FP-1

Further Pure Math 1

Roots of Polynomials.
Notes

Suresh Goel.
(Director)
Alliance World School.
Noida, Delhi-NCR.
INDIA.

FP$_1$

Roots of Polynomials

Notes

Formulae of Algebra:

1. $(a+b)^2 = a^2+b^2+2ab \Rightarrow a^2+b^2 = (a+b)^2 - 2ab$
2. $(a-b)^2 = a^2+b^2-2ab \Rightarrow a^2+b^2 = (a+b)^2 - 4ab$
3. $(a+b)^3 = a^3+b^3+3ab(a+b) \Rightarrow a^3+b^3 = (a+b)^3 - 3ab(a+b)$
   $= (a+b)(a^2+b^2-ab)$
4. $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$
   $\Rightarrow a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca)$
5. $(a+b+c)^3 = a^3+b^3+c^3+3(a+b+c)(ab+bc+ca) - 3abc$
   $\Rightarrow a^3+b^3+c^3 = (a+b+c)^3 - 3(a+b+c)(ab+bc+ca) + 3abc.$
   or $a^3+b^3+c^3 = (a+b+c)[a^2+b^2+c^2-(ab+bc+ca)] + 3abc,$
   $= (a+b+c)[(a+b+c)^2 - 3(ab+bc+ca)] + 3abc$
6. $a^4+b^4+c^4 = (a^2+b^2+c^2)^2 - 2(a^2b^2+b^2c^2+c^2a^2)$
   $= (a^2+b^2+c^2)^2 - 2[(ab+bc+ca)^2 - 2(a+b+c).abc]$
   $= [(a+b+c)^2 - 2(ab+bc+ca)]^2 - 2[(ab+bc+ca)^2 - 2abc(a+b+c$

Note 1 Symmetric functions of the roots of a poly. eqn:

(i) $\alpha+\beta$, $\alpha^2+\beta^2$, $\alpha\beta+\beta\alpha$, are all symmetric function for if α and β are interchanged the function remain the same.

(ii) $\alpha^2+\beta^2+\gamma^2$ or $\alpha\beta+\beta\gamma+\gamma\alpha$ are also symmetric.

Note 2. Let $\alpha, \beta, \gamma, ---$ are the roots of a polynomial equation and S_n denotes,

$$S_n = \alpha^n+\beta^n+\gamma^n+ - - -$$

Then
$$S_1 = \alpha+\beta+\gamma - - - = \Sigma\alpha$$
$$S_2 = \alpha^2+\beta^2+\gamma^2 - - - = \Sigma\alpha^2$$
$$S_3 = \alpha^3+\beta^3+\gamma^3 - - - = \Sigma\alpha^3$$

FP1 # Roots of Polynomials

Date ______

Notes

- Polynomial: $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$ is polynomial of degree n.
  Here n is a positive integer; x is a va...
  $a_0, a_1, a_2, \ldots a_n$ are constant real numbers.

- Polynomial Equation of degree n:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

a polynomial equation of degree n has atmost n real roots.

- Linear Equation: An equation of deg. 1.
  $ax+b=0$ is linear equation, $a \neq 0$
  Example: $2x+3=0$
- Quadratic Equation: "degree 2"
  $ax^2+bx+c=0$, $a \neq 0$
  Example, $2x^2+5x-7=0$; $x^2-3=0$, $5x^2=0$

- Cubic Equation: "degree 3"
  $ax^3+bx+cx+d=0$ $a \neq 0$
  Example: $5x^3-3x^2+x-7=0$

- Quartic Equation (or Biquadratic Equ^n):
  $ax^4+bx^3+cx^2+dx+e=0$; Deg → 4, $a \neq 0$
  Example: $\frac{3}{2}x^4-5x^3+2x+3=0$

§ Roots of Polynomial Equation:

α is root of a polynomial equation if x is replaced by α, makes is true.

Example: 3 is a root of $x^2-5x+6=0$
as $3^2-5\times3+6=0$ is True. ✓

Roots of Polynomials
Notes

- **Relation between the roots and coefficients of terms of a Quadratic Equation:**

Given a quadratic equation $ax^2+bx+c=0$ —— ①

Let α and β are the roots of ①

$\therefore\ a(x-\alpha)(x-\beta) \equiv ax^2+bx+c$

or $x^2-(\alpha+\beta)x+\alpha\beta \equiv x^2+\frac{b}{a}x+\frac{c}{a}$

Comparing the coefficients on both sides of the equations

Sum of roots $\alpha+\beta = -\frac{b}{a}$ —— ②

Product of roots $\alpha\beta = \frac{c}{a}$ —— ③

- **Alternatively:**

Using quadratic formula: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Let $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$ ---(i)

$\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$ ---(ii)

add (i) & (ii) $\alpha+\beta = -\frac{b}{a}$ ✓

Multiplying (i) & (ii) (and simplify) $\alpha\beta = \frac{c}{a}$ ✓

- **Example 1.**

Given a quadratic equation $2x^2-5x+3=0$

Find the sum and product of its roots.

Solution: $a=2,\ b=-5,\ c=3$

Sum of roots $\alpha+\beta = -\frac{b}{a} = \frac{-(-5)}{2} = \frac{5}{2}$ ✓

Product of roots $\alpha\beta = \frac{c}{a} = \frac{3}{2}$ ✓

Alternate method:

$2x^2-5x+3=0$

$\Rightarrow (x-1)(2x-3)=0$

roots are $1, 3/2$

Sum $= 1+3/2 = \frac{5}{2}$ ✓

Product of roots $= 1\times\frac{3}{2} = \frac{3}{2}$ ✓

(FP$_1$)

• Example 2 (conversely) (i) Find a quadratic equation whose roots α and β are given as 7 and -4 respectively.
(ii) Form another quad. equation whose roots are $\alpha+1$ and $\beta+1$.
(iii) " " " " " " " " α^2 and β^2

Solution: (i) Sum of roots $\alpha+\beta = 7+(-4) = 3$ --- (i)

Product of root $\alpha\beta = 7\times(-4) = -28$ --- (ii)

$\therefore$ Required quad. equation, $x^2-(\alpha+\beta)x+\alpha\beta=0$

or $x^2-3x+(-28)=0$ from (i) & (ii)

or $x^2-3x-28=0$ ✓

(ii) Now given new roots $\alpha+1$ and $\beta+1$

Sum of new roots $= (\alpha+1)+(\beta+1)$

$= (\alpha+\beta)+2 = 3+2 = 5$ ✓ from (i) --- (iii)

and Product of new roots $= (\alpha+1)(\beta+1)$

$= (\alpha+\beta)+\alpha\beta+1 = 3+(-28)+1$

$= -24$ ✓ from (i) & (ii) --- (iv)

$\therefore$ Req. quad. equ^n

$x^2-(\text{Sum of new roots})x+(\text{Product of new root})=0$

or $x^2-5x-24=0$ ✓

Verify:

Given roots $\alpha=7$, $\beta=-4$

New roots

$\alpha+1=8$, ✓ $\beta+1=-3$ ✓

$\therefore$ quad equ^n

$x^2-(\text{Sum})x+\text{Prod}=0$

$x^2-(8+(-3))x+8\times(-3)=0$

$x^2-5x-24=0$

(iii) Now the new root are α^2 and β^2

Sum of new roots $= \alpha^2+\beta^2$

$= (\alpha+\beta)^2-2\alpha\beta$ from (i)

$= 3^2-2\times(-28) = 65$ ✓

Product of new root $= \alpha^2\times\beta^2 = (\alpha\beta)^2$

$= (-28)^2 = 784$

$\therefore$ Required quad. equ^n:

$x^2-(\text{Sum of new roots})+\text{Product}=0$

$x^2-65x+784=0$ ✓

(iii) Alt: new roots α^2, β^2 are 7^2 & $(-4)^2$ or 49 & 16,

Sum $= 65$ and Product $= 784$ $\therefore$ $x^2-65x+784=0$ ✓

Scanned with CamScanner

Roots of Polynomials
Notes

- **Forming new equations using "Substitution method".**

Example 3. Given a quadratic equation $x^2-3x-28=0$ has roots α and β, form a new quad. equation (i) whose roots are α^2 and β^2 (ii) roots are $\alpha+1$ and $\beta+1$

Solution: Given quad eqn $x^2-3x-28=0$ ⟵ ①

Given roots are α and β.

(i) To form a new eqn whose roots are α^2 and β^2

Put $y=x^2 \Rightarrow x=\sqrt{y}$ in equation ①

we get, $(\sqrt{y})^2-3\sqrt{y}-28=0$

$\Rightarrow y-3\sqrt{y}-28=0$

$\Rightarrow 3\sqrt{y}=28-y$

Squaring both sides $9y=(28-y)^2$

$\Rightarrow 9y=784+y^2-56y$

$\Rightarrow y^2-65y-784=0$ ✓ Note: [same as example 2 (iii)]

(ii) New roots are $\alpha+1$ and $\beta+1$ put $y=x+1 \Rightarrow x=y-1$

Put $x=y-1$ in equation ①

$(y-1)^2-3(y-1)-28=0 \Rightarrow y^2-2y+1-3y+3-28=0$

$\Rightarrow y^2-5y-24=0$ ✓

we may replace y by x. [Note: same as example 2(ii)]

∴ Required equation is

$x^2-5x-24=0$ ✓

Cubic Equations:

General form: $ax^3+bx^2+cx+d=0$ —— (1)

Let α, β and γ are roots of equation.

$\Rightarrow a(x-\alpha)(x-\beta)(x-c) = ax^3+bx^2+cx+d$

$\cancel{a}[x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma] = \cancel{a}(x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a})$

Comparing the Coeff. on both sides we get:

$\Sigma\alpha = \alpha+\beta+\gamma = -\frac{b}{a}$ --- (i) ✓

$\Sigma\alpha\beta = \alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a}$ --- (ii) ✓

and $\Sigma\alpha\beta\gamma = \alpha\beta\gamma = -\frac{d}{a}$ --- (iii) ✓

Conversely:

Given α, β, γ are the roots of a cubic equation,

Required cubic equ^n is:

$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma=0$ ✓ —— (2)

Note: $\Sigma\alpha^2\beta = (\alpha^2\beta+\alpha\beta^2+\beta^2\gamma+\beta\gamma^2+\gamma^2\alpha+\gamma\alpha^2)$

Example 4: The cubic equation $2x^3-3x^2+4x-10=0$ has roots α, β and γ. (i) Find the value of $(\alpha+1)(\beta+1)(\gamma+1)$

(ii) Find the value of $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$

$\boxed{W-17/11/Q4}$

Solution:

For the given cubic equ^n $\alpha+\beta+\gamma = -\frac{b}{a} = 3/2$ ✓ —(i), $\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a} = \frac{4}{2} = 2$ ✓ —(ii)

and $\alpha\beta\gamma = -\frac{d}{a} = \frac{10}{2} = 5$ ✓

(i) $(\alpha+1)(\beta+1)(\gamma+1)$

$= (\alpha+\beta+\gamma)+(\alpha\beta+\beta\gamma+\gamma\alpha)+\alpha\beta\gamma+1$

$= \frac{3}{2}+2+5+1 = 9\frac{1}{2}$ ✓

from (i),(ii)(iii)

(ii) $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$

$(\frac{3}{2}-\alpha)(\frac{3}{2}-\beta)(\frac{3}{2}-\gamma)$ from (i)

$= \frac{27}{8} - \frac{9}{4}(\alpha+\beta+\gamma) - \alpha\beta\gamma + \frac{3}{2}(\alpha\beta+\beta\gamma+\alpha\beta)$

from (i),(ii)(iii) $= \frac{27}{8} - \frac{9}{4}\times\frac{3}{2} - 5 + \frac{3}{2}\times 2 = \underline{-2}$ ✓

• **Example 5:** The cubic equation $z^3 - z^2 - z - 5 = 0$ has roots α, β and γ.

(a) Show that the value of $\alpha^3 + \beta^3 + \gamma^3$ is 19. --[4]

(b) Find the value of $\alpha^4 + \beta^4 + \gamma^4$ --[2]

(c) Find a cubic equation with roots $\alpha+1$, $\beta+1$ and $\gamma+1$ give your answer in the form, $px^3 + qx^2 + rx + s = 0$ where p, q, r and s are constants to be determined. --[3]

Solution Given $z^3 - z^2 - z - 5 = 0$ — (1) $\boxed{SP-20/01/Q4}$

(a) $\alpha+\beta+\gamma = -(-1) = 1$, $\Sigma\alpha\beta = -1$, $\alpha\beta\gamma = -(-5) = 5$ ✓

$$\alpha^3+\beta^3+\gamma^3 = (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) + 3\alpha\beta\gamma$$
$$= (\alpha+\beta+\gamma)\left[(\alpha+\beta+\gamma)^2 - 3(\alpha\beta+\beta\gamma+\gamma\alpha)\right] + 3\alpha\beta\gamma$$
$$= 1\left[1^2 - 3(-1)\right] + 3\times 5 = \underline{19} ✓ — (2)$$

Alternate method: from (1)

$$z^3 = z^2 + z + 5$$
$$\Rightarrow \alpha^3+\beta^3+\gamma^3 = \Sigma\alpha^3 = \Sigma\alpha^2 + \Sigma\alpha + \Sigma 5$$
$$= \left[(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)\right] + (\alpha+\beta+\gamma) + 5\times 3$$
$$= \left[1^2 - 2\times(-1)\right] + 1 + 15 = 19 ✓ — (2)$$

(b) $\alpha^4+\beta^4+\gamma^4 = \left[(\Sigma\alpha)^2 - 2\Sigma\alpha\beta\right]^2 - 2\left[(\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma \times \Sigma\alpha\right]$

$= \left[1^2 - 2\times(-1)\right]^2 - 2\left[(-1)^2 - 2\times 5(1)\right]$ [On Page 1 formula 6]

$= (3)^2 - 2(-9) = \underline{27}$ ✓

Alternate method:

from (1) $z^3 = z^2 + z + 5 \Rightarrow z^4 = z^3 + z^2 + 5z$

$\Rightarrow \alpha^4+\beta^4+\gamma^4 = \Sigma\alpha^4 = \Sigma\alpha^3 + \Sigma\alpha^2 + 5\Sigma\alpha$

$= 19 + 3 + 5\times 1$

$= \underline{27}$ ✓

fr (2) & (3)

Consider $\Sigma\alpha^2 = \alpha^2+\beta^2+\gamma^2$

$= (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$\therefore \Sigma\alpha^2 = 1^2 - 2\times(-1) = 3$ — (3)

(c) New roots are $(\alpha+1)$, $(\beta+1)$ and $(\gamma+1)$.

Put $x = z+1 \Rightarrow z = (x-1)$ in (1)

fr (1) $(x-1)^3 - (x-1)^2 - (x-1) - 5 = 0$

$\Rightarrow x^3 - 4x^2 + 4x - 6 = 0$ ✓

- Example 6: Find the cubic equation with roots α, β and γ, such that: $\alpha+\beta+\gamma = 3$ — ①
$\alpha^2+\beta^2+\gamma^2 = 1$ — ②
$\alpha^3+\beta^3+\gamma^3 = -30$ — ③
giving your answer in the form $x^3+px^2+qx+r=0$ where p, q and r are integers to be found. ---[6]
[W-16 | 11 | Q2]

Solution: The required cubic equ^n will be:
$$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma=0 \quad ④$$

Now $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\gamma\beta+\gamma\alpha)$
fr ① & ② $\quad 1 = 3^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$
$\Rightarrow (\alpha\beta+\beta\gamma+\gamma\alpha) = 4$ — ⑤
we know: $(\alpha^3+\beta^3+\gamma^3) - 3\alpha\beta\gamma = (\Sigma\alpha)(\Sigma\alpha^2 - \Sigma\alpha\beta)$
fr ③, ⑤ $\quad -30 - 3\alpha\beta\gamma = 3[1-4]$
$\Rightarrow \alpha\beta\gamma = -7$ — ⑥
fm ④ Req. cubic equ^n.
$$x^3-3x^2+4x+7=0$$

- **Quartic Equation (or Biquadratic Equation):**
$$ax^4+bx^3+cx^2+dx+e=0$$
Let the roots of this quartic equ^n are α, β, γ and δ
Then $\quad x^4-(\alpha+\beta+\gamma+\delta)x^3+(\Sigma\alpha\beta)x^2-(\Sigma\alpha\beta\gamma)x+\alpha\beta\gamma\delta=0$ —

$$\Sigma\alpha = -\frac{b}{a}$$

$$\Sigma\alpha\beta = \frac{c}{a}$$

$$\Sigma\alpha\beta\gamma = -\frac{d}{a}$$

and $\quad \alpha\beta\gamma\delta = \dfrac{e}{a}$

• Example 7: The roots of the quartic equation, $x^4+4x^3+2x^2-4x+1=0$ are α, β, γ and δ.

Find the value of: (i) $\alpha+\beta+\gamma+\delta$

(ii) $\alpha^2+\beta^2+\gamma^2+\delta^2$

(iii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}$

(iv) $\frac{\alpha}{\beta\gamma\delta}+\frac{\beta}{\alpha\gamma\delta}+\frac{\gamma}{\alpha\beta\delta}+\frac{\delta}{\alpha\beta\gamma}$

Using substitution $y=x+1$, find a quartic equation in y. Solve this quartic equation and hence find the roots of the equation,

$x^4+4x^3+2x^2-4x+1=0$. —① $\boxed{W\text{-}14 / 11 / Q11}$

Solution: (i) $\alpha+\beta+\gamma+\delta = -\frac{b}{a} = -4$ ✓

(ii) $\alpha^2+\beta^2+\gamma^2+\delta^2 = (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\beta\delta+\alpha\gamma)$

$= (-4)^2 - 2\times 2 = 12$ ✓

(iii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta} = \frac{\Sigma\alpha\beta\gamma}{\alpha\beta\gamma\delta} = \frac{-(4)}{1} = 4$ ✓

(iv) $\frac{\alpha}{\beta\gamma\delta}+\frac{\beta}{\alpha\gamma\delta}+\ +\ = \frac{\alpha^2+\beta^2+\gamma^2+\delta^2}{\alpha\beta\gamma\delta} = \frac{12}{1} = 12$ ✓

Now Put $y=x+1 \Rightarrow x=y-1$ Put in ①

$(y-1)^4+4(y-1)^3+2(y-1)^2-4(y-1)+1=0$

$\Rightarrow (1-4y+6y^2-4y^3+y^4)+4(-1+3y-3y^2+y^3)-4y+4 + 2(y^2-2y+1)+1=0$

$\Rightarrow y^4-4y^2+4=0$ ✓ — ②

$\Rightarrow (y^2-2)^2=0 \Rightarrow y^2-2=0 \Rightarrow y=\pm\sqrt{2}$

∴ Roots of the new quartic equation ② are:

$\sqrt{2}, \sqrt{2}, -\sqrt{2}, -\sqrt{2}$

Now $x=y-1$ so the roots of the original Eqn ① are $(\sqrt{2}-1), (\sqrt{2}-1), (-\sqrt{2}-1), (-\sqrt{2}-1)$ ✓

- **Example 8:** The quartic equation $x^4 - px^2 + qx - r = 0$ where p, q and r are real constants, has two pairs of equal roots, show that $p^2 + 4r = 0$ and state the value of q. --[6]
$\boxed{S-15/13/Q1}$

Solution: Let the roots are α, α, β and β

Sum of roots $\Sigma\alpha = 2\alpha + 2\beta = 0 \Rightarrow \beta = -\alpha$ --(i)

$\Sigma\alpha\beta = \alpha^2 + 4\alpha\beta + \beta^2 = -p$ --(ii)

$\Sigma\alpha\beta\gamma = 2\alpha^2\beta + 2\alpha\beta^2 = -q$ --(iii)

$\Sigma\alpha\beta\gamma\delta = \alpha^2\beta^2 = -r$ --(iv)

from (1) put $\beta = -\alpha$ in (ii) and (iv)

$$p = 2\alpha^2 \text{ and } \alpha^4 = -r \Rightarrow p^2 + 4r = 0 \checkmark$$

also put $\beta = -\alpha$ in (iii)

$$-2\alpha^2\beta + 2\alpha\beta^2 = -q \Rightarrow q = 0 \checkmark$$

Roots of Polynomials

Notes

- **Sum of integral power (equal) of the roots of a polynomial:**

Given a polynomial of degree n,

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n \quad (1)$$

Let the roots of this polynomial are $\alpha, \beta, \gamma, \ldots$, and

$$\boxed{S_n = \alpha^n + \beta^n + \gamma^n + \cdots} \quad (*)$$

$S_1 = \Sigma\alpha = \alpha+\beta+\gamma+\cdots$
$S_2 = \Sigma\alpha^2 = \alpha^2+\beta^2+\gamma^2+\cdots$
$S_3 = \Sigma\alpha^3 = \alpha^3+\beta^3+\gamma^3+\cdots$

for (1)

$$a S_n + a_1 S_{n-1} + a_2 S_{n-2} + \cdots + n\cdot a_n = 0 \quad (2)$$

To find $S_1, S_2, S_3, \ldots$, we may write from (1)

$$a_0S_1 + a_1 = 0 \Rightarrow S_1 = -\frac{a_1}{a_0} \quad (2)$$

$$a_0S_2 + a_1S_1 + 2a_2 = 0 \Rightarrow S_2 = -\left(\frac{a_1}{a_0}S_1 + \frac{2a_n}{a_0}\right) \quad (3)$$

and $a_0S_3 + a_1S_2 + a_2S_1 + 3a_3 = 0$

Put the values of S_1 & S_2 we get S_3 — (0)

- **Example 9:** Given α, β, are the roots of a quad. equation $x^2-5x+6=0$ — (1)

find (i) $\alpha+\beta$
(ii) $\alpha^2+\beta^2$
(iii) $\alpha^3+\beta^3$ (iv) $\alpha^4+\beta^4$

Solution: $S_1 - 5 = 0$ from (1)

$\Rightarrow \Sigma\alpha = \alpha+\beta = S_1 = 5$ — (2)

Again from (1) $S_2 - 5S_1 + 2\times6 = 0$

$S_2 - 5\times5 + 12 = 0 \Rightarrow S_2 = 13$ — (3)

But To find S_3, multiply (1) by x [α is roots]

$x^3 - 5x^2 + 6x = 0$

$\Rightarrow S_3 = 5S_2 - 6S_1 = 5\times13 - 6\times5$

$= 35$ — (4)

To find S_4, multiply (1) by x^2

$x^4 - 5x^3 + 6x^2 = 0$

$\Rightarrow S_4 - 5S_3 + 6S_2 = 0$

$S_4 - 5\times35 + 6\times13 = 0 \Rightarrow S_4 = 97$ — (5)

Alternate method

$\alpha+\beta = -\frac{b}{a} = 5$, $\alpha\beta = \frac{c}{a} = 6$

(ii) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$
$= 5^2 - 2\times6 = 13$

(iii) $\alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$
$= 5^3 - 3\times6\times5$
$= 35$

(iv) $\alpha^4+\beta^4 = (\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2$
$= 13^2 - 2\times6^2 = 97$

$\because \begin{cases}\alpha^3 - 5\alpha^2 + 6\alpha = 0 \\ \beta^3 - 5\beta^2 + 6\beta = 0\end{cases}$

add $(\alpha^3+\beta^3) - 5(\alpha^2+\beta^2) + 6(\alpha+\beta) = 0$

or $S_3 - 5S_2 + 6S_1 = 0$

- **Example 10:** The cubic equation $z^3 - z^2 - z - 5 = 0$ has roots α, β and γ SP-20/01/Q4

(a) Show that the value of $\alpha^3+\beta^3+\gamma^3$ is 19. --[4]

(b) Find the value of $\alpha^4+\beta^4+\gamma^4$. --[2]

Solution:

Denote $\begin{cases} S_1 = \alpha+\beta+\gamma \\ S_2 = \alpha^2+\beta^2+\gamma^2 \\ S_3 = \alpha^3+\beta^3+\gamma^3 \\ S_4 = \alpha^4+\beta^4+\gamma^4 \end{cases}$

Given $z^3 - z^2 - z - 5 = 0$ —(1)

$\therefore$ $S_1 - 1 = 0 \Rightarrow S_1 = 1$ —(2)

Again $S_2 - S_1 + 2(-1) = 0$

$S_2 - 1 - 2 = 0 \Rightarrow S_2 = 3$ —(3)

Also fr(1) $S_3 - S_2 - S_1 + 3(-5) = 0 \Rightarrow S_3 - 3 - 1 - 15 = 0$

$\Rightarrow S_3 = 19$ —(4)

Now to get S_4, multiply (1) by z

$z^4 - z^3 - z^2 - 5z = 0$

$\Rightarrow S_4 - S_3 - S_2 - 5S_1 = 0$

$\Rightarrow S_4 - 19 - 3 - 5\times 1 = 0 \Rightarrow S_4 = 27$ —(5)

- **Example 11:** The roots of the cubic equation $x^3 - 5x^2 + 13x - 4 = 0$ are α, β, γ. W-18/12/Q1

(i) Find $\alpha^2+\beta^2+\gamma^2$ --[3]

(ii) Find the value of $\alpha^3+\beta^3+\gamma^3$ --[2]

Solution: Given

$x^3 - 5x^2 + 13x - 4 = 0$ —(1)

$\begin{cases} S_1 = \alpha+\beta+\gamma \\ S_2 = \alpha^2+\beta^2+\gamma^2 \\ S_3 = \alpha^3+\beta^3+\gamma^3 \end{cases}$

fr(1) $S_1 - 5 = 0 \Rightarrow S_1 = 5$ —(2)

fr(1) $S_2 - 5S_1 + 13\times 2 = 0$

$\Rightarrow S_2 - 5\times 5 + 26 = 0 \Rightarrow \underline{S_2 = -1}$ —(3)

Again fr(1)

$S_3 - 5S_2 + 13S_1 - 4\times 3 = 0$

$S_3 - 5\times(-1) + 13\times 5 - 12 = 0$

$\Rightarrow \underline{S_3 = -58}$

FP1

• **Example 12:** The roots of the equation $x^4 - 3x^2 + 5x - 2 = 0$ are α, β, γ and δ.
and S_n denotes $\alpha^n + \beta^n + \gamma^n + \delta^n$
Show that $S_{n+4} - 3S_{n+2} + 5S_{n+1} - 2S_n = 0$
Find the values of
(i) S_2 and S_4
(ii) S_3 and S_5 $\boxed{W-12/11/Q11}$
Hence find the value of:
$\alpha^2(\beta^3+\gamma^3+\delta^3)+\beta^2(\gamma^3+\delta^3+\alpha^3)+\gamma^2(\delta^3+\alpha^3+\beta^3)+\delta^2(\alpha^3+\beta^3+\gamma^3)$

Solution Given $x^4 - 3x^2 + 5x - 2 = 0$ —— ①
multiply ① by $x^n \Rightarrow x^{n+4} - 3x^{n+2} + 5x^{n+1} - 2x^n = 0$ ②

α is a root of ② $\Rightarrow \alpha^{n+4} - 3\alpha^{n+2} + 5\alpha^{n+1} - 2\alpha^n = 0$ — ③
β " " " $\Rightarrow \beta^{n+4} - 3\beta^{n+2} + 5\beta^{n+1} - 2\beta^n = 0$ — ④
γ " " " $\Rightarrow \gamma^{n+4} - 3\gamma^{n+2} + 5\gamma^{n+1} - 2\gamma^n = 0$ — 5
δ " " " $\Rightarrow \delta^{n+4} - 3\delta^{n+4} + 5\delta^{n+1} - 2\delta^n = 0$ — ⑥

add ③, ④, ⑤ & ⑥ $\Rightarrow S_{n+4} - 3S_{n+2} + 5S_{n+1} - 2S_n = 0$ ⑦

(i) from ① $S_1 - 0 = 0 \Rightarrow S_1 = 0$ —— ⑧
also from ① $S_2 + 0 \times S_1 - 3 \times 2 = 0 \Rightarrow S_2 - 6 = 0 \Rightarrow S_2 = 6$ — ⑧
Again from ⑦ Put $n=0$; $S_4 - 3S_2 - 2 \times 4 = 0 \Rightarrow S_4 - 3\times 6 - 8 = 0$
$\Rightarrow S_4 = 26$ ✓ —— ⑨

from ① $\cdot S_3 + 0 \times S_2 - 3S_1 + 5 \times 3 = 0$
$\rightarrow S_3 - 3 \times 0 + 15 = 0 \Rightarrow S_3 = -15$ ✓ —— ⑩

Multiply ① by x; $x^5 - 3x^3 + 5x^2 - 2x = 0$
$\Rightarrow S_5 - 3S_3 + 5S_2 - 2S_1 = 0$
$\Rightarrow S_5 - 3\times(-15) + 5 \times 6 - 0 = 0$
$S_5 = -75$ —— ⑪

Now $\alpha^2(\beta^3+\gamma^3+\delta^3) + \beta^2(\gamma^3+\delta^3+\alpha^3) + \cdots + \cdots$
$= \alpha^2(S_3 - \alpha^3) + \beta^2(S_3 - \beta^3) + \gamma^2(S_3 - \gamma^3) + \delta^2(S_3 - \delta^3)$
$= S_3(\alpha^2+\beta^2+\gamma^2+\delta^2) - (\alpha^5+\beta^5+\gamma^5+\delta^5)$
$= S_3 \times S_2 - S_5$
$= -15 \times 6 - (-75) = -90 + 75 = -15$ ✓