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Further Pure Math_1

Proof by Induction Notes

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Proof by Induction Notes

- **The Principle of Mathematical Induction (PMI):**

Suppose there is a given statement $P(n)$ involving the positive integer n such that:
(i) The statement is true for $n=1$, or $P(1)$ is true, and
(ii) If the statement is true for $n=k$ (assumed $P(k)$ is true) where k is some positive integer, then if we prove that the statement is true for $n=k+1$, i.e. If $P(k)$ is true implies $P(k+1)$ is true.
The $P(n)$ is true for all positive integers n.

- Example 1. Prove using PMI for all positive integers n.

$$1^3+2^3+3^3+ \cdots n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Proof: Let the given statement be denoted by $P(n)$.
(i) for $n=1$, $1^3 = \left(\frac{1(1+1)}{2}\right)^2 \Rightarrow 1=1$

$\therefore$ $P(1)$ is true ✓

(ii) Let $P(k)$ is true, i.e for $n=k$, where k is some positive integer,

$\therefore$ let $1^3+2^3+3^3+\cdots+k^3 = \left(\frac{k(k+1)}{2}\right)^2$ is true —— ①

Consider $P(k+1)$ to be proved

$$1^3+2^3+3^3+\cdots+k^3+(k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2 \quad \text{—— ②}$$

add $(k+1)^3$ on both sides of ①

$$\Rightarrow 1^3+2^3+3^3+\cdots+k^3+(k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{(k+1)^2 \cdot [k^2+4(k+1)]}{4}$$

$$= \frac{(k+1)^2 \cdot (k+2)^2}{4}$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)^2 = \text{R.H.S of ②}$$

Hence $P(k+1)$ is true whenever $P(k)$ is true and $P(1)$ is true. Using PMI. $P(n)$ is true for all positive integers n.

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- Example 2. Prove by using PMI

$$\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\frac{1}{7\cdot9}+\cdots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

Proof: Let the given statement to be proved be denoted by $P(n)$, where n is a positive integer.

(i) for $n=1$, $\frac{1}{3\cdot5}=\frac{1}{3\cdot(2\times1+3)} \Rightarrow \frac{1}{15}=\frac{1}{15}$ True.

$\therefore$ $P(1)$ is true.

(ii) Let the statement is true for $n=k$, i.e $P(k)$ is true for some positive integer k. or

Let $\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\frac{1}{7\cdot9}+\cdots+\frac{1}{(2k+1)(2k+3)}=\frac{k}{3(2k+3)}$ — ①

consider $P(k+1)$ to be proved:

$\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\cdots+\frac{1}{(2k+1)(2k+3)}+\frac{1}{(2k+3)(2k+5)}=\frac{(k+1)}{3(2k+5)}$ — ②

add $\frac{1}{(2\cdot+3)(2k+5)}$ on both sides of ①

we get:

$$\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\cdots+\frac{1}{(2k+1)(2k+3)}+\frac{1}{(2k+3)(2k+5)}=\frac{k}{3(2k+3)}+\frac{1}{(2k+3)(2k+5)}$$

$$=\frac{k(2k+5)+3}{3(2k+3)(2k+5)}$$

$$=\frac{2k^2+5k+3}{3(2k+3)(2k+5)}$$

$$=\frac{(k+1)(2k+3)}{3(2k+3)(2k+5)}$$

$$=\frac{(k+1)}{3(2k+5)}=\text{RHS of ②}$$

$\therefore$ $P(k+1)$ is true for $P(k)$ is true and $P(1)$ is true.

$\therefore$ Using PMI, $P(n)$ is true of all positive integers n. $n\geq1$

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• **Example 3;** Using Principle of Mathematical Induction (PMI) prove: $10^{2n-1}+1$ is divisible by 11.

Proof; Let the given statement is denoted by $P(n)$.

(i) for $n=1$, $10^{2\times1-1}+1$
$= 11$ which is divisible by 11.
$\therefore$ P(1) is true ✓

(ii) Let P(k) is true for some positive integer k.
i.e. $(10^{2k-1}+1)$ is divisible by 11. — ①
or $(10^{2k-1}+1)=11m$ — ①' where m is some positive integer.

Consider P(k+1),
$$10^{2(k+1)-1}+1$$
$$= 10^2 \cdot 10^{2k-1}+1$$
$$= (99+1)\,10^{2k-1}+1$$
$$= 99\cdot10^{2k-1}+(10^{2k-1}+1)$$
$$= 99\cdot10^{2k-1}+11m \quad \text{frm ①}$$
$$= 11\cdot(9\cdot10^{2k-1}+m)$$

$\therefore$ P(k+1) is (divisible by 11) True when P(k) is true and P(1) is true.
$\therefore$ Using PMI P(n) is true for all positive integers n.

Example 4; Prove that. Using PMI
$2\cdot7^n+3\cdot5^n-5$ is divisible by 24.

Proof; Let the statement be denoted by P(n), for any positive integer n. $P(n): (2\times7^n+3\cdot5^n-5)$ — ①

(i) for $n=1$, $P(1): 2\times7^1+3\times5^1-5$
$= 14+15-5 = 24$ divisible by 24.
$\therefore$ P(1) is True ✓

(ii) Let P(k) is true, ie divisible by 24
or $(2\cdot7^k+3\cdot5^k-5) = 24l$ [where l is a positive Int] ①

Consider P(k+1),
$$2\cdot7^{k+1}+3\cdot5^{k+1}-5 = 2\times7\times7^k+3\times5\cdot5^k-5$$
$$= 7[24l-3\cdot5^k+5]+15\cdot5^k-5 \quad \text{frm ①}$$
$$= 7\times24l-21\cdot5^k+35+15\cdot5^k-5$$

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(Example 4):
$$= 7\times 24l - 6\times 5^k + 30$$
$$= 7\times 24l - 6(5^k - 5)$$
$$= 7\times 24l - 6(4p) \quad [\because (5^k - 5) \text{ is a}$$
or $P(k+1)$ $$= 24(7l - p) \quad \text{multiple of } 4]$$
$$= 24r \; [r = 7l - p \text{ is some positive integer}]$$

$\therefore$ The expression on R.H.S of ① is divisible by 24. Thus $P(k+1)$ is true whenever $P(k)$ is $P(k)$. Hence $P(n)$ is true for all positive integer n.

- **Example 5:** Prove that $2^n > n$, for all positive integer n.

Proof: Consider $P(n)$: $2^n > n$ —— ①

for $n = 1$, $2^1 > 1$ is true $\Rightarrow P(1)$ is true.

Let us assume for $n = k$, for some positive integer k. $P(k)$ is true $\Rightarrow$ $2^k > k$ —— ②

Consider $P(k+1)$ to be proved, $2^{k+1} > k+1$

multiply both sides of ② by 2.
$$2\times 2^k > 2k$$
$$\Rightarrow 2^{k+1} > k + k > k+1 \quad (\because k \geq 1)$$
$$\Rightarrow 2^{k+1} > (k+1)$$

$\therefore$ $P(k+1)$ is true given in ① $P(k)$ is true,
$\therefore$ Using PMI we conclude that $P(n)$ is true for all positive integer $n \geq 1$.

- **Example 6:** If $u_1 = 1$ and $u_{n+1} = \frac{1}{(n+1)} u_n$, $n \geq 1$
Then show that: $u_{n+1} = \frac{1}{(n+1)!}$

Proof: Let $P(n)$: $u_{n+1} = \frac{1}{(n+1)!}$ —— ①

for given $u_{n+1} = \frac{1}{(n+1)} u_n$ —— ② & $u_1 = 1$

for $n = 1$, $u_2 = \frac{1}{(1+1)} \times 1 = \frac{1}{2}$ also from ① $u_2 = \frac{1}{2!} = \frac{1}{2}$]
from ②

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(Example 6 →) $\therefore$ P(1) is true ✓.

Let for a positive integer $n = k$

P(k) is True $\Rightarrow u_{k+1} = \frac{1}{(k+1)!}$ —— ③

$$\boxed{\text{Now consider } P(k+1) \text{ to be proved i.e. } u_{(k+1)+1} = \frac{1}{(k+1+1)!} \text{ or } u_{k+2} = \frac{1}{(k+2)!}}$$

Now $n = k+1$ from ② $u_{(k+1+1)} = \frac{1}{(k+1+1)} \times u_{k+1}$

$= \frac{1}{(k+2)} \times \frac{1}{(k+1)!}$ from ③

$= \frac{1}{(k+2)!}$

which R.H.S of P(k+1)

$\therefore$ P(k+1) is true for P(k) is true, and P(1) is true.

$\therefore$ Using PMI the statement P(n) is true for all positive integers $n \geq 1$.

• Example 7: Let $u_1 = u_2 = 5$ and $u_{n+1} = u_n + 6u_{n-1}$, $n \geq 2$,
Prove by mathematical induction:

$$u_n = 3^n - (-2)^n \text{ if } n \geq 1$$

Proof: Given $u_1 = u_2 = 5$ and $u_{n+1} = u_n + 6u_{n-1}$ —— ①

Let the P(n) is $u_n = 3^n - (-2)^n$ —— ②

Now for $n = 1$ from ② $u_1 = 3^1 - (-2)^1 = 5$ is True given $u_1 = 5$, $\therefore$ P(1) is True ✓

Now let P(n) is true for k any positive integer.

i.e. $P(k) = 3^k - (-2)^k$ —— ③ Let True.

$$\boxed{\text{Consider } P(k+1) \text{ To be Proved, } P(k+1) = 3^{k+1} - (-2)^{k+1} \text{ —— ④}}$$

for $n = k+1$, from ① $u_{(k+1)+1} = u_{k+1} + 6u_{(k+1)-1}$

$\Rightarrow u_{k+1} = u_{k+2} - 6u_k$

$= (3^{k+2} - (-2)^{k+2}) - 6(3^k - (-2)^k)$ from ③

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(Example 7). $P(k+1) = 9\cdot3^k - 4(-2)^k - 6\cdot3^k + 6(-2)^k$

$= 3\cdot3^k + 2(-2)^k$

$= 3^{k+1} - (-2)(-2)^k$

$= 3^{k+1} - (-2)^{k+1}$ = True from (4)

Using ∴ $P(k+1)$ is true for $P(k)$ is true and $P(1)$ is true.
∴ PMI, $P(n)$ is true from all positive integers $n \geq 1$

• **Example 8:** A sequence is defined by $u_{n+1} = 3u_n + 2$; $u_1 = 2$
Prove by induction, $u_n = 3^n - 1$

Proof: Given $u_1 = 2$ and $u_{n+1} = 3u_n + 2$ —— (1)

To prove let $P(n)$: $u_n = 3^n - 1$ —— (2)

for $n=1$ in (2) $u_1 = 3^1 - 1 = 2$ given ∴ $P(1)$ is True

Now let for $n = k$ some positive integer let
from (2) $P(k)$ is True ⇒ $u_k = 3^k - 1$ —— (3)

Now consider $P(k+1)$ to be proved from (2)
$u_{k+1} = 3^{k+1} - 1$ —— (4)

Now for $n = k+1$ from (1)

$u_{(k+1)+1} = 3u_{k+1} + 2$

$\Rightarrow 3u_{(k+1)} = u_{(k+2)} - 2$

$= (3^{k+2} - 1) - 2$ from (2)

$\Rightarrow 3u_{k+1} = 3\cdot3^{k+1} - 3$

$\Rightarrow u_{k+1} = 3^{k+1} - 1$

True for (4)

hence $P(k+1)$ is true whenever $P(k)$ is true.
∴ Using PMI, $P(n)$ is true for all positive integers $n \geq 1$.

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• Example 9: It is given that $u_r = r \times r!$ for $r = 1, 2, 3, \cdots$
Let $S_n = u_1 + u_2 + u_3 + \cdots + u_n$. Write down the values of
$2! - S_1$, $3! - S_2$, $4! - S_3$, $5! - S_4$, ---
Conjecture a formula for S_n.
Prove, by mathematical induction, a formula for S_n, for all positive integer n. $\boxed{W-14/11/Q3}$

Proof: $u_r = r \times r!$, $r = 1, 2, 3$
and $S_n = u_1 + u_2 + u_3 + \cdots + u_n$

Now $2! - S_1 = 2! - u_1 = 2! - 1 \times 1! = 1$
$3! - S_2 = 3! - S_2 = 3! - (u_1 + u_2) = 3! - (1 \times 1! + 2 \times 2!) = 1$
$4! - S_3 = 4! - S_3 = 4! - [1 \times 1! + 2 \times 2! + 3 \times 3!] = 1$
⋮

$\Rightarrow (n+1)! - S_n = 1 \Rightarrow \underbrace{S_n = (n+1)! - 1}_{P(n)}$ —— ①

for $n = 1$, $S_1 = u_1 = 1 \times 1! = 1$ & from ① $S_1 = (1+1)! - 1 = 2! - 1 = 1$
$\therefore$ P(1) is True

Now let P(n) is True for $n = K$. $\Rightarrow S_K = (K+1)! - 1$ from ①
$\Rightarrow u_1 + u_2 + \cdots + u_K = (K+1)! - 1$ —— ②

$\boxed{\text{Consider } P(K+1) \text{ to be proved. } S_{K+1} = (K+2)! - 1}$ —— ③

Now for $n = K+1$, $S_{K+1} = (u_1 + u_2 + \cdots u_K) + u_{K+1}$
$= (K+1)! - 1 + (K+1)(K+1)!$ $\{\because u_r = r \times r!\}$
$= (K+1)! [1 + K + 1] - 1$
$= (K+2) \cdot (K+1)! - 1$
$= (K+2)! - 1$ is True from ③

$\therefore$ P(K+1) is True for given P(K) is True,
and P(1) is True
Using PMI, P(n) is True for all positive integers. $n \geq 1$.

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