FP1

Further Pure Math_1

Matrices Notes_1

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Matrices
Notes.

- **Introduction:**

In a class there are three friends Simran, Akriti and Joyce, they have 15, 10 and 13 note books and 6, 2 and 5 pens respectively. We can represent this information in a rectangular array as follows:

(i)
$$A = \begin{array}{c} \\ \text{Simran} \\ \text{Akriti} \\ \text{Joyce} \end{array} \begin{array}{c} \begin{array}{cc} \text{Note Books} & \text{Pens} \end{array} \\ \begin{pmatrix} 15 & 6 \\ 10 & 2 \\ 13 & 5 \end{pmatrix}_{3\times 2} \end{array}$$

A is called a matrix of order 3×2 (3 by 2)
3 rows and 2 columns.

The same information can also be expressed as:

(ii)
$$B = \begin{array}{c} \\ \text{Note Books} \\ \text{Pens} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Simran} & \text{Akriti} & \text{Joyce} \end{array} \\ \begin{pmatrix} 15 & 10 & 13 \\ 6 & 2 & 5 \end{pmatrix}_{2\times 3} \end{array}$$

Matrix B has order 2×3,
2 rows and 3 columns.

- **General form of a Matrix**

A matrix of Order $m\times n$.

$$A_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$$

here $a_{2\times 3}$ is an element in second row and third column.

- **Squared Matrices:** A matrix having the same number of rows and columns.

$$P = \begin{pmatrix} 2 & 3 \\ 5 & -7 \end{pmatrix}_{2\times 2}$$

is a matrix of Order 2.

$$Q = \begin{pmatrix} 4 & 2 & -3 \\ 0 & 1 & 9 \\ 5 & 7 & 6 \end{pmatrix}_{3\times 3}$$

Q is squared matrix of Order 3.

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- **Diagonal Matrix:** A squared matrix in which all non-diagonal elements are zero.

(i) Diagonal matrix of Order 2.

$$E = \begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix}_{2\times 2}$$

(ii) Diagonal matrix Order 3.

$$F = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 13 \end{pmatrix}_{3\times 3}$$

- **Row Matrix:** Only one row,

$$G = \begin{pmatrix} 2 & -5 & 7 \end{pmatrix}_{1\times 3} \quad ; \quad H = \begin{pmatrix} 1 & 6 \end{pmatrix}_{1\times 2}$$

- **Column Matrix:** Only one column.

$$I = \begin{pmatrix} 2 \\ -3 \end{pmatrix}_{2\times 1} \quad ; \quad J = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}_{3\times 1}$$

- **Zero Matrix:** A matrix whose all the elements are zero.

$$K = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{2\times 2} \qquad L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{2\times 3}$$

- **Identity Matrix:** A square matrix in which all the diagonal element are 1 each and the rest are zero.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}$$

Identity matrix of order 2

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3}$$

Identity matrix order 3.

- **Equal Matrices:** Two matrices are equal if
  (i) they have the same order
  (ii) corresponding elements are equal.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}_{2\times 3} = B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}_{2\times 3}$$

(i) A and B both have the same order 2×3

(ii) and $\left.\begin{matrix} a_{11} = b_{11} \\ a_{21} = b_{21} \end{matrix}\right\}$ $\left.\begin{matrix} a_{12} = b_{12} \\ a_{22} = b_{22} \end{matrix}\right\}$ and $\left\{\begin{matrix} a_{13} = b_{13} \\ a_{23} = b_{23} \end{matrix}\right.$

- **Scalar Multiplication:** Given $A = \begin{pmatrix} 2 & -3 \\ 5 & 0 \end{pmatrix} \Rightarrow 4A = 4\times\begin{pmatrix} 2 & -3 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 8 & -12 \\ 20 & 0 \end{pmatrix}$

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• Example 1: Find the values of x and y

Given $\begin{pmatrix} 2 & x \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ y & 5 \end{pmatrix} \Rightarrow \begin{matrix} x = -5 \\ y = -3 \end{matrix}$

• **Addition and Subtraction of Matrices:**

(i) Both the matrices should be of the same order

(ii) add (subtract) the corresponding element.

Example 2(i)

$$\begin{pmatrix} 2 & -4 \\ 5 & 0 \end{pmatrix}_{2\times2} + \begin{pmatrix} 1 & 6 \\ -2 & 7 \end{pmatrix}_{2\times2} = \begin{pmatrix} 2+1 & -4+6 \\ 5+(-2) & 0+7 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 3 & 7 \end{pmatrix}_{2\times2}$$

(ii)

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}_{3\times1} - \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}_{3\times1} = \begin{pmatrix} 2-(-1) \\ 1-5 \\ 3-0 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}_{3\times1}$$

• **Addition of Matrices – Properties** Given matrices of the same order.

(i) Commutative $A+B = B+A$

(ii) Associative $(A+B)+C = A+(B+C)$

• **Multiplication of two Matrices:**

Condition:

$$A_{m\times n} \times B_{n\times p} = C_{m\times p}$$

Two matrices A and B can only be multiplies if the number of columns in the first matrix is same as the number of row in the second.

• Example 3(i) Multiply the element of first row of first with the element of first column of second matrix.

$$\begin{pmatrix} a & b & c \end{pmatrix}_{1\times3} \times \begin{pmatrix} p \\ q \\ r \end{pmatrix}_{3\times1} = (ap+bq+cr)_{1\times1}$$

Row → Column

$$\begin{pmatrix} 2 & -3 & 4 \end{pmatrix}_{1\times3} \times \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}_{3\times1} = (2\times1+(-3\times0)+4\times5)$$
$$= (2+0+20)$$
$$= (22)_{1\times1}$$

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Multiplication of Matrices.

Example 3(ii)

$$\begin{pmatrix} a & b & c \end{pmatrix}_{1\times3} \times \begin{pmatrix} p & d \\ q & e \\ r & f \end{pmatrix}_{3\times2} = \begin{pmatrix} ap+bq+cr & ad+be+cf \end{pmatrix}_{1\times2}$$

or

$$\begin{pmatrix} 2 & -3 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 3 \\ 0 & -5 \\ 5 & 2 \end{pmatrix}$$ Row to column

$$= \begin{pmatrix} 2\times1+(-3\times0)+4\times5 & 2\times3+(-3)(-5)+4\times2 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & 29 \end{pmatrix}_{1\times2}$$ ✓

3(iii)

$$\begin{pmatrix} a & b & c \\ l & m & n \end{pmatrix}_{2\times3} \times \begin{pmatrix} p & d \\ q & e \\ r & f \end{pmatrix}_{3\times2} = \begin{pmatrix} ap+bq+cr & ad+be+cf \\ lp+mq+nr & ld+me+nf \end{pmatrix}_{2\times2}$$

(Repeat the process with the second row)

or

$$\begin{pmatrix} 2 & -3 & 4 \\ -1 & 2 & 3 \end{pmatrix}_{2\times3} \times \begin{pmatrix} 1 & 3 \\ 0 & -5 \\ 5 & 2 \end{pmatrix}_{3\times2} = \begin{pmatrix} 2\times1+(-3)\times0+4\times5 & 2\times3+(-3)\times(-5)+4\times2 \\ (-1\times1)+2\times0+(3\times5) & (-1\times3)+2(-5)+3\times2 \end{pmatrix}$$

(Repeat the process with second row)

$$= \begin{pmatrix} 22 & 29 \\ 14 & -7 \end{pmatrix}_{2\times2}$$ ✓

- **Matrix multiplication is not commutative in general:**

$$A_{2\times2} \times B_{2\times2} = C_{2\times2}$$

$$B_{2\times2} \times A_{2\times2} = D_{2\times2}$$

$$AB \neq BA \text{ in general.}$$

- But Matrix multiplication is associative: $(AB)C = A(BC)$ ✓

- **Determinant of a squared Matrix:**

To each squared matrix there corresponds a value, Determinant of matrix A is denoted by det A or |A|.

Given a squared matrix of order 2, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then $\det A$ or $|A| = (ad - bc)$ ✓

- Example 4 (i) $A = \begin{pmatrix} 2 & -3 \\ 5 & 7 \end{pmatrix} \Rightarrow |A| = 2\times 7 - 5\times(-3) = 29$ ✓

(ii) $B = \begin{pmatrix} -1 & 3 \\ 4 & 6 \end{pmatrix} \Rightarrow \det B = (-1)\times 6 - 3\times 4 = -18$ ✓

(iii) $C = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow |I| = 1\times 1 - 0\times 0 = 1$ ✓

- **Det. of order 3:**

Given a squared matrix A of order 3×3.

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

We evaluate the value of det A, by expanding with first row (or first column)

(Keep the sign ± in mind)

$$\det A = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Example 5 Evaluate: Expanding with first row:

$$A = \begin{pmatrix} -1 & 6 & -2 \\ 2 & 1 & 1 \\ 4 & 1 & -3 \end{pmatrix} \Rightarrow \det A = -1 \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} - 6 \begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix} + (-2) \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix}$$

$$= -(-3-1) - 6(-6-4) - 2(2-4)$$

$$= 4 + 60 + 4 = 68$$ ✓

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- Example 6: Given a matrix M, find the value of det M.

$$M=\begin{pmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 2 & 5 & 2 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Expanding with first column;

$$\det M = 1\times\begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} - 4\times\begin{vmatrix} -3 & 2 \\ 5 & 2 \end{vmatrix} + 2\times\begin{vmatrix} -3 & 2 \\ -1 & 2 \end{vmatrix}$$

$$= 1\times(-2-10) - 4(-6-10) + 2\times(-6+2)$$

$$= -12 + 64 - 8$$

$$= \underline{44}$$

- Singular Matrix:

A squared matrix A is said to be singular if det A = 0 [if det A ≠ 0 then A is non-Singular matrix]

Example 7(i) Given $A=\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix} \Rightarrow \det A = 3\times4 - 6\times2$

$= 6-6$

$=0$

∴ Matrix A is singular.

(ii) Given a matrix $M=\begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & 2 \\ 3 & 2 & 3 \end{pmatrix}$

Check if the matrix M is Singular or Non-Singular.

find $\det M = 2\times\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix}$

$= 2(9-4) + 1(3-6) + 1\times(2-9)$

$= 10 - 3 - 7$

Det M = 0

∴ given matrix M is Singular.

- Scaler Multiplication: Given a Matrix A and real number k. Then

$A=\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \Rightarrow kA\begin{pmatrix} ka & kb & kc \\ kd & ke & kf \end{pmatrix}$ k.A is obtained by multiplying each element of A by k.

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- **Inverse of a matrix of second order:**

Given a square matrix A of second order $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
Then another square matrix B of the same order, such that:

$$A \cdot B = I \quad \text{---(1)} \qquad (I \text{ is a unit matrix}) \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Inverse of A: $A^{-1} = B$ ✓ ---(2)

Let $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$

from (1) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

equating the corresponding elements

$$\begin{cases} ap+br=1 \\ cp+dr=0 \end{cases} \text{ and } \begin{cases} aq+bs=0 \\ cq+ds=1 \end{cases}$$

Solving we get $p = \frac{d}{(ad-bc)}$, $q = \frac{-b}{(ad-bc)}$

$r = \frac{-c}{(ad-bc)}$, $s = \frac{a}{(ad-bc)}$

from (2)

$$\therefore A^{-1} = B = \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} \frac{d}{(ad-bc)} & \frac{-b}{(ad-bc)} \\ \frac{-c}{(ad-bc)} & \frac{a}{ad-bc} \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$\left[A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad |A| = (ad-bc) \right.$

or $A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ ✓

Note: A^{-1} exists only for non-singular matrix. $|A| \neq 0$

[Note:
Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Then to get A^{-1} (i) Interchange a & d
(ii) Change the signs of b & c]

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- **Example 8:** Given $M = \begin{pmatrix} 3 & 1 \\ -11 & -2 \end{pmatrix}$ Find M^{-1}

Solution: $|M| = 3\times(-2) - (-11)\times 1 = 5$ ✓

$M = \begin{pmatrix} 3 & 1 \\ -11 & -2 \end{pmatrix}$

$\therefore M^{-1} = \frac{1}{5}\begin{pmatrix} -2 & -1 \\ 11 & 3 \end{pmatrix}$

Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then $A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ ✓

To verify $MM^{-1} = I$

$MM^{-1} = \begin{pmatrix} 3 & 1 \\ -11 & -2 \end{pmatrix} \times \frac{1}{5}\begin{pmatrix} -2 & -1 \\ 11 & 3 \end{pmatrix}$

$= \frac{1}{5}\begin{pmatrix} -6+11 & -3+3 \\ 22-22 & 11-6 \end{pmatrix}$

$= \frac{1}{5}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

- **Note** If A and B are two square matrices of the same order, $|A| \neq 0$, $|B| \neq 0$ Then

$(AB)^{-1} = B^{-1} \cdot A^{-1}$

- **Example 9:**

Given two invertible matrics A and B, find A^{-1}, B^{-1}. and hence find $(AB)^{-1}$! $A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 6 \\ 3 & 2 \end{pmatrix}$

Solution: $A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} \Rightarrow |A| = 3\times5 - 7\times2 = 1$.

$\therefore A^{-1} = \frac{1}{1}\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$ — (1) ✓

and $B = \begin{pmatrix} 4 & 6 \\ 3 & 2 \end{pmatrix} \Rightarrow |B| = 4\times2 - 3\times6 = -10$ ✓

Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then $A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$\therefore B^{-1} = \frac{1}{-10}\begin{pmatrix} 2 & -6 \\ -3 & 4 \end{pmatrix}$ — (2) ✓

$\therefore (AB)^{-1} = B^{-1} \cdot A^{-1}$

$= \frac{-1}{10}\begin{pmatrix} 2 & -6 \\ -3 & 4 \end{pmatrix} \times \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$

$= -\frac{1}{1}\begin{pmatrix} 10+42 & -4-18 \\ -15-28 & 6+12 \end{pmatrix}$

$= \frac{-1}{10}\begin{pmatrix} 52 & -22 \\ -43 & 18 \end{pmatrix}$ — (3) ✓

Verify: $(AB)^{-1} = B^{-1} \cdot A^{-1}$

$AB = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}\begin{pmatrix} 4 & 6 \\ 3 & 2 \end{pmatrix}$

$AB = \begin{pmatrix} 18 & 22 \\ 43 & 52 \end{pmatrix}$ — (4)

$|AB| = 18\times52 - 43\times22$
$= -10$

$\therefore$ from (4)

$(AB)^{-1} = \frac{-1}{10}\begin{pmatrix} 52 & -22 \\ -43 & 18 \end{pmatrix}$ — (5) ✓

from (3) & (5)

$(AB)^{-1} = B^{-1} \cdot A^{-1}$

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- **Transpose of a Matrix:**

Given a matrix $A_{m\times n}$ then its transpose denoted by A^T (or A') is obtained by interchanging its rows and column, and its order will be $n\times m$

Given $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}_{m\times n}$

Then

Transpose of $A = A^T = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}_{n\times m}$

Note: $\underline{(AB)^T = B^T \cdot A^T}$

- **Example 10.** Given $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}$

Verify $(AB)^T = B^T \cdot A^T$.

Solution.

$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix} \Rightarrow B^T = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$

Now $AB = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1+6 & 4+15 \\ 2+8 & 8+20 \end{pmatrix}$

$= \begin{pmatrix} 7 & 19 \\ 10 & 28 \end{pmatrix}$

$B^T \cdot A^T = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1+6 & 2+8 \\ 4+15 & 8+20 \end{pmatrix}$

$\Rightarrow B^T \cdot A^{-1} = \begin{pmatrix} 7 & 10 \\ 19 & 28 \end{pmatrix}$ — (2)

$\Rightarrow (AB)^T = \begin{pmatrix} 7 & 10 \\ 19 & 28 \end{pmatrix}$ — (1)

from (1) & (2) $(AB)^T = B^T \cdot A^T$

- **Minors and Cofactors of the elements of a matrix:**

Given $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_{3\times3}$

§ **Minor of element a_{11}**
denoted by $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = (a_{22}.a_{33} - a_{32}.a_{23})$ — (1)

is the value of determinant obtained by deleting the element of row and column through a_{11}

and

Minor of $a_{21} = M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = (a_{12}.a_{33} - a_{32}.a_{13})$ — (2)

§ **Cofactor of an element:**
is the value of corresponding minor with proper sign.

Sign of $a_{ij} = (-1)^{i+j} = \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

C_{11} = Cofactor of $a_{11} = +M_{11} = (a_{22}a_{33} - a_{32} * a_{23})$ for (1)

Cofactor of $a_{21} = C_{21} = -M_{21} = -(a_{12}.a_{33} - a_{32}.a_{13})$ for (2)

Example 11: Write the values of minors and cofactors of each element of the first column.

$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$A = \begin{pmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{pmatrix}$

(1) Minor of a_{11} or $M_{11} = \begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} = -12$ ✓

Cofactor $C_{11} = +(-12) = -12$ ✓

Minor of (4) $a_{21} = M_{21} = \begin{vmatrix} -3 & 2 \\ 5 & 2 \end{vmatrix} = -6 - 10 = -16$ ✓

∴ Cofactor $C_{21} = -M_{21} = -(-16) = 16$ ✓

Minor of a_{31} (or 3) $= M_{31} = \begin{vmatrix} -3 & 2 \\ -1 & 2 \end{vmatrix}$

$M_{31} = -6 - (-1)\times2 = -4$ ✓

∴ Cofactor $C_{31} = +(-4) = -4$ ✓

- <u>Adjoint of a Square Matrix:</u>

(or Adjugate) Given a square matrix A.

<u>Adj A</u> is the transpose of a matrix, whose element are the cofactors of the respective element of given matrix A.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow \text{Adj}A = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}^T$$

§ <u>Example</u> 12. Given a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Find adj A.

$\begin{pmatrix} + & - \\ - & + \end{pmatrix}$

Solution: Cofactor of element a, $C_a = d$

Cofactor of element b, $C_b = -c$

Cofactor of element c, $C_c = -b$

Cofactor of element d, $C_d = a$

$$\therefore \text{Adj } A = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T$$

$$= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{—— (1)}$$

- Now conder:

$$A \cdot \text{Adj}A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= (ad-bc) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$|A| = (ad-bc)$

$$A \cdot \text{Adj}A = |A| \cdot I \Rightarrow A \cdot \left(\frac{\text{Adj}A}{|A|}\right) = I$$

$$\boxed{\therefore A^{-1} = \frac{1}{|A|} \cdot \text{Adj}A,}$$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Given

$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & a \\ -c & -b \end{pmatrix}$

§ (See Page 7)

• **Adjoint (or Adjugate) of a square matrix of third order:**

Given $A \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Then $\text{Adj } A = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$

If Minor of $a_{21} = M_{21}$

here cofactor of $a_{21} = C_{21} = -M_{21}$ $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

• **Example 13:** Given a matrix $A = \begin{pmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{pmatrix}$

(i) Find Adj A

(ii) Find A^{-1}

(iii) Verify $A \cdot A^{-1} = I$

(iv) $|\text{Adj} A| = |A|^2$

Solution:

(i) $A = \begin{pmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{pmatrix}$

$\text{Adj} A = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$ —— (1) $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$C_{11} = +\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} = 2-(3) = -1$ ✓

$C_{12} = -\begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} = -(4+4) = -8$ ✓

$C_{13} = +\begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix} = -6-4 = -10$ ✓

$C_{21} = -\begin{vmatrix} -2 & 3 \\ -3 & 2 \end{vmatrix} = -(-4+9) = -5$ ✓

$C_{22} = +\begin{vmatrix} 3 & 3 \\ 4 & 2 \end{vmatrix} = (6-12) = -6$

$C_{23} = -\begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix} = -(-9+8) = 1$ ✓

$C_{31} = +\begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} = (2-3) = -1$ ✓

$C_{32} = -\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix} = -(-3-6) = 9$ ✓

$C_{33} = +\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = (3+4) = 7$ ✓

(continued →)

FP1

Matrices
Notes

~~(Continued→)~~

Example 13 (i)

$$\text{Adj}\,A = \begin{pmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{pmatrix}^T$$

$$= \begin{pmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{pmatrix} \checkmark \quad (2)$$

(ii)

$$A^{-1} = \frac{1}{|A|}\,\text{Adj}\,A$$

$$A = \begin{pmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{pmatrix}$$

$$\det A = a_{11}\cdot C_{11} + a_{12}C_{12} + a_{13}\cdot C_{13}$$
$$= 3(-1) + (-2)(-8) + 3(-10)$$
$$= -3 + 16 - 30 = -17 \quad (3)$$

$$\Rightarrow A^{-1} = \frac{1}{-17}\begin{pmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{pmatrix}$$

$$\text{or} = \frac{-1}{17}\begin{pmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{pmatrix}$$

(iii) Consider.

$$A\cdot A^{-1} = -\frac{1}{17}\begin{pmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{pmatrix} \times \begin{pmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{pmatrix}$$

$$= -\frac{1}{17}\begin{pmatrix} -3+16-30 & -15+12+3 & -3-18+21 \\ -2-8+10 & -10-6-1 & -2+9-7 \\ -4+24-20 & -20+18+2 & -4-27+14 \end{pmatrix}$$

$$= -\frac{1}{17}\begin{pmatrix} -17 & 0 & 0 \\ 0 & -17 & 0 \\ 0 & 0 & -17 \end{pmatrix}$$

$$= -\frac{1}{17} \times -17 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$\therefore\ A\cdot A^{-1} = I \checkmark$

(Continued→)

(We may verify $AA^{-1} = A^{-1}A = I$)

(continued)

Example 13(iv) To show $|Adj\, A| = |A|^2$ for given matrix A of order 3

from ③ $|A| = -17$ — ③

Now from ②

$$Adj\, A = \begin{pmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{pmatrix}$$

$|Adj\, A| = -1(-6\times7 - 1\times9) - (-5)(-8\times7 - (-10)\times9) - 1(-8\times1 - (-10)(-6))$

$= -1(-51) + 5(34) - 1(-68)$

$= 289$

$= (-17)^2$ — ④

from ③ and ④ $|Adj\, A| = |A|^2$

Note:

Give A is a matrix of order n, Then

$|Adj\, A| = |A|^{(n-1)}$

Note:

(i) $Adj\, AB = (Adj\, B)(Adj\, A)$

(ii) $|AB| = |A||B|$

(iii) $|A^{-1}| = \frac{1}{|A|}$ →

Verify (iii) "from Example 8"

$M = \begin{pmatrix} 3 & 1 \\ -11 & -2 \end{pmatrix} \Rightarrow |M| = 5$ — ①

$M^{-1} = \frac{1}{5}\begin{pmatrix} -2 & -1 \\ 11 & 3 \end{pmatrix}$

$= \begin{pmatrix} -\frac{2}{5} & -\frac{1}{5} \\ \frac{11}{5} & \frac{3}{5} \end{pmatrix}$

$\Rightarrow |M^{-1}| = -\frac{2}{5}\times\frac{3}{5} - \frac{11}{5}\times\left(-\frac{1}{5}\right)$

$= -\frac{6}{25} + \frac{11}{25} = \frac{1}{5}$ — ②

from ① & ② $|M^{-1}| = \frac{1}{|M|}$

Example 14 (i) $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(i) A, B, C are matrices of order 2.

Given $BAC = I$

• Show $A = B^{-1}C^{-1}$

Solution:

$BAC = I$

$(B^{-1}B)A(CC^{-1}) = B^{-1}IC^{-1}$

$\Rightarrow A = B^{-1}C^{-1}$ ✓

(ii) $AX = B$ A, B and X are matrices of the same order

Then $X = A^{-1}B$

(iii) Given a squared matrix A, which satisfy $x^2 - 6x + 17 = 0$ find A^{-1}, A is of order 2.

$\Rightarrow A^2 - 6A + 17I_2 = 0$

$A^{-1}.A^2 - 6A^{-1}A + 17A^{-1}.I_2 = 0$

$\Rightarrow A - 6I + 17A^{-1} = 0 \Rightarrow A^{-1} = -\frac{1}{17}(A - 6I_2)$

Some Properties of Determinants:

(i) Given $\text{Det}A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$k \cdot \text{Det}A = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

multiply to the elements of an one row (or column)

$$= \begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix} \quad \text{— (1)}$$

$$= k(\text{Det }A)$$

(ii) Given A is a matrix of order 3.
Then. $\text{Det}(kA)$

$$= \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_1 & kc_2 \\ ka_3 & kb_2 & kc_3 \end{vmatrix}$$

$$\underline{\text{Det}(kA) = k^3 \cdot \text{Det}A}$$

(iii) If any two rows (or columns) are identical, then its value $= 0$

$$\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0 \quad \text{as } C_1 = C_2$$

Example 14. without expanding find the value of det.

$$D = \begin{vmatrix} 4 & 8 & 12 \\ 1 & 2 & 3 \\ 7 & -5 & 1 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 7 & -5 & 1 \end{vmatrix} = 4 \times 0 = 0$$

Taking 4 common from each element of R_1 | Then $R_1 = R_2$ $\Rightarrow D = 0$

• Example 15 (i) Find the values of x, y and z

Given. $\begin{pmatrix} 4 & 3 \\ x & 5 \end{pmatrix} = \begin{pmatrix} y & z \\ 1 & 5 \end{pmatrix}$

$\Rightarrow x=1, \; y=4$ and $z=3$.

(ii) $2\begin{pmatrix} x & 5 \\ 7 & y-3 \end{pmatrix} + \begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 15 & 14 \end{pmatrix}$.

$\Rightarrow \begin{pmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 15 & 14 \end{pmatrix}$

$\Rightarrow 2x+3=7$ and $2y-4=14$

$\Rightarrow x=2$ ✓ $y=9$ ✓

(iii) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

Sol.

$\Rightarrow (2-20) = 2x^2-24 \Rightarrow 2x^2=6$

$\Rightarrow x = \pm\sqrt{3}$

(iv) Evaluate:

$$\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 6\times17 & 6\times3 & 6\times6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

$$= 6\begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

$= 6\times0$ as $R_1 = R_3$

$= 0$ ✓