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Further Pure Maths 1

Polar Coordinates Notes

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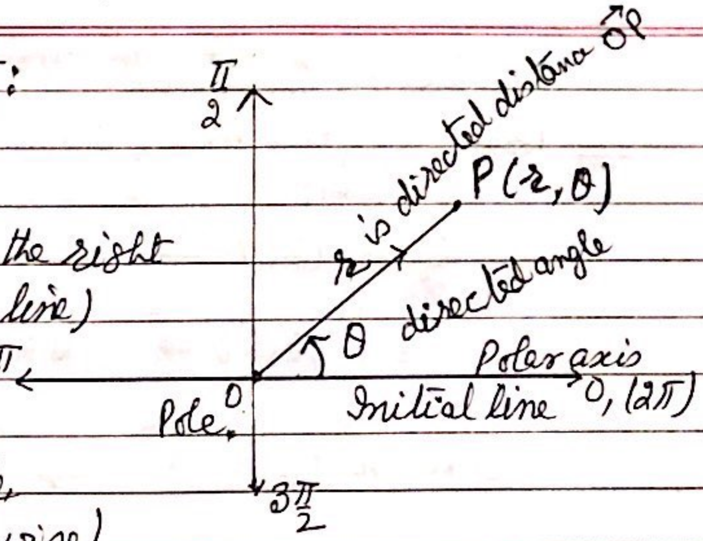
Polar Coordinates Notes

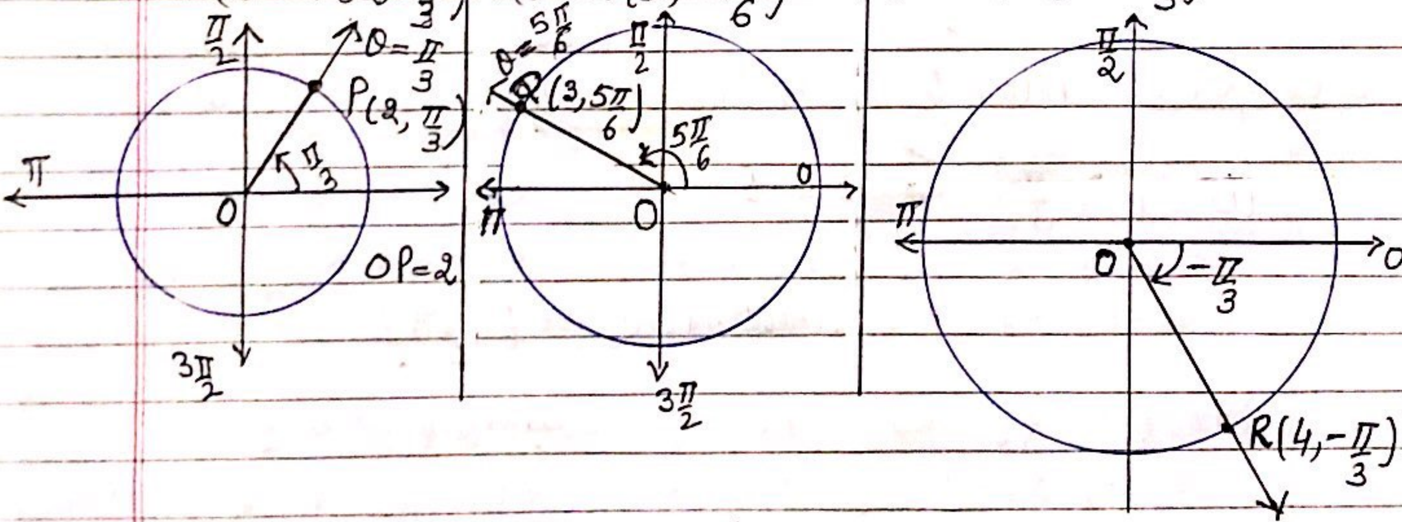
- Polar coordinates of a point:

Here 'O' is Pole.
Horizonal line through O, in the right of it is polar axis (or initial line)
r is the directed distance from O to P.
and θ is the directed angle, counterclockwise (anticlockwise) from polar axis to segment OP.



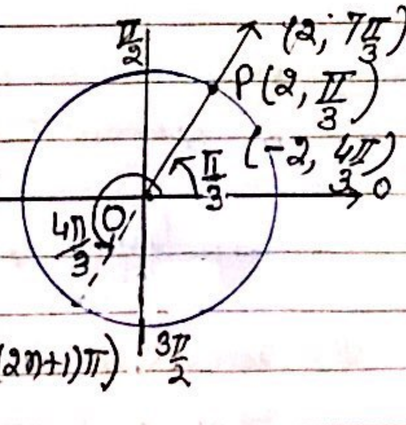
Example 1. Plot the following points in polar coordinates

(i) $P(2, \frac{\pi}{3})$ (ii) $Q(3, \frac{5\pi}{6})$ (iii) $R(4, -\frac{\pi}{3})$



Note: Each point (x, y) in Cartesian coordinate system has a unique representation,
But in Polar coordinate system $P(r, \theta)$ can be represented in multiple ways.

$$P(r,\theta) = (r, \theta + 2\pi) = (-r, \theta + \pi)$$



In general: $P(r,\theta) = (r, \theta \pm 2n\pi)$ or $= (-r, \theta \pm (2n+1)\pi)$

Example 1 (i) $P(2, \frac{\pi}{3}) = (2, \frac{7\pi}{3})$ or $(-2, \frac{4\pi}{3})$ etc.

(ii) $Q(3, \frac{5\pi}{6}) = (3, -\frac{7\pi}{6})$ (iii) $R(4, -\frac{\pi}{3}) = (4, \frac{5\pi}{3})$

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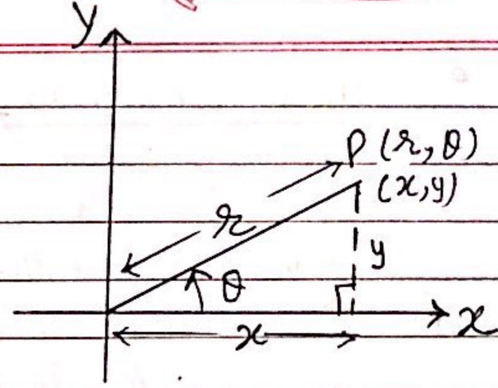
Coordinate conversion



Cartisian Coordinates $P(x, y)$ to Polar coordinate $P(r, \theta)$ and conversly:

$$\frac{x}{r} = \cos\theta \Rightarrow x = r\cos\theta$$

$$\frac{y}{r} = \sin\theta \Rightarrow y = r\sin\theta$$

and $x^2 + y^2 = r^2 \Rightarrow$ or $\begin{cases} \sin\theta = x/r \\ \cos\theta = \frac{y}{r} \end{cases}$

$$\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\frac{y}{x}$$

Note. $\begin{cases} \text{for angle } \tan^{-1}x, \text{ choose the} \\ \text{correct angle as per the} \\ \text{quadrant } P(x,y) \text{ lies in.} \end{cases}$

Example2: Polar to rectangular (cartesian) coordinates.

(i) $\left(2, \frac{\pi}{3}\right) \Rightarrow \begin{cases} x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\times\frac{1}{2} = 1 \\ y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\times\frac{\sqrt{3}}{2} = \sqrt{3} \end{cases}$

$r = 2, \ \theta = \frac{\pi}{3}$

$\therefore$ Rect. coord. are $\underline{(1, \sqrt{3})}$

(ii) $\left(4, -\frac{\pi}{3}\right) \Rightarrow \begin{cases} x = r\cos\theta = 4\cos\left(-\frac{\pi}{3}\right) = 4\times\frac{1}{2} = 2 \\ y = r\sin\theta = 4\sin\left(-\frac{\pi}{3}\right) = 4\times-\frac{\sqrt{3}}{2} = -2\sqrt{3} \end{cases}$

$r = 4, \ \theta = -\frac{\pi}{3}$

$\therefore$ Rect Coord. $(x, y) = \underline{(2, -2\sqrt{3})}$

Example3: Rectangular coordinates Polar coordinate;

(i) Given $(1, \sqrt{3})$, $r^2 = \sqrt{x^2+y^2} = \sqrt{1^2+(\sqrt{3})^2} = 4$

$\Rightarrow r = 2$ ✓

$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{1} \Rightarrow \theta = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$ ✓ (as $(1, \sqrt{3})$ lies in Ist quad.)

$\therefore$ Polar Coord. $(r, \theta) = \underline{\left(4, \frac{\pi}{3}\right)}$ ✓

(ii) Given $(2, -2\sqrt{3}]$;

$\Rightarrow r^2 = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{16} \Rightarrow r = 4$

$\tan\theta = \frac{y}{x} = -\frac{2\sqrt{3}}{2} = -\sqrt{3} \Rightarrow \theta = \tan^{-1}(-\sqrt{3}) = -\tan^{-1}\sqrt{3} = -\frac{\pi}{3}$

$\therefore$ Polar Coord $(r, \theta) = \underline{\left(4, -\frac{\pi}{3}\right)}$ ✓ $\begin{cases} \text{as } (2, -2\sqrt{3}) \text{ lies} \\ \text{in 4th quad.} \end{cases}$

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Converting Equation

- **Rectangular Equations to Polar Equations:**

Put $x = r\cos\theta$
$y = r\sin\theta$

Example 4 (i) $x^2+y^2=9$

$\Rightarrow (r\cos\theta)^2+(r\sin\theta)^2=9$

$\Rightarrow r^2(\cos^2\theta+\sin^2\theta)=9$

$\Rightarrow r^2=9$

$\Rightarrow \underline{r=3}$ ✓

$\left\{ \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \right.$

(ii) $y^3=x^2$

$\Rightarrow (r\sin\theta)^3=(r\cos\theta)^2$

$\Rightarrow r^3\sin^3\theta=r^2\cos^2\theta$

$\Rightarrow \underline{r=\cot^2\theta\cdot\operatorname{cosec}\theta.}$ ✓

(iii) $y^2-8x-16=0$

$r^2\sin^2\theta-8r\cos\theta-16=0$

using quad. formula.

b^2-4ac
$=(-8\cos\theta)^2-4\sin^2\theta\cdot(-16)$
$=64(\cos^2\theta+\sin^2\theta)$
$=64$

$$r=\frac{+8\cos\theta\pm\sqrt{64}}{2\sin^2\theta}$$

$$=\frac{8\cos\theta\pm8}{2(1-\cos^2\theta)}$$

$$=\frac{4(\cos\theta\pm1)}{(1+\cos\theta)(1-\cos\theta)}$$

$$\Rightarrow r=\frac{4}{1-\cos\theta} \quad \text{or} \quad r=\frac{-4}{1+\cos\theta}$$ ✓

(IV) $x^2+y^2-2ax=0$

$r^2-2ar\cos\theta=0$

$r(r-2a\cos\theta)=0$

$\Rightarrow r-2a\cos\theta=0$

$\Rightarrow \underline{r=2a\cos\theta}$ ✓

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- Polar Equations to Rectangular Equations:

Example 5 (i) $r = 4\sin\theta$

$\Rightarrow r^2 = 4r\sin\theta$ $\quad\because r^2 = x^2+y^2$

$\Rightarrow x^2+y^2 = 4y$ $\quad$ and $y = r\sin\theta$

or $\underline{x^2+y^2-4y=0}$ ✓

(ii) $r^2 = \cos\theta$

$\Rightarrow r^3 = r\cos\theta$ $\quad\because r = \sqrt{x^2+y^2}$

$(\sqrt{x^2+y^2})^3 = x$ $\quad x = r\cos\theta$

$\Rightarrow (x^2+y^2)^{3/2} = x$

$\Rightarrow x^2+y^2 = x^{2/3}$

$\Rightarrow \underline{x^2+y^2-x^{2/3}=0}$ ✓

(iii) $r = 2\sin 3\theta$

$r = 2(3\sin\theta - 4\sin^3\theta)$

$\Rightarrow r^4 = 6r^3\sin\theta - 8r^3\sin^3\theta$

$\Rightarrow (x^2+y^2)^2 = 6(x^2+y^2)\cdot y - 8y^3$

$\Rightarrow \underline{(x^2+y^2)^2 = 6x^2y-2y^3}$ ✓

(iv) $r = \dfrac{6}{2-3\sin\theta}$

$\Rightarrow 2r = 6+3r\sin\theta$

$\Rightarrow 2\cdot\sqrt{x^2+y^2} = 6+3y$

$\Rightarrow 4(x^2+y^2) = (6+3y)^2$

$\Rightarrow 4x^2+4y^2 = 36+36y+9y^2$

$\Rightarrow \underline{4x^2-5y^2-36y-36=0}$ ✓

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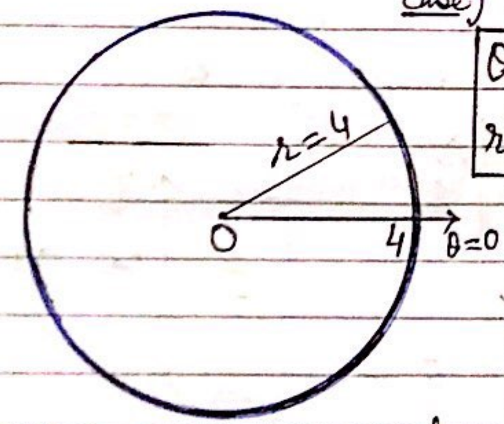
Polar Coordinates

- **Sketching curves with polar equations:**

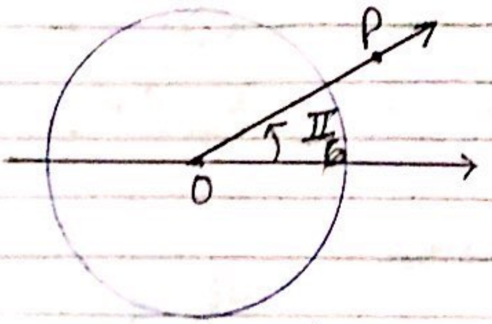
Sketch the graph of the following Polar Equations:

Example 6 (i) $r = 4$ (Particular Case)



Angle θ may be any angle.
So $r = 4$ represents a circle of radius $= 4$ unit
(or $r = 4 \Rightarrow r^2 = 16$
$\Rightarrow x^2 + y^2 = 16$
is the equation of circle with centre at origin $(0,0)$ and rad $= 4$)

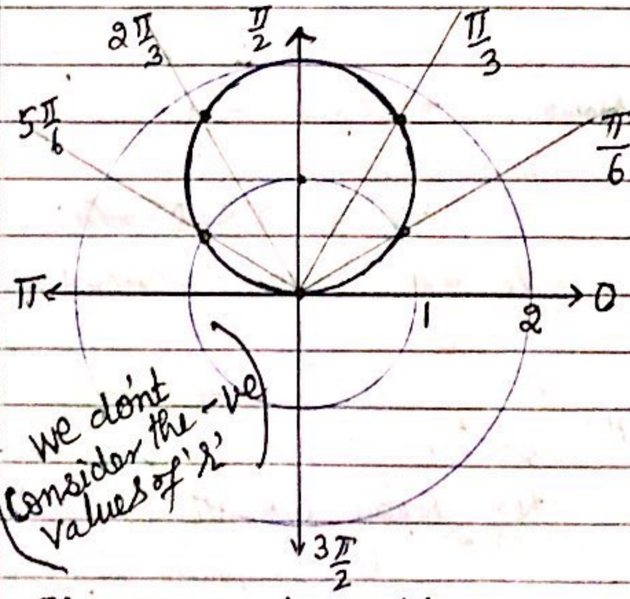
(ii) $\theta = \frac{\pi}{6}$ (Particular case)



$\theta = \frac{\pi}{6}$ represents a ray $\overrightarrow{OP}$ directed.

(iii) $r = 2\sin\theta \qquad 0 \le \theta \le 2\pi$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
r	0	1	$\sqrt{3}$	2	$\sqrt{3}$	1	0	-1	-2	-1	0



The required graph is a circle of radius 1 and centre $(x,y) = (0,2)$

Alternate method: convert the given polar equ^n $r = 2\sin\theta$ to rectangular coord. equ^n.

$r = 2\sin\theta$
$\Rightarrow r^2 = 2r\sin\theta$
$\Rightarrow x^2 + y^2 = 2y$
$\Rightarrow x^2 + y^2 - 2y = 0$ ✓ $\quad \left\{ \begin{array}{l} c = 0 \\ g = 0 \\ f = -1 \end{array} \right.$

which represent a circle with centre $(-g, -f) = (0, 1)$ ✓
radius $= \sqrt{g^2 + f^2 - c} = \sqrt{0^2 + (-1)^2 - 0}$
$r = 1$ ✓

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To. Draw the graph of Polar Equations

Symmetry of Graph in Polar Coordinates:

(i) Given a polar equation, $f(r, \theta) = f(r, \pi - \theta)$
The graph is symmetrical about $\theta = \frac{\pi}{2}$

(ii) $f(r,\theta) = f(r, -\theta)$ then graph is symmetrical about polar axis.

(iii) $f(r,\theta) = f(r, \pi + \theta)$, the graph is symmetrical about pole.

Example 7 (i)

(See example 6 (iii) - Page 5)

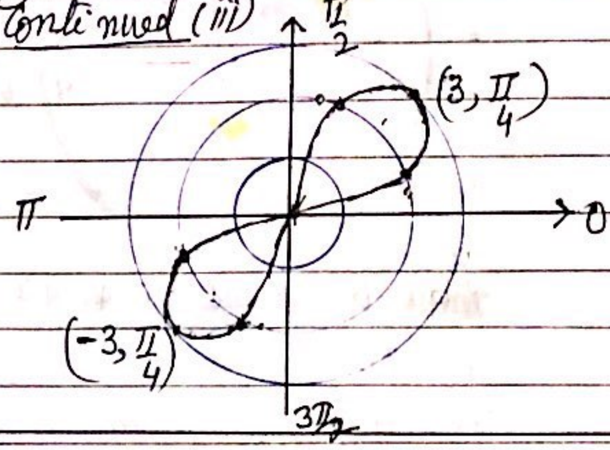
$r = f(\sin\theta)$: $r = 2\sin\theta$ — (I)

Consider $r = 2\sin(\pi - \theta)$

$\Rightarrow r = 2\sin\theta$ — (II)

the graph is symmetrical about $\theta = \frac{\pi}{2}$ ✓

(X) Continued (iii)



(ii) Sketch the graph of $r = 1 + 2\cos\theta$: $0 \le \theta \le 2\pi \Rightarrow$ "$1 \le 1 + 2\cos\theta \le 3$"

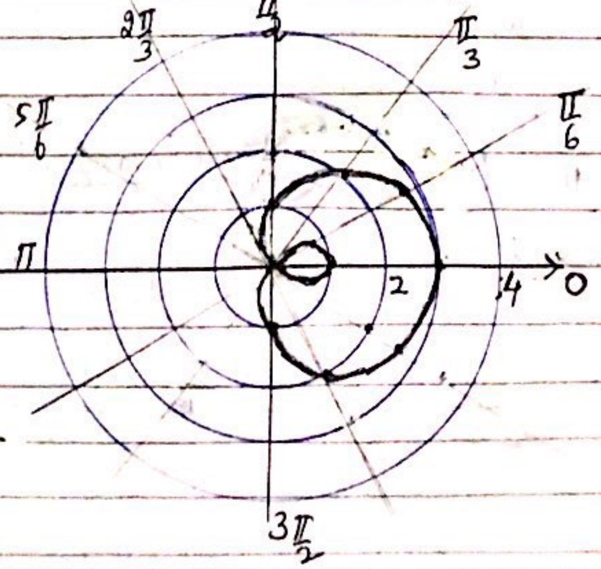
here Replace θ by $(-\theta) \Rightarrow r = 1 + 2\cos(-\theta) = 1 + 2\cos\theta$, No change

$r = f(\cos\theta)$ $\therefore \Rightarrow$ the graph is symmetrical about Polar axis.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	3	$1+\sqrt{3}$	2	1	0	$1-\sqrt{3}$	-1

-0.7 → 0.7; -1 → $+1$

(we never take the -ve value of r)



(iii) Sketch the Graph of $r^2 = 9\sin 2\theta$ — (1)

Now consider $r^2 = 9\sin 2(\pi + \theta)$

$= 9\sin(2\pi + 2\theta)$

$= 9\sin 2\theta$ — (2)

for (1) & (2) Polar equation (1) has a Symmetry about the pole

(X) (Continued)

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{4}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	0	$\pm\frac{3}{\sqrt{2}}$	± 3	$\pm\frac{3}{\sqrt{2}}$	0

$r = \pm 3\sqrt{\sin 2\theta}$

Max value of $|r| = 3$ when $\theta = \frac{\pi}{4}$

Zeros of r, $r = 0$ when $\theta = 0, \frac{\pi}{2}$

$\sin 2\theta < 0$, the eqn has no soln

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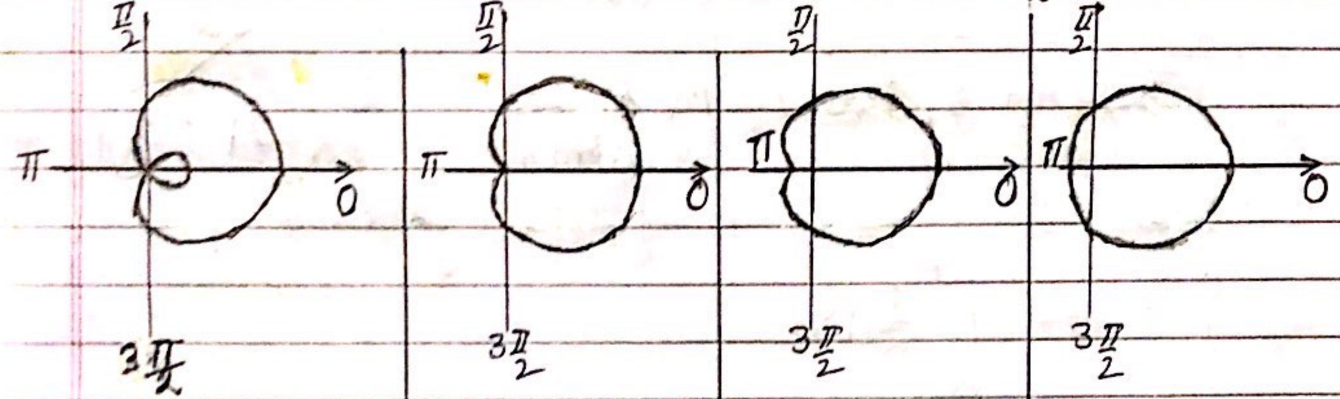
Graph of Polar Equations

Some Special Polar Graphs

- $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ $(a > 0;\ b > 0)$

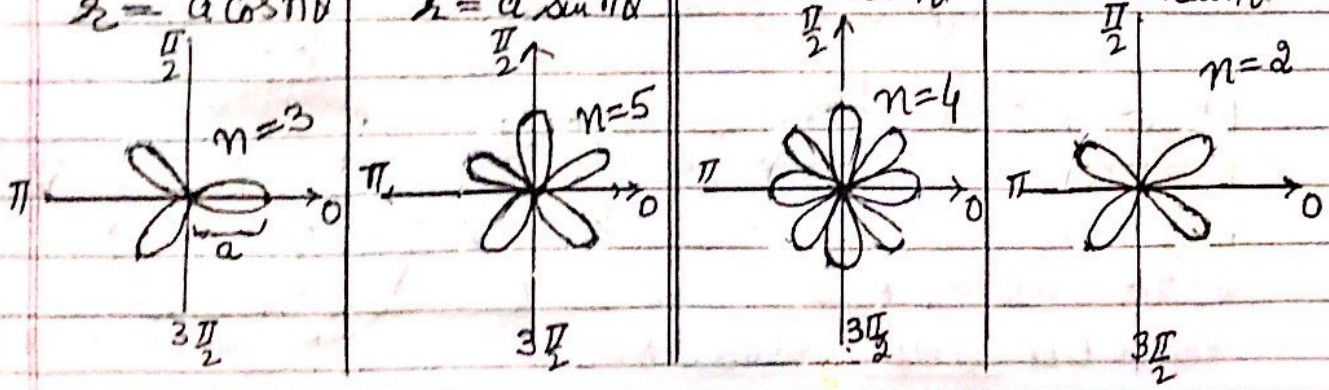


$\frac{a}{b} < 1$	$\frac{a}{b} = 1$	$1 < \frac{a}{b} < 2$	$\frac{a}{b} \geq 2$
Limacon with inner loop.	Cardioid (heart-shaped)	Dimpled limacon	Convex limacon

- Rose curves

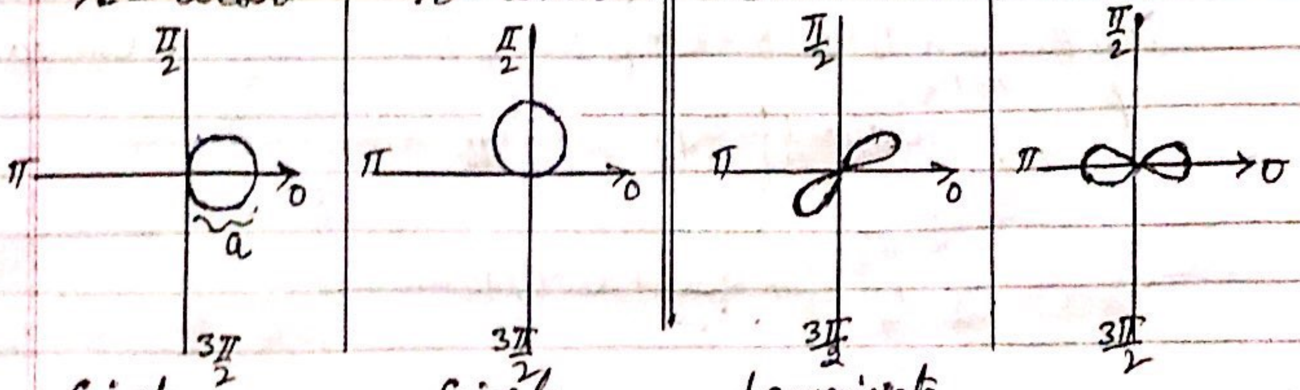
n petals when n is odd. $2n$ petals when n is even $(n \geq 2)$

$r = a\cos n\theta$	$r = a\sin n\theta$	$r = a\cos n\theta$	$r = a\sin n\theta$



- Circles and Lemniscates:

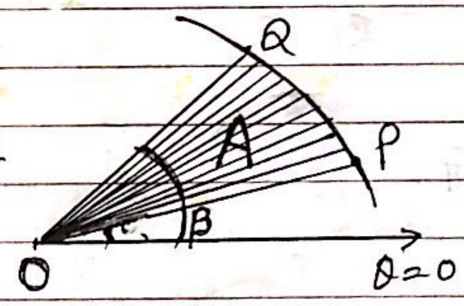
$r = a\cos\theta$	$r = a\sin\theta$	$r^2 = a^2\sin 2\theta$	$r^2 = a^2\cos 2\theta$



Circle	Circle	Lemniscate	

- **The area enclosed by a polar curve $r = f(\theta)$**

Given a polar curve $r = f(\theta)$



The area of region OPQ, bounded by a polar curve and two straight lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta.$$

Example 8. The curve C has polar equation $r = a(1+\sin\theta)$, where a is a positive constant and $0 \le \theta < 2\pi$

(i) Draw a sketch of C.

(ii) Find the exact value of the area of the region enclosed by C and the half lines $\theta = \frac{1}{3}\pi$ and $\theta = \frac{2}{3}\pi$

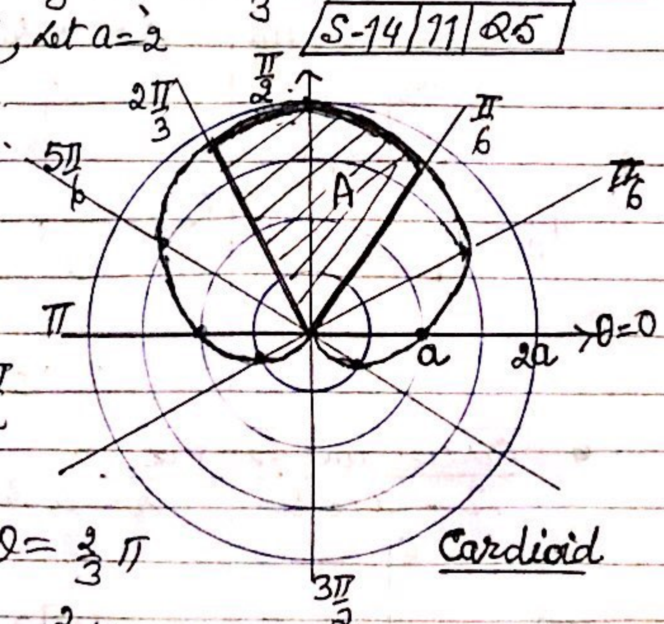
S-14 / 11 / Q5

Solution: $r = a(1+\sin\theta)$, $0 \le \theta \le 2\pi$, Let $a = 2$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
r	a	$\frac{3a}{2}$	2a	$\frac{3a}{2}$	a	$\frac{a}{2}$	0	$\frac{a}{2}$	a

$r = f(\theta) = f(\pi - \theta)$ [$\because \sin(\pi-\theta) = \sin\theta$]

Curve C is symmetrical about $\theta = \frac{\pi}{2}$



Area enclosed by the curve and the half lines $\theta = \frac{1}{3}\pi$ to $\theta = \frac{2}{3}\pi$

$$A = \frac{1}{2}\int_{\pi/3}^{2\pi/3} r^2 \, d\theta = \frac{1}{2}\int_{\pi/3}^{2\pi/3} a^2 (1+\sin\theta)^2 \, d\theta$$

$$= \frac{a^2}{2}\int_{\pi/3}^{2\pi/3} (1 + 2\sin\theta + \sin^2\theta)\, d\theta =$$

$$= \frac{a^2}{2}\int_{\pi/3}^{2\pi/3} \left(1 + 2\sin\theta + \left(\frac{1-\cos 2\theta}{2}\right)\right) d\theta$$

$$A = \frac{a^2}{2}\left[\theta - 2\cos\theta + \frac{\left(\theta - \frac{\sin 2\theta}{2}\right)}{2}\right]$$

$$= \frac{a^2}{2}\left[\frac{3\theta}{2} - 2\cos\theta - \frac{\sin 2\theta}{4}\right]_{\pi/3}^{2\pi/3}$$

$$= a^2\left(\frac{\pi}{4} + 1 + \frac{1}{8}\sqrt{3}\right)$$

Example 9: The curve C has polar equation $r = 2 + 2\cos\theta, 0 \le \theta \le \pi$,

(i) Sketch the graph of C.

(ii) Find the area of the region R enclosed by C and initial line.

(iii) The half line $\theta = \frac{1}{5}\pi$ divides R into two parts find the area of each part, correct to 3 decimal places. [S-12/11/Q4]

Solution: $r = 2 + 2\cos\theta$

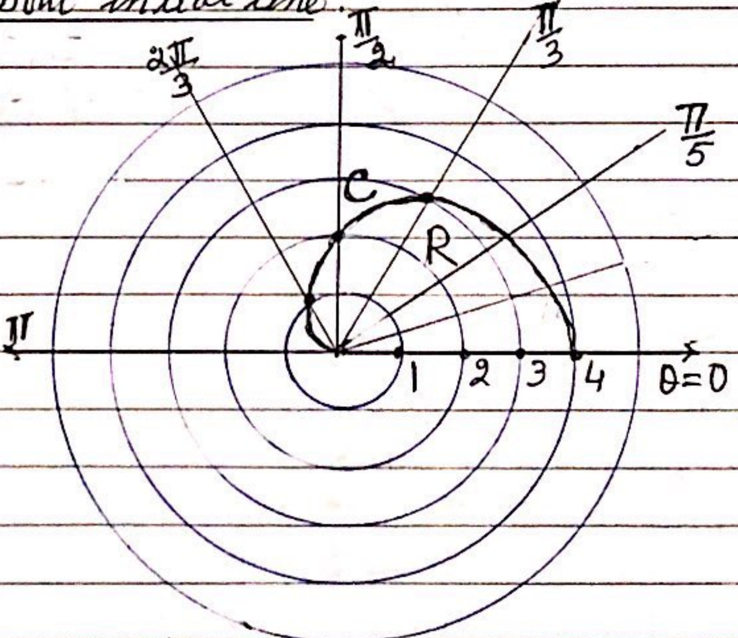
$0 \le \theta \le \pi$

[replace θ by $-\theta$

$\Rightarrow r = 2 + 2\cos(-\theta) = 2 + 2\cos\theta$]

[hence the graph will be symmetrical about initial line.]

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	4	3	2	1	0



Area enclosed between the curve C and initial line.

$$= \frac{1}{2}\int_0^{\pi} \{2(1+\cos\theta)\}^2 d\theta$$

$$= 2\int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta) d\theta$$

$$= 2\int_0^{\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \int_0^{\pi} (3 + 4\cos\theta + \cos 2\theta) d\theta \quad \text{—(1)}$$

$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$A_R = \underline{3\pi} \checkmark$$

Now area enclosed by $\theta = \frac{\pi}{5}$ and the initial line: from (1)

$$A_1 = \int_0^{\frac{\pi}{5}} (3 + 4\cos + \cos 2\theta) d\theta$$

$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/5}$$

$$= [1.8849 + 2.35114 + 0.47552]$$

$$= \underline{4.712} \checkmark$$

Area of of the second part

$$A_2 = A - A_1$$

$$= 3\pi - 4.712$$

$$= \underline{4.713} \checkmark$$

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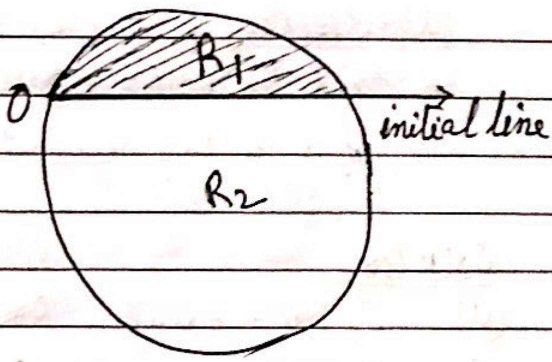
Example 10. The diagram shows a sketch of the loop whose polar equation is:

$$r = 2(1-\sin\theta)\sqrt{\cos\theta} \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

where O is the pole.

(a) Show that the area enclosed by the loop is $16/3$.

(b) Show that the initial line divides the area enclosed by the loop in the ratio 1:7.



Solution:

$$r = 2(1-\sin\theta)\sqrt{\cos\theta} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$\text{Area of the loop} = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2\, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(2(1-\sin\theta)\sqrt{\cos\theta}\right)^2 d\theta$$

$$= \frac{1}{2}\times 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\sin\theta)^2 \cdot \cos\theta\, d\theta \quad \text{—(1)}$$

Put $\sin\theta = u$
diff $\cos\theta\, d\theta = du$
$\theta = \frac{\pi}{2} \Rightarrow u = 1$
$\theta = -\frac{\pi}{2} \Rightarrow u = -1$

$$= 2\int_{-1}^{1} (1-u)^2\, du$$

$$A = 2\left[\frac{(1-u)^3}{-3}\right]_{-1}^{1}$$

$$= -\frac{2}{3}\left[0 - \frac{8}{3}\right]$$

$$A = \frac{16}{3} \checkmark$$

from (1) Area R_1

$$2\int_{0}^{\frac{\pi}{2}} (1-\sin\theta)^2 \cos\theta\, d\theta$$

Using the same substitution
$\theta = 0 \Rightarrow u = 0$

$$= 2\int_{0}^{1} (1-u)^2\, du$$

$$R_1 = 2\left[\frac{(1-u)^3}{-3}\right]_0^1$$

$$R_1 = 2\left[0 + \frac{1}{3}\right] = \frac{2}{3} \checkmark$$

$$\therefore R_2 = A - R_1 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3} \checkmark$$

$$\therefore R_1 : R_2 = \frac{2}{3} : \frac{14}{3} = 1:7 \checkmark$$

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Example 11: The curves C_1 and C_2 have polar equations.

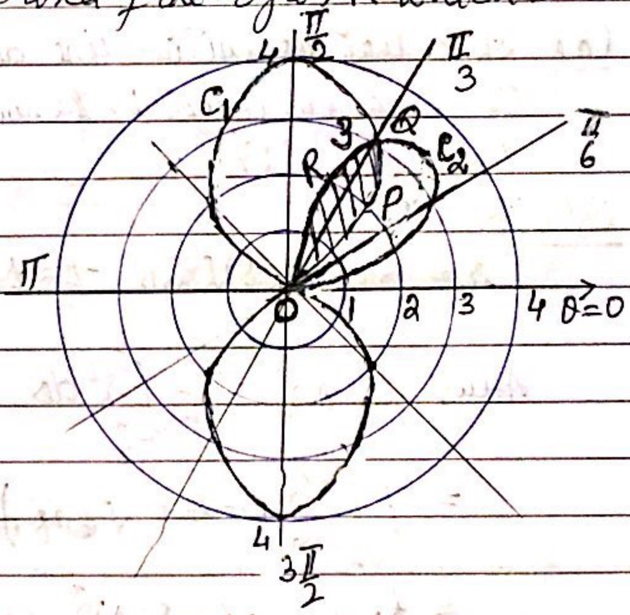
C_1: $r = 4\sin^2\theta$; $0 \le \theta < 2\pi$

C_2: $r = (2\sqrt{3})\sin 2\theta$; $0 \le \theta < \frac{1}{2}\pi$

(a) Sketch C_1 and C_2 on the same diagram.

(b) Find the polar coordinates of all points of intersection of C_1 and C_2.

(c) Find, to two decimal places, the area of the region R which is inside both C_1 and C_2.



Solution

(a) $C_1 : r = 4\sin^2\theta$, $0 \le \theta < 2\pi$

C_1 is symmetrical about $\theta = \frac{\pi}{2}$ also about initial line

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	1	2	3	4

C_2: $r = (2\sqrt{3})\sin 2\theta$, $0 \le \theta < \frac{1}{2}\pi$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	3	$2\sqrt{3}$	3	0

(b) for Intersection of C_1 and C_2.

$4\sin^2\theta = 2\sqrt{3}\cdot\sin 2\theta$

$\Rightarrow 4\sin^2\theta = 2\sqrt{3}\times 2\sin\theta\cos\theta$

$\sin^2\theta - \sqrt{3}\sin\theta\cos\theta = 0$

$\sin\theta(\sin\theta - \sqrt{3}\cos\theta) = 0$

$\sin\theta = 0$ or $\tan\theta = \sqrt{3}$

$\Rightarrow \theta = 0$ or $\theta = \frac{\pi}{3}$

$\therefore$ Point of intersection from (1)

(r,θ) ; $\underline{(0,0)}$ and $\underline{\left(3, \frac{\pi}{3}\right)}$ ✓

(iii)

Area of region R.

$R =$ area OPQO + area OQRO

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2_{(C_1)}\, d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} r^2_{(C_2)}\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} (4\sin^2\theta)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 12\sin^2 2\theta\, d\theta$$

$$= \frac{1}{2}\int 16\cdot\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$$

$$+ \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 12\times(1-\cos^2 2\theta)\, d\theta$$

(continued →)

FP1

(Continued →)

Example 11 (iii)

$$R = 2\int_0^{\frac{\pi}{3}} (1-2\cos 2\theta+\cos^2 2\theta)\,d\theta + 6\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(1-\left(\frac{1+\cos 4\theta}{2}\right)\right)d\theta$$

$$= 2\int \left(1-2\cos 2\theta + \left(\frac{1+\cos 4\theta}{2}\right)\right)d\theta + 3\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1-\cos 4\theta)\,d\theta$$

$$= \int (2-4\cos 2\theta + 1 + \cos 4\theta)\,d\theta + 3\left[\theta - \frac{\sin 4\theta}{4}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \int_0^{\frac{\pi}{3}} (3-4\cos 2\theta+\cos 4\theta)\,d\theta + 3\left[\left(\frac{\pi}{2}-0\right)-\left(\frac{\pi}{3}-\frac{1}{4}\left(-\frac{\sqrt{3}}{2}\right)\right)\right]$$

$$= \left[3\theta - \frac{4\sin 2\theta}{2} + \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{3}} + 3\left[\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right]$$

$$= \left[\left(\pi - 2\sin\frac{2\pi}{3} + \frac{1}{4}\sin\frac{4\pi}{3}\right) - 0\right] + \frac{\pi}{2} - \frac{3\sqrt{3}}{8}$$

$$= \left(\pi - \sqrt{3} - \frac{\sqrt{3}}{8}\right) + \text{---}$$

$$= \left(\pi - \frac{9\sqrt{3}}{8}\right) + \frac{\pi}{2} - \frac{3\sqrt{3}}{8}$$

$$= \frac{3\pi}{2} - \frac{12\sqrt{3}}{8}$$

$$= \underline{2.11}\ \checkmark$$