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Further Pure Maths 1

Proof by Induction Exercise

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Proof by induction

Date

1. It is given that, $\phi(n) = 5^n(4n+1) - 1$, for $n = 1, 2, 3, \ldots$
Prove, by mathematical induction that $\phi(n)$ is divisible by 8, for every positive integer n. ---[7]
[SP-20|01|Q2]

2. (i) Prove by mathematical induction that, for $z \neq 1$ and all positive integers n,
$$1 + z + z^2 + \cdots + z^{n-1} = \frac{z^n - 1}{z - 1}$$ --[5]
[S-19|11|Q8(i)]

3. Prove by mathematical induction that, $3^{3n} - 1$ is divisible by 13 for every positive integer n. --[5]
[S-19|13|Q1]

4. It is given that $f(n) = 2^{3n} + 8^{n-1}$. By simplifying $f(k) + f(k+1)$, or otherwise, prove by mathematical induction that $f(n)$ is divisible by 9, for every positive integer n. --[6]
[S-18|11|Q2]

5. For the sequence $u_1, u_2, u_3, \ldots$, it is given that $u_1 = 8$ and
$$u_{r+1} = \frac{5u_r - 3}{4} \quad \text{For all } r.$$
(i) Prove by mathematical induction that,
$$u_n = 4\left(\frac{5}{4}\right)^n + 3$$
for all positive integer n. --[5]
(ii) Deduce the set of value of x for which the infinite series, $(u_1 - 3)x + (u_2 - 3)x^2 + \cdots + (u_r - 3)x^r + \cdots$ is convergent. --[2]
(iii) Use the result given in part (i) to find surds a and b such that,
$$\sum_{n=1}^{N} \ln(u_n - 3) = N^2 \ln a + N \ln b.$$ --[3]
[S-18|13|Q9]

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6. The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 < 3$ and for $n \geq 1$,

$$u_{n+1} = \frac{4u_n + 9}{u_n + 4}$$

(i) By considering $3 - u_{n+1}$, or otherwise, prove by mathematical induction that $u_n < 3$ for all positive integers n. --[5]

(ii) Show that $u_{n+1} > u_n$ for $n \geq 1$ --[3]

[W-18 | 11 | Q3]

7. Prove, by mathematical induction, that $5^n + 3$ is divisible by 4 for all non-negative integer n. --[5]

[S-17 | 11 | Q2]

8. Prove, by mathematical induction, that

$$\sum_{r=1}^{n} r \ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(n+1)^n}{n!}\right)$$

for all positive integer n. --[6] [S-17 | 13 | Q3]

9. Prove by mathematical induction that, for all positive integers n, $10^n + 3 \times 4^{n+2} + 5$ is divisible by 9. --[6]

[S-16 | 11 | Q3]

10. It is given that a diagonal of a polygon is a line joining two non-adjacent vertices. Prove, by mathematical induction, that the n-sided polygon has $\frac{1}{2} n(n-3)$ diagonal, when $n \geq 3$. --[6] [S-16 | 13 | Q2]

11. Using factorials show that, $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ --[2]

Hence Prove by mathematical induction that:

$(a+x)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \ldots \binom{n}{r} a^{n-r} \cdot b^r + \ldots + \binom{n}{n} x^n$

for all positive integer n. --[4]

[W-16 | 11 | Q4]

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12. The sequence $a_1, a_2, a_3, \ldots$ is such that $a_1 > 5$ and $a_{n+1} = \frac{4a_n}{5} + \frac{5}{a_n}$ for every positive integer n. Prove by mathematical induction that $a_n > 5$ for every positive integer n. --[5]
Prove also that $a_n > a_{n+1}$ for every positive integer n.
$\boxed{S-15/11/Q3}$

13. Prove by mathematical induction that, for all positive integers n, $\sum_{r=1}^{n} \frac{1}{(2r)^2 - 1} = \frac{n}{(2n+1)}$ --[6]
State the value of $\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1}$. --[1] $\boxed{S-15/13/Q3}$

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Answers on Page 4 onwards:

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Answers

$\phi(n) = 5^n(4n+1)-1$,

1. $\phi(1) = 5\times5-1 = 24$ which is div by 8. hence $P(1)$ is True.

Assume P_k is true for some positive integer k, i.e. $\phi(k) = 8l$ — (1)

Consider $\phi(k+1)-\phi(k)$

$= 5^{k+1}(4k+5)-1 - 5^k(4k+1)+1$

$= 5^k(16k+24) = 8\cdot5^k(2k+3)$

$\therefore \phi(k+1)-\phi(k) = 8m$ (let)

$\phi(k+1) - 8l = 8m$ (fr (1))

$\phi(k+1) = 8(l+m)$

$\therefore$ $\phi(n)$ is True for all n, using PMI.

Alternate method:

$\phi(1) = 5\times5-1 = 24$ which is div. by 8

$\therefore$ $P(1)$ is True

now let $P(k)$ is True for some positive integer k, $\Rightarrow \phi(k) = 8l$ — (1)

Now $\phi(k+1) = 5^{k+1}(4k+5)-1$

$= 5(4k\cdot5^k)+25\times5^k-1$

fr (1) $= 5(8l-5^k+1)+25\times5^k-1$

$= 40l+20\cdot5^k+4$

$= 40l+4(5^{k+1}+1)$

$= 40l+4\times(2m)$

$= 8(5l+m)$ ($\because (5^{k+1}+1)$ is even)

So $\phi(k+1)$ is divisible by 8,

$\therefore$ Using PMI $\phi(n)$ is true for all positive integers.

2. for $n=1$, $1 = \frac{z^1-1}{z-1}$ is True

$\therefore$ $\phi(1)$ is True

Now let for $n=k$, let

$1+z+\dots+z^{k-1} = \frac{z^k-1}{z-1}$ is true. — (1)

add z^k on both sides of (1)

$1+z+\dots+z^{k-1}+z^k = \frac{z^k-1}{z-1}+z^k$

$= \frac{z^{k+1}-1}{z-1}$

so true for $n=k+1$

$H_k \to H_{k+1}$, hence using PMI, true for all positive integers.

3. for $n=1$, $3^3-1 = 26$ is divisible by 13. hence $\phi(1)$ is true.

let for $n=k$

$(3^{3k}-1)$ is divisible by 13. — (1)

Consider for $n=k+1$.

$3^{3(k+1)}-1 = 27\cdot3^{3k}-1$

$= (26+1)3^{3k}-1$

$= (3^{3k}-1)+26\cdot3^{3k}$

The first term is div by 13 from (1) and second term is clearly div. so their sum.

$\therefore$ $H_k \Rightarrow H_{k+1}$, using PMI the statement is true for all positive integers.

4. $\phi(1) = 9$ divisible by 9,

$\phi(1)$ is true.

Assume $\phi(k)$ is true

or $(2^{3k}+8^{k-1})$ is div by 9 — (1)

Now consider $f(k+1)+f(k)$

$= 2^{3k+3}+8^k+2^{3k}+8^{k-1}$

$= 2^3\cdot2^{3k}+8\cdot8^{k-1}+2^{3k}+8^{k-1}$

$= 9(2^{3k}+8^{k-1})$ is div by 9

$\therefore$ $f(k+1)$ is div by 9 when $f(k)$

$\therefore$ Using PMI, the statement is true for all positive int n.

Answers

5.(i) $P_n : u_n = 4\left(\frac{5}{4}\right)^n + 3$

let $n = 1, \Rightarrow 4\left(\frac{5}{4}\right)^1 + 3 = 8 \Rightarrow$ P_1 is true.

let P_k is true for some k, then

$u_{k+1} = \frac{1}{4}\left[5\left(4\left(\frac{5}{4}\right)^k + 3\right) - 3\right]$

$= 4\left(\frac{5}{4}\right)^{k+1} + 3$

$\therefore P_k \Rightarrow P_{k+1}$, $\therefore$ using PMI P_n is true for all positive integers.

(ii) $(u_n - 3)x^n = 4x^n\left(\frac{5}{4}\right)^n$

$= 4\left(\frac{5x}{4}\right)^n \Rightarrow r = \frac{5x}{4}$

so the series is convergent for $-1 < \frac{5x}{4} < 1 \Rightarrow -\frac{4}{5} < x < \frac{4}{5}$

(iii) $\sum_{n=1}^{N} \ln(u_n - 3) = \sum_{n=1}^{N} \ln\left(4\left(\frac{5}{4}\right)^n\right)$

$= \ln\frac{5}{4}\sum_{n=1}^{N} n + \sum_{n=1}^{N} \ln 4$

$= \frac{1}{2}N(N+1)\ln\frac{5}{4} + N\ln 4$

$= N^2 \ln\frac{\sqrt{5}}{2} + N\ln(2\sqrt{5})$

$\Rightarrow a = \frac{\sqrt{5}}{2}, \; b = 2\sqrt{5}$

6.(i) $u_1 < 3$ (given)

Assume that $u_k < 3$

Then $3 - u_{k+1} = 3 - \frac{4u_k + 9}{u_k + 4} = \frac{-u_k + 3}{u_k + 4} > 0$

$\Rightarrow u_{k+1} < 3$

$\therefore$ by PMI, $u_n < 3$ for all $n \geq 1$

(ii) $u_{n+1} - u_n = \frac{4u_n + 9}{u_n + 4} - u_n = \frac{-u_n^2 + 9}{u_n + 4}$

So $u_n < 3 \Rightarrow u_{n+1} - u_n \geq 0$

7. let P_n denotes $5^n + 3$ is divisible by 4.

for $P_1 \rightarrow 5^1 + 3 = 8$ is divisible by 4,

$\therefore$ P_1 is true.

let P_k is true for positive int k.

$\Rightarrow 5^k + 3$ is divisible by 4

Consider P_{k+1} is $\Rightarrow 5^k + 3 = 4\alpha$ —(1)

$5^{k+1} + 3 = 5 \cdot 5^k + 3$

$= 5(4\alpha - 3) + 3$

$= 20\alpha - 12 = 4(5\alpha - 3)$

using (1) and $4(5\alpha - 3)$ is divisible by 4

$\Rightarrow P_{k+1}$ is true for P_k is true

$\therefore$ using PMI, P_n is true for all $n \geq 1$.

8. for $n = 1$, $1 \cdot \ln 2 = \ln 2 = \ln\left(\frac{2}{1!}\right) \Rightarrow P_1$ is ✓ True

assume for some positive integer k, $\sum_{r=1}^{k} r\ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(k+1)^k}{k!}\right)$ —(1)

Consider P_{k+1},

$\sum_{r=1}^{k+1} r\ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(k+1)^k}{k!}\right) + (k+1)\ln\left(\frac{k+2}{k+1}\right)$

$= \ln\frac{(k+1)^k \cdot (k+2)^k}{k! \cdot (k+1)^{k+1}}$

$= \ln\frac{(k+1)^k \cdot (k+2)^k}{(k+1)!\,(k+1)^k}$

$= \ln\frac{(k+2)^k}{(k+1)!}$

$\therefore P_k \Rightarrow P_{k+1}$

$\therefore$ using PMI we conclude that P_n is true for all positive $n \geq 1$.

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Answers

9. for $n=1$, $10+192+5$
$= 207 = 9 \times 23 \Rightarrow P_1$ is true

Let P_K is True for some posi. Int
$\Rightarrow 10^K + 3 \times 4^{K+2} + 5 = 9\alpha$ — (1)

Consider $P_{K+1} - P_K$
$= 10^K(10-1) + 3 \cdot 4^{K+2}(4-1)$
$= 9(10^K + 4^{K+2})$
$= 9\beta.$

$\therefore P(K+1) - P(K) = 9\beta$
$\rightarrow P(K+1) - 9\alpha = 9\beta$ fr (1)
$\Rightarrow P(K+1) = 9(\beta + \alpha)$

$\therefore P_K \Rightarrow P_{K+1}$ True
$\therefore$ Using PMI, P_n is True for all positive integers.

10. with $n=3$, $\frac{1}{2} n(n-3) = 0$
A triangle has no diagonal.
$\therefore P_3$ is True

Now P_K is True, ie, a K sided polygon has $\frac{1}{2}K(K-3)$ diagonals for $K \geq 3$ — (1)

Note: "Adding an extra vertex, a further $(K-1)$ diagonals can be drawn"

$\therefore$ No. of diagonal for $(K+1)$ vertices
$= \frac{1}{2}K(K-3) + (K-1) = \frac{K^2 - 3K + 2K - 2}{2}$
$= \frac{1}{2}(K+1)(K-2)$
$= \frac{1}{2}(K+1)[(K+1)-3]$
$= P_{K+1}$

$\therefore P_{K+1}$ is True for P_K is True
$\therefore$ using PMI, the given statement is true for all positive integer $n \geq 3$.

11. $\binom{n}{r-1} + \binom{n}{r}$

$= \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!}$

$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{n-r+1} + \frac{1}{r}\right]$

$= \frac{n!}{(r-1)!(n-r)!} \times \frac{(n+1)}{r(n+1-r)}$

$= \frac{(n+1)!}{r!(n+1-r)!} = \binom{n+1}{r}$ — (1)

Now $(a+x)^1 = \binom{1}{0}a + \binom{1}{1}x = a + x$
$\therefore P(1)$ is True.

Now let P_K is true $\Rightarrow$
$(a+x)^K = \binom{K}{0}a^K + \binom{K}{1}a^{K-1}x + \cdots$
$\cdots + \binom{K}{r}a^{K-r}x^r + \binom{K}{K}x^K$

multiply by $(a+x)$ on both side
the coeff of $x^{K-r+1} = \binom{K}{r-1} + \binom{K}{r}$
$= \binom{K+1}{r}$ fr (1)

$\Rightarrow P_{K+1}$ is True for P_K is True
$\therefore$ PMI P_n is True for all positive integers $n \geq 1$

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12. **Answers**

$a_1 > 5$ (given) $\Rightarrow P_1$ is True

Let P_K is True for some positive integer K. $\Rightarrow a_K = 5+\delta,\ \delta > 0$

Now consider $a_{K+1} - 5$

$= \frac{4a_K^2 + 25}{5a_K} - 5$

$= \frac{4a_K^2 + 25 - 25a_K}{5a_K} = \frac{(4a_K - 5)(a_K - 5)}{5a_K} > 0$

$\Rightarrow a_{K+1} > 5$

$\therefore P_K \Rightarrow P_{K+1}$. $\Rightarrow P_n$ is True for all positive int. $n \geq 1$.

Again $a_{K+1} - a_K = \frac{5}{a_K} - \frac{1}{5}a_K$

as $\frac{5}{a_K} < 1$ and $\frac{1}{5}a_K > 1$

$\Rightarrow a_{K+1} - a_K < 0$

$\Rightarrow a_{K+1} > a_K$

or $a_{n+1} > a_n$

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13. $P_K : \sum_{r=1}^{K} \frac{1}{(2r)^2 - 1} = \frac{K}{2K+1}$

Let is True for some 'k' integer.

Now $\frac{1}{2^2 - 1} = \frac{1}{3} = \frac{1}{2\times1+1} = P_1$ is ✓ True

Now $\frac{K}{2K+1} + \frac{1}{(2K+2)^2 - 1}$

$= \frac{K}{2K+1} + \frac{1}{(2K+1)(2K+3)}$

$= \frac{2K^2 + 3K + 1}{(2K+1)(2K+3)}$

$= \frac{(2K+1)(K+1)}{(2K+1)(2K+3)} = \frac{K+1}{2(K+1)+1}$

$\therefore P_K \Rightarrow P_{K+1}$, using PMI

P_n is True for all positive $n \geq 1$

(Continued →)

Now $\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1} =$

$= \lim_{n\to\infty} \frac{n}{2n+1}$ [use P_n]

$= \lim_{n\to\infty} \frac{1}{2 + \frac{1}{n}}$

$= \frac{1}{2}$ ✓ $\left[\lim_{n\to\infty} \frac{1}{n} = 0\right]$

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