FP-1

Further Pure Maths-1

Rational Functions and Graphs.

Exercise.

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Rational functions and graphs

1. The curve C has equation $y = \dfrac{2x^2 - 3x - 2}{x^2 - 2x + 1}$
   (a) State the equation of the asymptotes of C, --[2]
   (b) Show that $y \le \dfrac{25}{12}$ at all points on C. --[4]
   (c) Find the coordinates of any stationary points of C. --[3]
   (d) Sketch C, stating the coordinates of any intersection of C with the coordinate axes and the asymptotes. --[4]
   (e) Sketch the curve with equation $y = \left|\dfrac{2x^2 - 3x - 2}{x^2 - 2x + 1}\right|$ and find the set of values of x for which $\left|\dfrac{2x^2 - 3x - 2}{x^2 - 2x - 1}\right| < 2$ --[4]

   SP-20 | 01 | Q7

2. The curves C_1 and C_2 have equations,
   $$y = \frac{ax}{x+5} \quad \text{and} \quad y = \frac{x^2 + (a+10)x + 5a + 26}{x+5}$$
   respectively, where a is constant and $a > 2$,
   (i) Find the equations of the asymptotes of C_1. --[2]
   (ii) Find the equation of the oblique asymptote of C_2 --[2]
   (iii) Show that C_1 and C_2 do not intersect. --[2]
   (iv) Find the coordinates of the stationary points of C_2 --[3]
   (v) Sketch C_1 and C_2 on a single diagram. [you do not need to calculate the coordinates of any points where C_2 crosses the axes] --[3]

   S-19 | 11 | Q10

3. The curve C has equation, $y = \dfrac{x^2}{kx - 1}$,
   where k is a positive constant.
   (i) Obtain the equations of asymptotes of C. --[3]
   (ii) Find the coordinates of the stationary points of C. --[3]
   (iii) Sketch C. --[3]

   S-19 | 13 | Q6

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4) The curve C has equation $y = x^a$ for $0 \leq x \leq 1$, where a is a positive constant. Find, in terms of a, the coordinates of the centroid of region enclosed by C, the line $x = 1$ and the x-axis. ---[6]
(Not in the syllabus) [W-19/11/Q1]

5. The line $y = 2x+1$ is an asymptote of the curve C with equation, $y = \frac{x^2+1}{ax+b}$.
(i) Find the values of the constants a and b. ---[3]
(ii) State the equation of the other asymptote of C. --[1]
(iii) Sketch C. [Your sketch should indicate the coordinates of any points of intersection with the y-axis. You do not need to find the coordinates of any stationary points.] [W-19/11/Q4] ---[3]

6. The curve C has equation $y = \frac{x^2+b}{x+b}$ where b is a positive constant.
(i) Find the equations of the asymptotes of C. ---[3]
(ii) Show that C does not intersect the x-axis. --[1]
(iii) Justifying your answer, find the number of stationary points on C. --[2]
(iv) Sketch C. Your sketch should indicate the coordinates of any points of intersection with the y-axis. You do not need to find the coordinates of any stationary points. [S-18/11/Q6] ---[3]

7. The curve C has equation $y = \frac{x^2+7x+6}{x-2}$
(i) Find the coordinates of the points of intersection of C with the axes. ---[2]
(ii) Find the equation of each of the asymptotes of C. --[3]
(iii) Sketch C. --[3]
[S-18/13/Q4]

8. The curve C has equation, $y = \frac{x^2 + ax - 1}{x+1}$ where a is constant and $a > 1$.

(i) Find the equations of the asymptotes of C. --[3]

(ii) Show that C intersect x-axis twice. --[1]

(iii) Justifying your answer, find the number of stationary points on C. --[2]

(iv) Sketch C, stating the coordinates of its points of intersection with the y-axis. [W-18|11|Q6] --[3]

9. The curve C has equations $y = \frac{5x^2+5x+1}{x^2+x+1}$

(i) Find the equation of the asymptote of C. --[2]

(ii) Show that, for real value of x, $-\frac{1}{3} \leq y < 5$ --[4]

(iii) Find the coordinates of any stationary points of C. --[2]

(iv) Sketch C, stating the coordinates of any intersection with the y-axis. [W-18|12|Q9] --[2]

10. The curve C has equation, $y = \frac{x^2 - 3x + 6}{1 - x}$

(i) Find the equations of asymptotes of C. --[3]

(ii) Find the coordinates of the turning points of C. --[3]

(iii) Find the coordinates of any intersection with the coordinate axes. --[2]

(iv) Sketch C. [S-17|11|Q9] --[3]

11. The curve C has equation $y = \frac{3x-9}{(x-2)(x+1)}$

(i) Find the equations of the asymptotes of C. --[2]

(ii) Show that there is no point on C for which $\frac{1}{3} < y < 3$ --[4]

(iii) Find the coordinates of the turning points of C. --[3]

(iv) Sketch C [W-17|11|Q9] -[3]

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12. A curve C has equation $y = \frac{x^2}{x-2}$. Find the equations of the asymptotes of C. [3]
Show that there are no points on C for which $0 < y < 8$ --[4]
Sketch C, giving the coordinates of the turning points. --[3]
[S-16/11/Q7]

13. The curve C has equation $y = \frac{x-2}{x^2-9}$, show that $\frac{dy}{dx} < 0$ at all points on C. --[3]
State the equations of the asymptotes of C. ---[2]
Sketch C, showing the coordinates of any points of intersection with the coordinate axes. --[3]
[S-16/13/Q5]

14. The curve C has equation $y = \frac{4x^2 - 3x}{x^2+1}$, Verify that the equation of C may be written in the form $y = -\frac{1}{2} + \frac{(3x-1)^2}{2(x^2+1)}$ and also in the form $y = \frac{9}{2} - \frac{(x+3)^2}{2(x^2+1)}$, --[3]
Hence show, $-\frac{1}{2} \leq y \leq \frac{9}{2}$ --[2]

Without differentiating, write down the coordinates of the turning points of C. --[2]
State the equation of the asymptote of C. --[1]
Sketch the graph of C, stating the coordinates of the intersections with the coordinate axes and the asymptote. [S-15/13/Q10] --[3]

← X — X →

(Answers on the next page)

Answers

1. $y = \frac{2x^2-3x-2}{(x-1)^2}$ or $y = 2 + \frac{x-4}{(x-1)^2}$

(a) Vertical asymptote at $x=1$, and horizontal at $y=2$ ✓

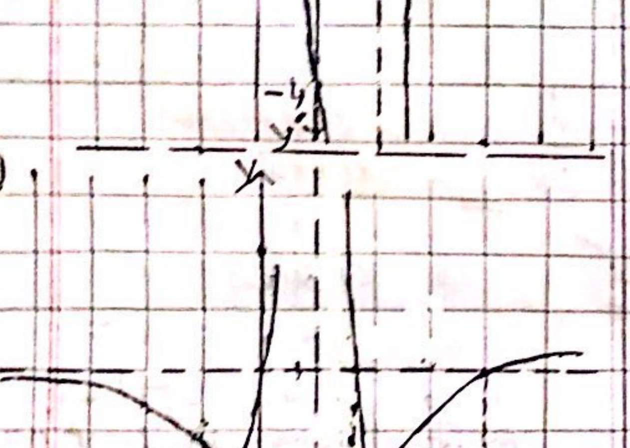
(b) Eqn of curve may be written as
$yx^2 - 2yx + y = 2x^2 - 3x - 2$
$\Rightarrow (y-2)x^2 - (2y-3)x + (y+2) = 0$
for real roots $B^2 - 4AC \geq 0$
$(2y-3)^2 - 4(y-2)(y+2) \geq 0$
$\Rightarrow -12y + 25 \geq 0 \Rightarrow y \leq \frac{25}{12}$ ✓

(c) $\frac{dy}{dx} = \frac{(x^2-2x+1)(4x-3) - (2x^2-3x-2)(2x-2)}{(x^2-2x+1)}$

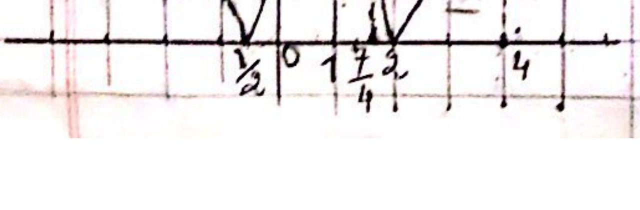
for turning points $\frac{dy}{dx} = 0 \Rightarrow x^2 - 8x + 7 = 0$
$\Rightarrow x = 7$ ($x=1$ is an asymptote)

Stationary point $\left(7, \frac{25}{12}\right)$ ✓

(d) Intercepts x-axis $\Rightarrow 2x^2 - 3x - 2 = 0$, $x = 2, -\frac{1}{2}$
Intercepts y axis $\Rightarrow y = -4$



(e)



1(e) continued: Find $0, \frac{7}{4}, 4$ as $|f(x)| < 2$, critical points.
for $x < 0$ or $\frac{7}{4} < x < 4$ ✓

2. $C_1: y = \frac{ax}{x+5}$ and $C_2: y = \frac{x^2 + (a+10)x + 5a + 26}{x+5}$

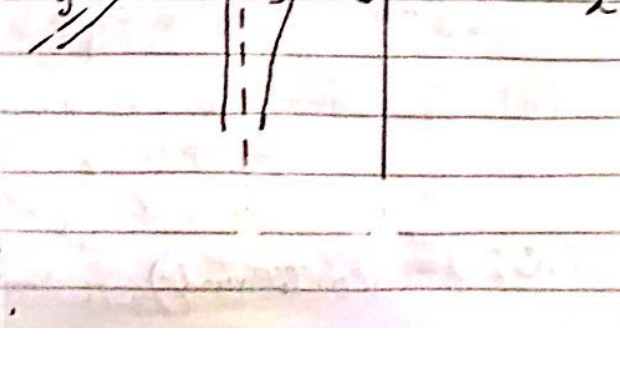
(i) $C_1: y = a + \frac{(-5a)}{x+5}$, $a > 2$
asymptotes are, $y = a$ & $x = -5$ ✓

(ii) $C_2: y = (x + a + 5) + \frac{1}{x^2 + (a+10)x + 5a + 26}$
$\therefore$ Oblique asymptote is: $y = x + a + 5$ ✓

(iii) for intersection.
$ax = x^2 + (a+10)x + 5a + 26$
$\Rightarrow x^2 + 10x + (5a+26) = 0$
$B^2 - 4AC = 10^2 - 4(5a+26)$
$= -(4 + 20a) < 0;\ a > 2$
$\therefore$ No Points of intersection.

(iv) diff for C_2, $\frac{dy}{dx} = 0$
$\Rightarrow (x+5)(2x+a+10) - (x^2+ax+10x+5a+26) = 0$
$\Rightarrow x^2 + 10x + 24 = 0 \Rightarrow x = -4, -6$
$\therefore$ Stationary points are $(-4, a+2)$ and $(-6, a-2)$ ✓

(v)



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Answers Continued→

3. $C: y = \dfrac{x^2}{kx-1}$; $k>0$

Vertical asymptote is → $x =$

(i) $y = \left(\dfrac{1}{k}x + \dfrac{1}{k^2}\right) + \dfrac{1}{k^2(kx-1)}$

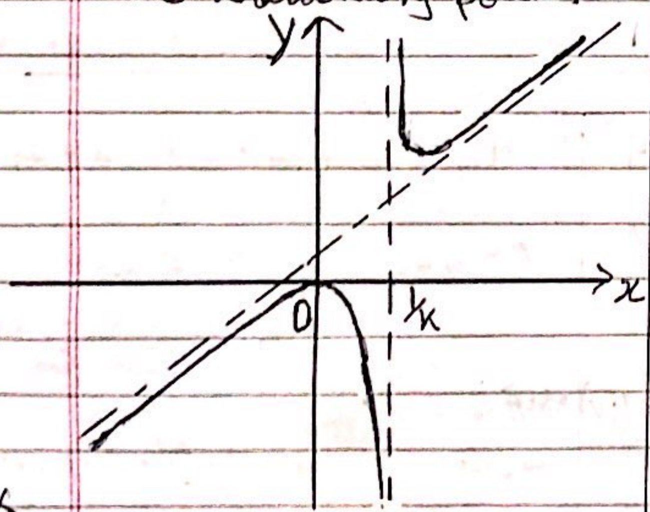
∴ oblique asymptote is, $y = k^{-1}x + k^{-2}$ ✓

(ii) $\dfrac{dy}{dx} = \dfrac{2x(kx-1) - kx^2}{(kx-1)^2} = 0$

$\Rightarrow kx^2 - 2x = 0 \Rightarrow x(kx-2) = 0$

$x = 0,\ 2k^{-1}$

∴ $(0,0)$, $(2k^{-1}, 4k^{-2})$ ✓ are stationary points.



4. $C: y = x^a ;\ 0 \le x \le 1$,
(Not in the syllabus)

5. $C: y = \dfrac{x^2+1}{ax+b}$; line $y = 2x+1$ — ①

(i) $C: y = \left(\dfrac{x}{a} - \dfrac{b}{a^2}\right) + \dfrac{a^2-b^2}{a^2(ax+b)}$ — ②

Here asymptote, $y = \dfrac{1}{a}\cdot x - \dfrac{b}{a^2}$ — ③

from ① & ③ $\dfrac{1}{a} = 2 \Rightarrow a = \dfrac{1}{2}$ ✓

and $-\dfrac{b}{a^2} = 1 \Rightarrow -b = \dfrac{1}{4} \Rightarrow b = -\dfrac{1}{4}$ ✓

(ii) other asymptote $ax+b=0$

$\Rightarrow \dfrac{1}{2}x - \dfrac{1}{4} = 0$

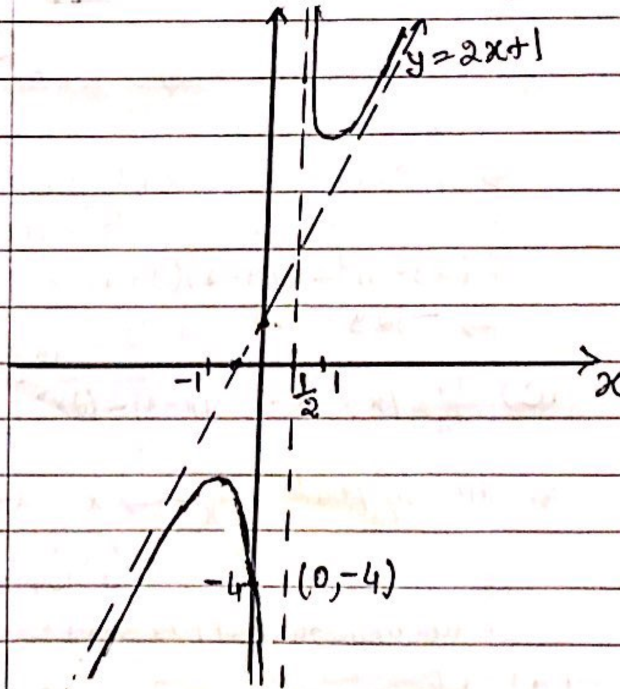
$\Rightarrow x = \dfrac{1}{2}$ ✓

∴ $C: y = (2x+1) + \dfrac{5}{(2x-1)}$ ✓ — ④

5(iii) from ② C intersect Y-axis at $(0,-4)$

from ④ $x=0 \Rightarrow y=-4$

$C: y = (2x+1) + \dfrac{5}{(2x-1)}$



6. $C: y = \dfrac{x^2+b}{x+b}$ — ① $b>0$

(i) $C: y = (x-b) + \dfrac{(b^2+b)}{(x+b)}$ — ②

The oblique asymptote is.
$y = x-b$ ✓
and $x = -b$ is asymptote parallel to Y-axis

(ii) In ① $y=0 \Rightarrow x^2+b=0$ has no real roots ⇒ C does not intersect X-axis.

(iii) $\dfrac{dy}{dx} = \dfrac{2x(x+b) - (x^2+b)}{(x+b)^2} = 0$

$\Rightarrow x^2 + 2bx - b = 0$

$B^2 - 4AC = 4b^2 + 4b > 0$

∴ Curve C has two stationary points.

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Answers

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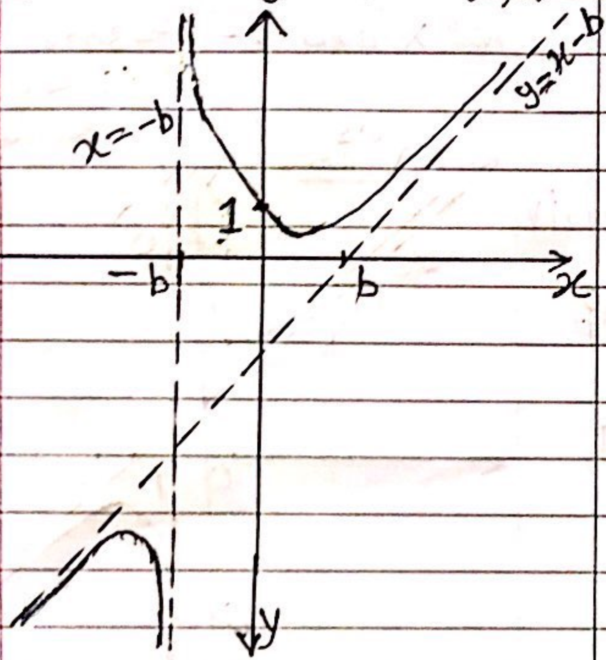
6(iv) $x=0$ in ① $\Rightarrow y=1$

$\therefore$ C intersects y-axis at $(0,1)$



7. C: $y = \dfrac{x^2+7x+6}{x-2}$

(i) Intersects x-axis $x^2+7x+6=0$
$\Rightarrow x=-1,-6$
$(-6,0), (-1,0)$ ✓

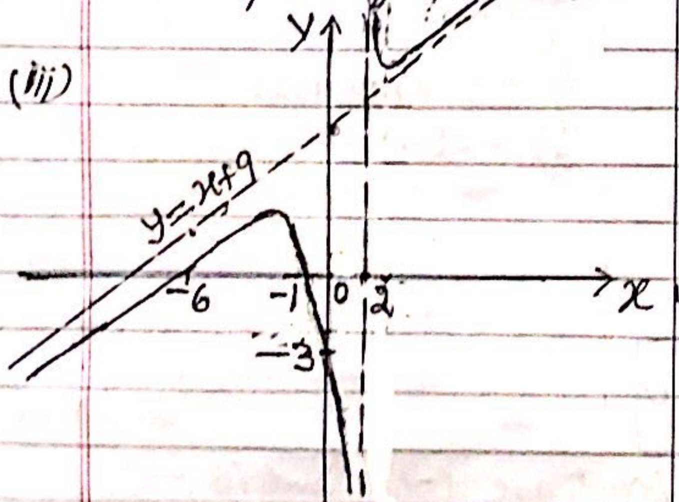
Intersects y-axis $(0,-3)$ ✓

(ii) $y = x+9 + \dfrac{24}{x-2}$

One asymptote parallel to y-axis is $x-2=0 \Rightarrow x=2$ ✓

and the Oblique asymptote is $\boxed{y=x+9}$ ✓

(iii)



8. C: $y = \dfrac{x^2+ax-1}{x+1}$, $a>1$

or $y=(x+a-1) - \dfrac{a}{x+1}$

(i) asymptotes are
$x=-1$; $y=x+a-1$ ✓

(iii) $\dfrac{dy}{dx} = \dfrac{(x+1)(2x+a) - (x^2+ax-1)}{(x+1)^2} = 0$

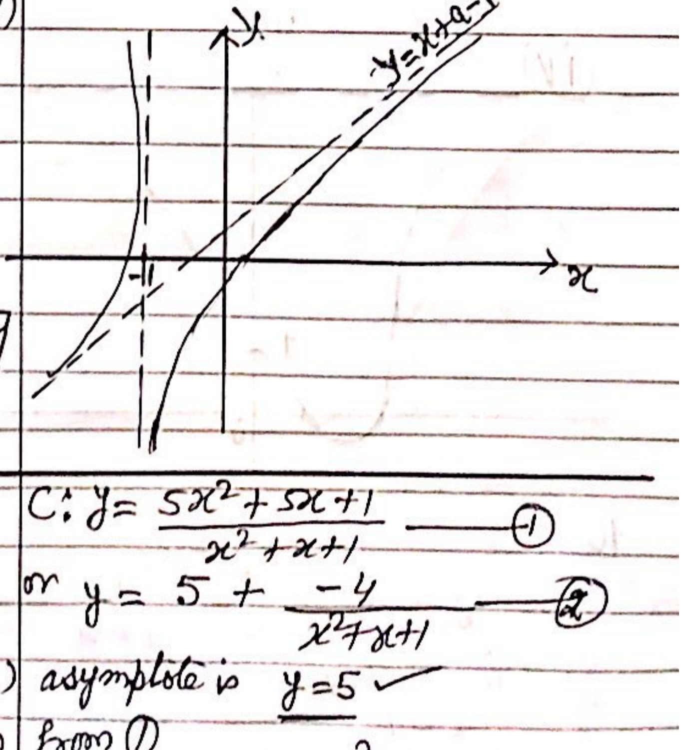
$\Rightarrow x^2+2x+a+1=0$

$B^2-4AC = 4-4(a+1)$
$= \cancel{4}-4a-\cancel{4}$
$= -4a<0$

$\therefore$ there are no stationary points.

(ii) C, intersects x-axis,
$y=0 \Rightarrow x^2+ax-1=0$
$B^2-4AC = a^2+4>0$
$\therefore$ C intersects x-axis twice.

(iv)



9. C: $y = \dfrac{5x^2+5x+1}{x^2+x+1}$ ——①

or $y = 5 + \dfrac{-4}{x^2+x+1}$ ——②

(i) asymptote is $y=5$ ✓

(ii) from ①
$yx^2+yx+y=5x^2+5x+1$
$\Rightarrow (y-5)x^2+(y-5)x+(y-1)=0$ —③

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(Continued →) **Answers** (Continued →)

9(ii) for real values of x,

$b^2 - 4ac > 0$

$\Rightarrow (y-5)^2 - 4(y-5)(y-1) \geq 0$

$\Rightarrow (y-5)(3y+1) \leqslant 0$

$\Rightarrow -\frac{1}{3} \leqslant y < 5$ [∵ $y=5$ is an asymptote]

(iii) $\frac{dy}{dx} = (x^2+x+1)(10x+5) - (5x^2+5x+1)\cdot x(2x+1) = 0$

$\Rightarrow 4(2x+1) = 0$

$\Rightarrow x = -\frac{1}{2}, y = -\frac{1}{3}$

∴ $(-\frac{1}{2}, -\frac{1}{3})$ is a stationary point.

(iv) C intersects y-axis for $x=0$

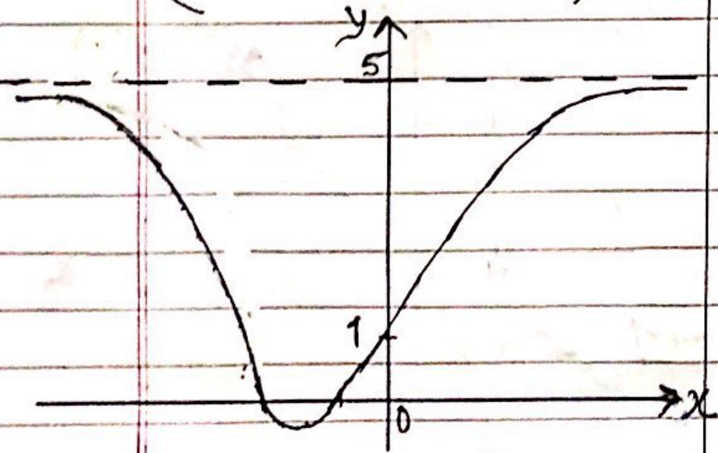
$\Rightarrow y = 1$, $(0,1)$

(and intersects x-axis for $y=0$

$5x^2+5x+1=0$

$x = \frac{-5 \pm \sqrt{5}}{10} = \frac{-5 \pm 2.24}{2}$

$= -3.6, -1.38$)



10 C: $y = \frac{x^2-3x+6}{(1-x)}$ ——(1)

or $y = -x+2+\frac{4}{(1-x)}$ ——(2)

(i) Asymptotes are,

$x=1$ and $y = 2-x$

(ii) $\frac{dy}{dx} = \frac{(1-x)(2x-3)+(x^2-3x+6)}{(1-x)^2} = 0$

$\Rightarrow (-x^2+2x+3) = 0$

$\Rightarrow x = -1, 3 \Rightarrow$ Turning-points are $(-1,5)$ and $(3,-3)$

10(iii) Points of intersection:

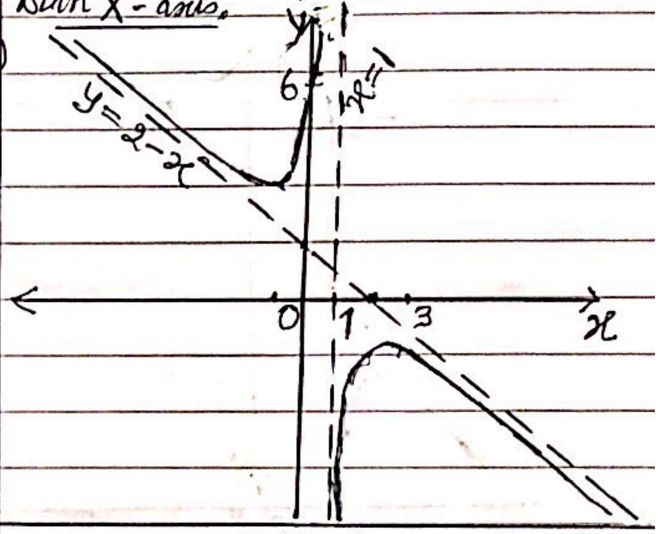
with y-axis: $(0,6)$;

with x-axis $y = x^2-3x+6 = 0$

$\Rightarrow b^2-4ac < 0$

NO Point of intersection with x-axis.

(iv)



11 C: $y = \frac{3x-9}{(x-2)(x+1)}$ ——(1)

(i) Vertical asymptotes at

$x=2$ and $x=-1$

(ii) for (1) $yx^2-(y+3)x+9-2y=0$

Now $B^2-4AC < 0$ for no point of intersection.

$\Rightarrow (y+3)^2 - 4y(9-2y) < 0$

$\Rightarrow 3y^2-10y+3 < 0$

$\Rightarrow (3y-1)(y-3) < 0$

$\Rightarrow \frac{1}{3} < y < 3$

(iii) for turning points:

$\frac{dy}{dx} = 3(x^2-x-2) - (3x-9)(2x-1) = 0$

$\Rightarrow (x-1)(x-5) = 0$

$x = 1, 5$

$\Rightarrow$ Turning points are $(1,3)$, $(5,\frac{1}{3})$

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Answers

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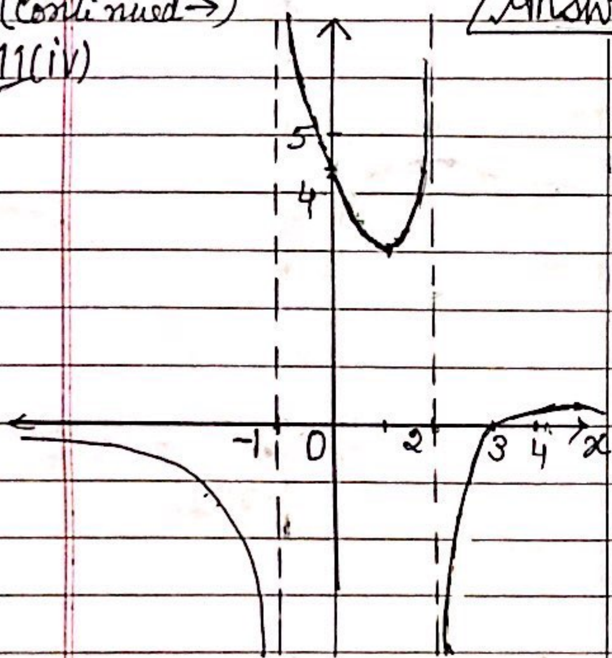
11(iv)



12 C: $y = \dfrac{x^2}{x-2}$ — ①

or $y = x+2 + \dfrac{4}{x-2}$ — ②

(i) Vertical asymptote is $x = 2$ ✓
oblique asymptote is $y = x+2$ ✓

(ii) from ① $x^2 - yx + 2y = 0$
for no real roots $B^2 - 4AC < 0$
$\Rightarrow y^2 - 8y < 0$
$y(y-8) < 0$
$\Rightarrow 0 < y < 8$ ✓

(iii) for turning points
$\dfrac{dy}{dx} = \dfrac{2x(x-2) - x^2 \times 1}{(x-2)} = 0$
$\Rightarrow x^2 - 4x = 0$
$\Rightarrow x(x-4) = 0 \Rightarrow x = 0, 4$

Turning points $(0,0)$, $(4,8)$

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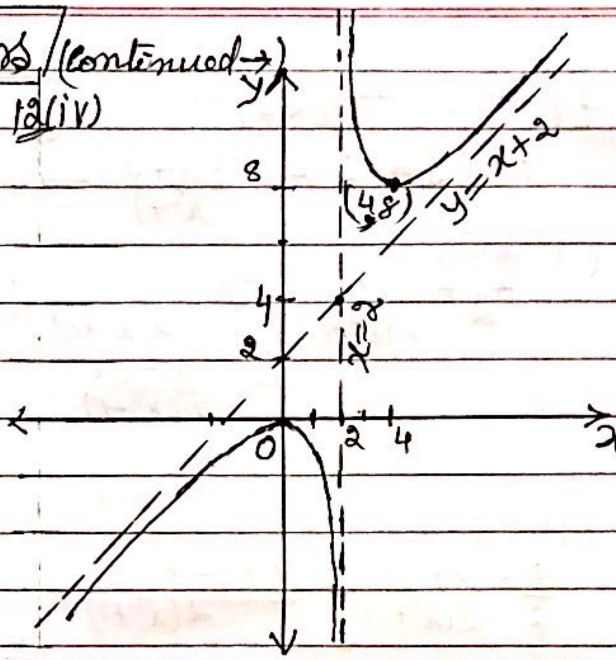
12(iv)



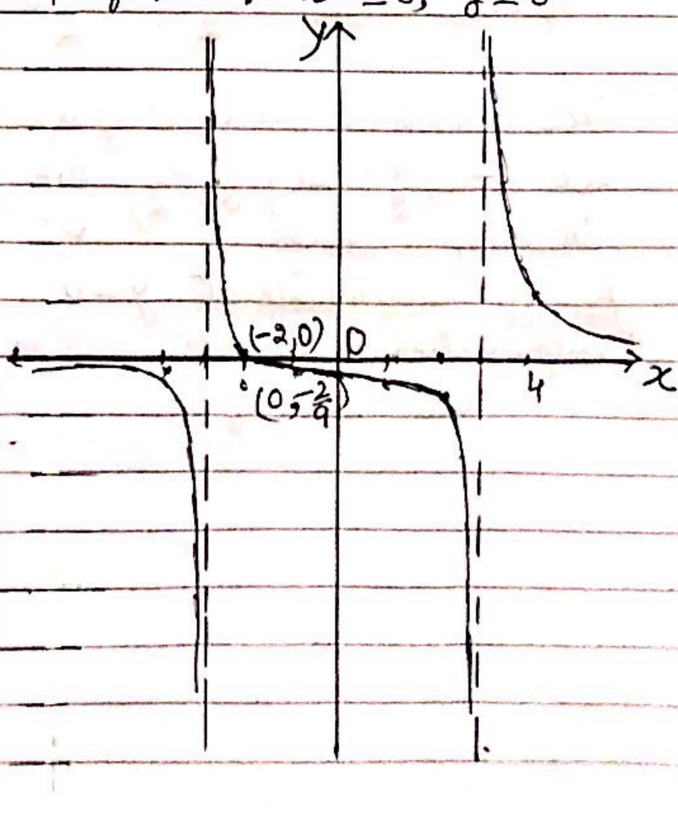
13 C: $y = \dfrac{x+2}{x^2-9}$ — ①

or $y = \dfrac{(x+2)}{(x-3)(x+3)}$ — ②

(i) $\dfrac{dy}{dx} = \dfrac{1 \times (x^2-9) - (x+2) \times 2x}{(x^2-9)^2}$

$= \dfrac{-[(x+2)^2+5]}{(x^2-9)^2} < 0$ ✓

ii Asymptotes: $x = \pm 3$; $y = 0$



Answers

14. C: $y = \frac{4x^2 - 3x}{x^2+1}$ — (1)

or $y = 4 - \frac{3x+4}{x^2+1}$ — (2)

(i) consider

$-\frac{1}{2} + \frac{(3x-1)^2}{2(x^2+1)} = \frac{-x^2-1+9x^2-6x+1}{2(x^2+1)}$

$= \frac{8x^2-6x}{2(x^2+1)}$

$= \frac{4x^2-3x}{(x^2+1)}$ ✓ — (3)

Again consider

$\frac{9}{2} - \frac{(x+3)^2}{2(x^2+1)} = \frac{9x^2+9-x^2-6x-9}{2(x^2+1)}$

$= \frac{8x^2-6x}{2(x^2+1)}$

$= \frac{4x^2-3x}{x^2+1}$ ✓ — (4)

Now from (3),

$\frac{(3x-1)^2}{2(x^2+1)} \geq 0 \Rightarrow y \geq -\frac{1}{2}$ — (5)

and from (4) $\frac{(x+3)^2}{2(x^2+1)} \geq 0 \Rightarrow y \leq \frac{9}{2}$ — (6)

from (5) and (6) $-\frac{1}{2} \leq y \leq \frac{9}{2}$ ✓

Turning points occur at equality so $(-3, \frac{9}{2})$ and $(\frac{1}{3}, -\frac{1}{2})$ are turning points. ✓

from (2) Asymptote at $y = 4$.

Intersection with axis;

$(0,0)$ and $(\frac{3}{4}, 0)$.