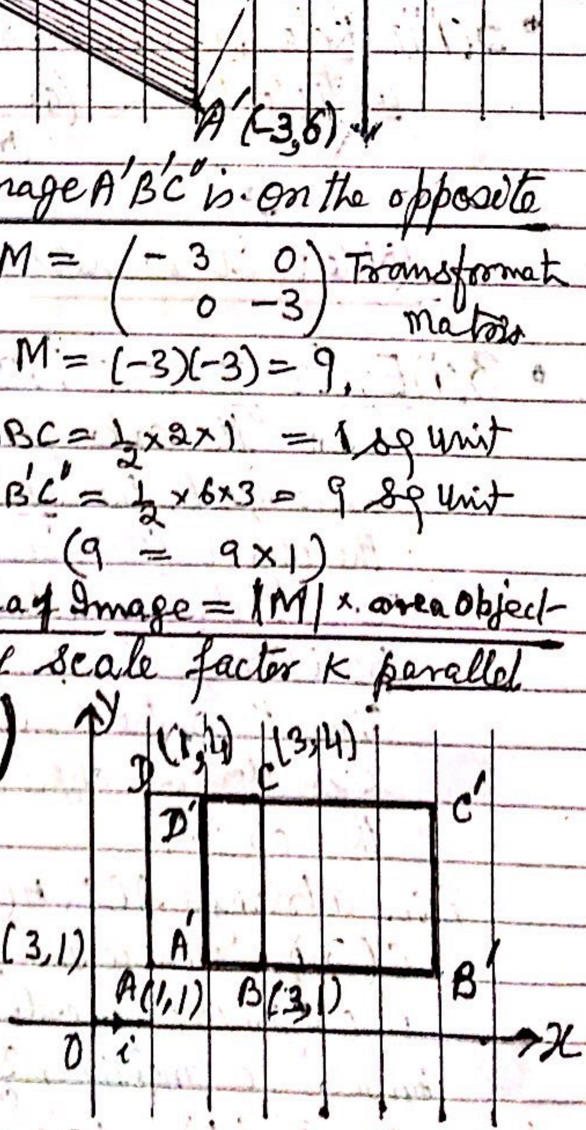
area of Image $\Delta A'B'C' = \frac{1}{2} \times 6 \times 3 = 9$ sq unit

$(9 = 9 \times 1)$

Observe: Area of Image $= |M| \times$ area Object

• 4(i) Stretch transformation of scale factor K parallel to the x-axis. is $\begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix}$

let $K=2$; $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$



Given a rect. ABCD; A(1,1), B(1,4), C(3,1) and D(3,4), Image $A'B'C'D'$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 4 & 4 \end{pmatrix}$$

$$\rightarrow \begin{array}{cccc} \begin{pmatrix} 2 & 6 & 6 & 2 \\ 1 & 1 & 4 & 4 \end{pmatrix} \\ A' \quad B' \quad C' \quad D' \end{array} \text{Image } A'B'C'D'$$

The distance relative to Y-axis is doubled.

Scanned with CamScanner

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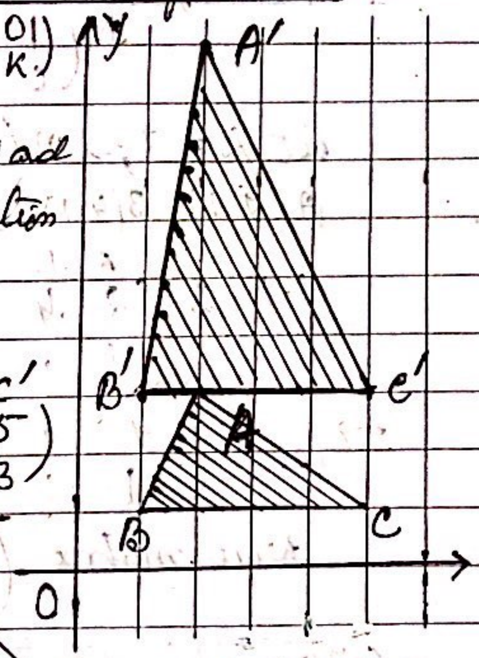
4(ii) <u>Stractch transformation of scale factor k parallel to the y-axis</u> is $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$

Let $k = 3$.

Given a triangle ABC, $A(2,3)$, $B(1,1)$ and $C(5,1)$ and stratch transformation matrix is $M\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 2 & 1 & 5 \\ 3 & 1 & 1 \end{pmatrix} \end{array} \rightarrow \begin{array}{c} \text{Image} \\ \begin{array}{ccc} A' & B' & C' \end{array} \\ \begin{pmatrix} 2 & 1 & 5 \\ 9 & 3 & 3 \end{pmatrix} \end{array}$$



<u>Verify</u>:

area of Image $A'B'C' = |M|$ area ABC

$$\left(\frac{1}{2} \times 4 \times 6\right) = \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} \times \frac{1}{2} \times 4 \times 2$$

$$12 = 3 \times 4 \quad \text{True}$$

<u>The distance relative to x-axis is tripled.</u>

• 5(i) <u>Shear Transformation parallel to Y-axis</u>:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ k \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow M = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$$

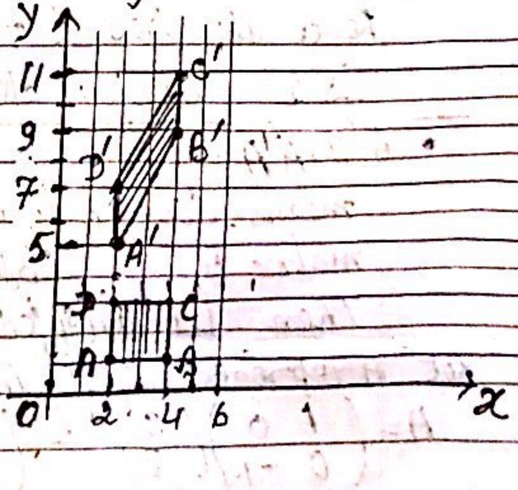
Let $k = 2$ $M = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, Consider a rectangle ABCD.

$A(2,1)$, $B(4,1)$, $C(2,3)$, $D(4,3)$

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 2 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix}$$

Image $A'B'C'D = \begin{pmatrix} 2 & 4 & 2 & 4 \\ 5 & 9 & 7 & 11 \end{pmatrix}$

$\therefore$ matrix $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ give a shearing effect in the y-direction.



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• 5(ii) Shear Transformation parallel to x-axis:

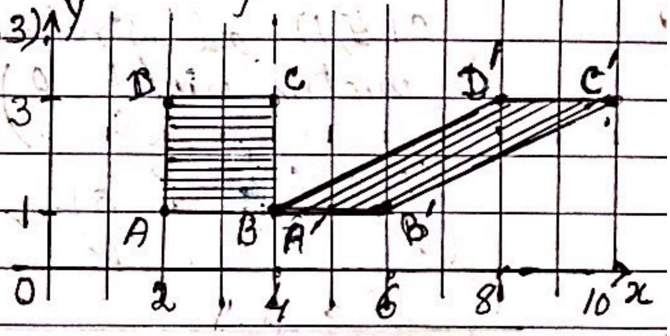
$M\begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix}$ will produce a shearing the x-direction.

Let $K = 2, \Rightarrow M\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ consider a rectangle ABCD.

A(2,1), B(4,1), C(4,3), D(2,3)

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 2 & 4 & 4 & 2 \\ 1 & 1 & 3 & 3 \end{pmatrix} \end{array}$$

$$\rightarrow \begin{pmatrix} 4 & 6 & 10 & 8 \\ 1 & 1 & 3 & 3 \end{pmatrix}$$



hence matrix $\begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix}$ gives a shearing effect in the x- direction.

• 6(i) Compound Transformations:

Example(i) Reflection in X- axis is given by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Find M^2. what transformation is represented by M^2?

Solution: $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\Rightarrow M^2 = M \times M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

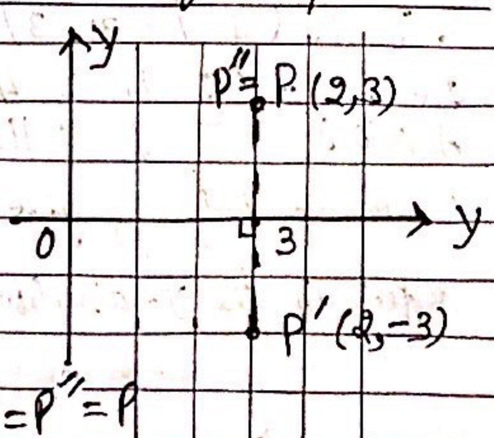
Identity matrix.

hence the result of performing a reflection twice is to return all points to their original positions.

• 6(ii) $A^{-1}A = I$

means Given a Transformation matrix A, Object $P = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Then $(A^{-1}A)(\text{object}) = \text{object}$



Let A represents Reflection in X-axis

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} P'$$

$$A^{-1} = -\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = P'' = P$$

$$(A^{-1})P' = P'' = P \checkmark$$

• 7 Invariant Points:

The points whose images under a transformation are same as the point (object), are called the invariant points.

(i) Origin is always an invariant point for $M\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a transformation matrix then.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \checkmark$$

(ii) In reflection transformations the points on the mirror line are invariant points.

Example: Reflection in x-axis the transformation matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, any point on x-axis $\begin{pmatrix} a \\ 0 \end{pmatrix}$

now: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} \checkmark$ hence invariant points.

(iii) Consider transformation matrix $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ to find the invariant point $\begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow 2x - y = x \Rightarrow x = y$$

$$x = y \Rightarrow x = y$$

Then $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 7 \end{pmatrix}$ are all invariant points. ✓

Example 1. Find the invariant points under the transformation: $\begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}$

Solution: Let the required points are (x, y) type

$$\Rightarrow \begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} 4x + y = x \Rightarrow 3x + y = 0 \\ 6x + 3y = y \Rightarrow 3x + y = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} y = -3x \\ \text{At } x = k \text{ anyreal} \\ y = -3k \end{matrix}$$

consistent

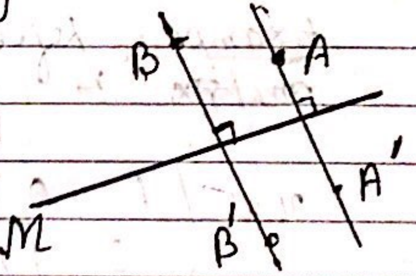
$\therefore$ Req points are of type $(k, -3k)$

- **Invariant Lines:**

A line is said to be invariant for a given transformation if the image of every point on line also lies on the same line.

Example 2(i). In reflection transformation the image of every point on the mirror line remains on the mirror line only, hence the mirror lines are invariant lines.

(ii) In reflection transformation the lines perp. to the mirror lines are also invariant lines.



Example 2 (ii) For matrix $M = \begin{pmatrix} 4 & 11 \\ 11 & 4 \end{pmatrix}$

Find the invariant lines of the transformation M.

Solution: Let the equation of invariant line is $y = mx + c$ —①

Let $\begin{pmatrix} x \\ y \end{pmatrix}$ be any point on the line ① and its image is at point $\begin{pmatrix} x' \\ y' \end{pmatrix}$

$$\therefore \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 11 \\ 11 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x + 11y \\ 11x + 4y \end{pmatrix}$$

$$\Rightarrow \begin{cases} x' = 4x + 11y & \text{—②} \\ y' = 11x + 4y & \text{—③} \end{cases}$$

now (x', y') lies on line ①; from ② & ③ in ①

$$\therefore \quad 11x + 4y = m(4x + 11y) + c$$

$$11x + 4(mx + c) = m[4x + 11(mx + c)] + c$$

$$\Rightarrow 11x + \cancel{4mx} + 4c = \cancel{4mx} + 11m^2x + 11mc + c$$

$$\Rightarrow 11m^2x - 11x + (11m - 3)c = 0$$

$$\Rightarrow 11x(m^2 - 1) + (11m - 3)c = 0$$

$$\Rightarrow m^2 - 1 = 0 \quad \text{or} \quad 11m - 3 = 0 \text{ or } c = 0$$

(continued→)

(continued)

(Example (ii))

$\Rightarrow \quad m^2 - 1 = 0$ and $11m - 3 = 0$ or $C = 0$

$\Rightarrow \quad m = \pm 1$ and $c = 0$

from ① Required invariant lines are

$$y = \pm x + c$$

$$\therefore \quad y = x \quad \text{or} \quad y = -x$$

Example 3: $A = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

(a) describe fully the single geometrical transformation U represented by the matrix A. [3]

The transformation V, represent by the 2×2 matrix B, is a reflection in the line $y = -x$

(b) write down the matrix B. [1]

Given that U followed by V is the transformation T, which is represented by matrix C.

(c) Find the matrix C. [2]

(d) Show that there is a real number k for which the point $(1, k)$ is invariant under T. [4]

Solution (a) Rotation through $120°$, anticlockwise, about origin $(0,0)$.

Rotation transformation in anticlockwise about origin $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

(b) Now $V = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(c) $VU = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = C$

(d) $\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix} \Rightarrow \begin{pmatrix} -\frac{\sqrt{3}}{2} + \frac{1}{2}k \\ \frac{1}{2} + \frac{\sqrt{3}}{2}k \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix}$

(continued→)

FP1

(Continued →)

Example 3(d): $\Rightarrow -\frac{\sqrt{3}}{2} + \frac{1}{2}k = 1$ and $\frac{1}{2} + \frac{\sqrt{3}}{2}k = k$

$\Rightarrow k = (2+\sqrt{3})$ and $k = \frac{1}{(2-\sqrt{3})} = (2+\sqrt{3})$

$\therefore$ Invariant point is $(1, 2+\sqrt{3})$

Example 4. Given $M = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$

(a) Show that M is non-singular. ---[2]

The hexagon R is transformed to the hexagon S by the transformation represented by the matrix M.

Given that the area of hexagon R is 5 sq. units.

(b) find the area of hexagon S. ---[1]

The matrix M represent an enlargement, with centre (0,0) and scale factor k, where $k > 0$, followed by a rotation anti-clockwise through an angle θ about (0,0).

(c) Find the value of k [2]

(d) Find the value of θ. [2]

Solution (a) $M = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$

$\det M = \begin{vmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{vmatrix} = 1+3 = 4 \neq 0$ $\therefore$ M is Non-singular

(b) Area S = $|M| \times$ area R

$= 4 \times 5 = 20$ sq unit

(c) Enlargement $= \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

Rotation $= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

$M = \begin{pmatrix} k\cos\theta & -k\sin\theta \\ k\sin\theta & k\cos\theta \end{pmatrix}$ —— (1)

for (1)

$\begin{pmatrix} k\cos\theta & -k\sin\theta \\ k\sin\theta & k\cos\theta \end{pmatrix} = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$

$\Rightarrow \begin{cases} k\cos\theta = 1 & \text{—— (2)} \\ k\sin\theta = \sqrt{3} & \text{—— (3)} \end{cases}$

sq and add. $k^2 = 4$

$k = 2$ or -2 ✗

$k = 2$

(d) for (2) & (3) $\tan\theta = \sqrt{3}$

$\Rightarrow \theta = 60^\circ$ or $\frac{\pi}{3}$

Example 5. The matrix A is given by, $A=\begin{pmatrix}5 & K\\ -3 & -4\end{pmatrix}$

(a) Find the value of K for which A is singular. ---[2]
It is given that $K=6$ so that $\longrightarrow A=\begin{pmatrix}5 & 6\\ -3 & -4\end{pmatrix}$
(b) Find the equations of invariant lines, through the origin, of the transformation in the X-Y plane represented by A. ---[6]
(c) The triangle DEF in the X-Y plane is transformed by A on triangle PQR
(i) Given that the area of triangle DEF is 10 cm², find the area of triangle PQR. --[2]
(ii) Find the matrix which transforms triangle PQR onto triangle DEF. $\boxed{\text{SP-20/01/Q5}}$ --[2]

Solution: Given $A=\begin{pmatrix}5 & K\\ -3 & -4\end{pmatrix}$

(a) A is singular $\Rightarrow |A|=0$

$$\Rightarrow \begin{vmatrix}5 & K\\ -3 & -4\end{vmatrix}=0 \Rightarrow 3K-20=0$$
$$\Rightarrow K=\frac{20}{3}$$

(b) Now $A=\begin{pmatrix}5 & 6\\ -3 & -4\end{pmatrix}$ for invariant line passes through origin
$\therefore y=mx$ —— (1)

Image $\begin{pmatrix}X\\ Y\end{pmatrix}=\begin{pmatrix}5 & 6\\ -3 & -4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$

$$=\begin{pmatrix}5x+6y\\ -3x-4y\end{pmatrix} \Rightarrow \left.\begin{matrix}X=5x+6y\\ Y=-3x-4y\end{matrix}\right\}$$

from (1) $-3x-4y=m(5x+6y)$
$$\Rightarrow -3x-4(mx)=m(5x+6mx)$$
$$\Rightarrow -3x-4mx=5mx+6m^2x$$
$$x(6m^2+9m+3)=0 \Rightarrow m=-1,-\tfrac{1}{2}$$

from (1) Equations of Invariant lines are:
$$y=-x\ ;\ y=-\tfrac{1}{2}x$$
$$\text{or } y=-x\ ;\ 2y+x=0$$

(Continued)

FP₁

(Continued →)

Example (c)(i) $\det A = \begin{vmatrix} 5 & 6 \\ -3 & -4 \end{vmatrix} = -20 - (-18) = -2$

Area of Image $PQR = \det A \times$ area of object DEF

$= 2 \times 10 = 20\,cm^2$ ✓

C(ii). the matrix which transform triangle PQR to triangle $DEF = A^{-1}$

$= \frac{1}{|A|}\begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}$

$= -\frac{1}{2}\begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} 2 & 3 \\ -3/2 & -\frac{5}{2} \end{pmatrix}$

$\because A = \begin{pmatrix} 5 & 6 \\ -3 & -4 \end{pmatrix}$

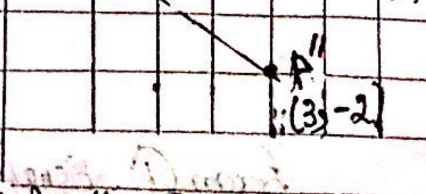
Interchange the diagonal element and change the signs of other elements to get A^{-1}

Example 6: The transformation represented by $C = \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}$ is equivalent to a single transformation B followed by a single transformation A. Give geometrical description of a pair of possible transformations B and A and state the matrices that represent them.

Solution: we need to find B followed by $A \Rightarrow AB = C$. — (1)

Consider a transformation matrix $B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ which represents a stretch of factor 3, parallel to the y-axis, followed by

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which represents a rotation of 90° clockwise about origin.



Now $AB = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix} = C$ ✓

Consider a point $P\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$B(P) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} P'$

$AB(P) = A(B(P)) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(P') = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} P''$ — (2)

Again $(AB)(P) = \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} P''$ — (3)   ∴ from (2) + (3) ✓

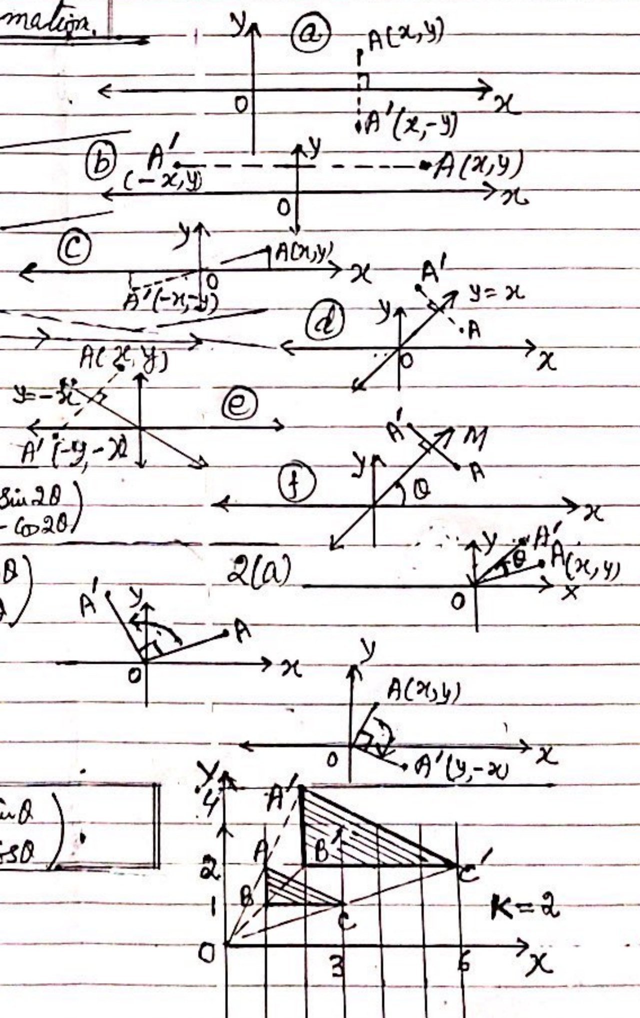
FP1

Matrices and Transformations.

Transformations	Object Column matrix	Image.	Matrix representing the transformation.
1(a) Reflection in X-axis (Mirror line $y=0$)	$A\begin{pmatrix}x\\y\end{pmatrix}$	$\begin{pmatrix}x\\-y\end{pmatrix}A'$	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$
(b) Reflection in Y-axis (Mirror line $x=0$)	$A\begin{pmatrix}x\\y\end{pmatrix}$	$\begin{pmatrix}-x\\y\end{pmatrix}A'$	$\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$
(c) Reflection in Origin $(0,0)$	$A\begin{pmatrix}x\\y\end{pmatrix}$	$\begin{pmatrix}-x\\-y\end{pmatrix}A'$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$
(d) Reflection in the line $y=x$.	$A\begin{pmatrix}x\\y\end{pmatrix}$	$\begin{pmatrix}y\\x\end{pmatrix}A'$	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$
(e) Reflection in the line $y=-x$	$A\begin{pmatrix}x\\y\end{pmatrix}$	$\begin{pmatrix}-y\\-x\end{pmatrix}A'$	$\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$
(f) Reflection in the line $y=x\tan\theta$.	$A\begin{pmatrix}x\\y\end{pmatrix}$	A'	$\begin{pmatrix}\cos 2\theta & \sin 2\theta\\\sin 2\theta & -\cos 2\theta\end{pmatrix}$
2 (a) Rotation through $\theta°$, anticlockwise centre of rotation origin $(0,0)$	$A\begin{pmatrix}x\\y\end{pmatrix}$	A'	$\begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix}$
(b) Rotation through $90°$, anticlockwise Centre of rotation-origin $(0,0)$	$A\begin{pmatrix}x\\y\end{pmatrix}$	$\begin{pmatrix}-y\\x\end{pmatrix}A'$	$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$
(c) Rotation through $90°$, clockwise, Centre of rotation-origin $(0,0)$	$A\begin{pmatrix}x\\y\end{pmatrix}$	$\begin{pmatrix}y\\-x\end{pmatrix}A$	$\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$
(d) Rotation transformation about origin in clockwise direction	$A\begin{pmatrix}x\\y\end{pmatrix}$	A'	$\begin{pmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{pmatrix}$
3. (i) Enlargement transformation Centre of enlargement is origin and scale factor $K>0$	$A\begin{pmatrix}x\\y\end{pmatrix}$	$\begin{pmatrix}Kx\\Ky\end{pmatrix}$	$\begin{pmatrix}Kx & 0\\0 & Kx\end{pmatrix}$

(ii) $K<0$

Then the image will be on the opposite side of origin.



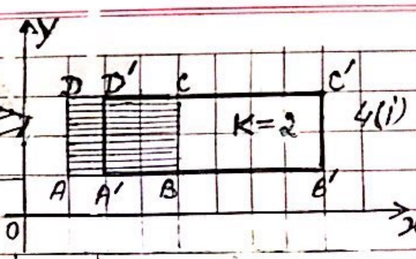
FP1

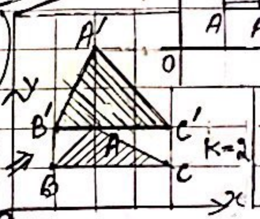
Matrices and Transformations

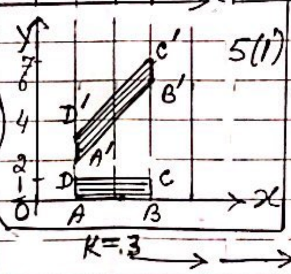
Transformations	Object Column matrix.	Image	Matrix representing the transformation
4(i) Stretch transformation of scale factor k, parallel to x-axis.	$A\begin{pmatrix} x \\ y \end{pmatrix}$	$\begin{pmatrix} kx \\ y \end{pmatrix} A'$	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
4(ii) Strech of scale factor k, parallel to y-axis.	$A\begin{pmatrix} x \\ y \end{pmatrix}$	$\begin{pmatrix} x \\ ky \end{pmatrix} A'$	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
5(i) Shear transformation parallel to y-axis	$A\begin{pmatrix} x \\ y \end{pmatrix}$	$\begin{pmatrix} x \\ kx+y \end{pmatrix} A'$	$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$
5(ii) Shear transformation parallel to x-axis	$A\begin{pmatrix} x \\ y \end{pmatrix}$	$\begin{pmatrix} x+ky \\ y \end{pmatrix} A'$	$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

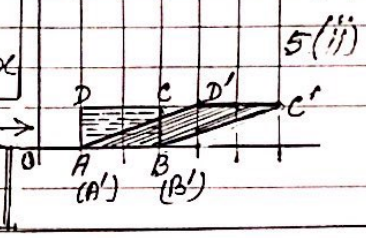
(Example → $k=2$): $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 2 & 6 & 7 & 3 \end{pmatrix}$

A B C D → A′B′C′D′









6. Area of Image = |M|· Area of object, here M is the transformation matrix.

7(i) Invariant Points: The points for which the image is same as the object for a given transformation → $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

(ii) Invariant Lines for a given transformation: If the image of every point on this line lies on the same line.

for every point on line ①

(x', y') lies on ①

Invariance Line – $y = mx + c$ — ①

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$