F.P$_1$

Further Pure Maths-1

Summation Of Series Exercise.

Suresh Goel
(Director)
Alliance World School.
Noida, Delhi-NCR.
INDIA

Summation of Series

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1.(a) Given that $f(r) = \frac{1}{(r+1)(r+2)}$, Show that.

$$f(r-1) - f(r) = \frac{2}{r(r+1)(r+2)} \quad \text{---[2]}$$

(b) Hence find, $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$ ---[3]

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$ ---[1]

SP-20/01/Q1

2. Let $u_n = \frac{4\sin(n-\frac{1}{2})\sin\frac{1}{2}}{\cos(2n-1)+\cos 1}$

(i) Using the formula for $\cos P \pm \cos Q$, Show that

$$u_n = \frac{1}{\cos n} - \frac{1}{\cos(n-1)} \quad \text{---[2]}$$

(ii) Use method of difference to find $\sum_{n=1}^{N} u_n$ --[2]

(iii) Explain why the infinite series,

$u_1 + u_2 + u_3 + \cdots$ does not converge. ---[1]

S-19/11/Q2

3.(i) Use the method of differences to show that:

$$\sum_{r=1}^{N} \frac{1}{(3r+1)(3r-2)} = \frac{1}{3} - \frac{1}{3(3N+1)} \quad \text{---[4]}$$

(ii) Find the limit, as $N \to \infty$, of

$$\sum_{r=N+1}^{N^2} \frac{N}{(3r+1)(3r-2)} \quad \text{---[4]}$$

S-19/13/Q4

4. Let $S_N = \sum_{r=1}^{N} (5r+1)(5r+6)$ and

$$T_N = \sum_{r=1}^{N} \frac{1}{(5r+1)(5r+6)}.$$

(i) Use standard results from the formula list MF-10

To show $S_N = \frac{1}{3}N(25N^2+90N+83)$ ---[3]

(continued)

4. (ii) Use the method of differences to express T_N in terms of N. ---[4]

(iii) Find $\lim_{N\to\infty} (N^{-3} \cdot S_N \cdot T_N)$ ---[2]

[W-19/13/Q5]

5. Let $S_n = \sum_{r=1}^{n} (-1)^{r-1} \cdot r^2$.

(i) Use the standard result for $\sum_{r=1}^{n^2} r^2$, given in the list of formulae (MF-10)

Show that $S_{2n} = -n(2n+1)$ ---[4]

(ii) State the value of $\lim_{n\to\infty} \frac{S_{2n}}{n^2}$ and $\lim_{n\to\infty} \frac{S_{2n+1}}{n^2}$ --[4]

[S-18/11/Q5]

6. (i) Verify that, $\frac{n(e-1)+e}{n(n+1)e^{n+1}} = \frac{1}{ne^n} - \frac{1}{(n+1)e^{n+1}}$ ---[1]

Let $S_N = \sum_{n=1}^{N} \frac{n(e-1)+e}{n(n+1)e^{n+1}}$

(ii) Express S_N in terms of N and e --[2]

Let $S = \lim_{n\to\infty} S_N$

(iii) Find the least value of N such that

$(N+1)(S-S_N) < 10^{-3}$ --[3]

[S-18/13/Q2]

7. (i) By considering $(2r+1)^2-(2r-1)^2$, use the method of difference to prove that:

$$\sum_{r=1}^{n} r = \frac{1}{2} n(n+1) \quad --[3]$$

(ii) By considering $(2r+1)^4-(2r-1)^4$, use the method of differences and the result given in part (i),

$$\sum_{r=1}^{n} r^3 = \frac{1}{4} n^2(n+1)^2 \quad --[5]$$

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The sums S and T are defined as follows:

$$S = 1^3 + 2^2 + 3^3 + \cdots + (2N)^3 + (2N+1)^3$$

$$T = 1^3 + 3^3 + 5^3 + \cdots + (2N-1)^3 + (2N+1)^3.$$

(iii) Use the result given in part (ii) to show that $S = (2N+1)^2 \cdot (N+1)^2$ --[1]

(iv) Hence or otherwise, find an expression in terms of N for T, factorising your answer as far as possible. --[2]

(v) Deduce the value of $\frac{S}{T}$ as $N \to \infty$. --[2]

$\boxed{W\text{-}18 / 11 / Q11}$

8. Let $S_N = \sum_{r=1}^{N} (3r+1)(3r+4)$ and

$$T_N = \sum_{r=1}^{N} \frac{1}{(3r+1)(3r+4)}$$

(i) Use standard results from the list of formula (MF 10) to show: $S_N = N(3N^2+12N+13)$ ---[3]

(ii) Use the method of differences to show that:

$$T_N = \frac{1}{12} - \frac{1}{3(3N+4)}$$ --[3]

(iii) Deduce that $\frac{S_N}{T_N}$ is an integer. ---[2]

(iv) Find $\lim_{N\to\infty} \frac{S_N}{N^3 \cdot T_N}$ $\boxed{W\text{-}18 / 12 / Q7}$ --[2]

9. It is given that $\sum_{r=1}^{n} u_r = n^2(2n+3)$, where n is a positive integer.

(i) Find $\sum_{r=n+1}^{2n} u_r$ --[2]

(ii) Find u_r. --[3]

$\boxed{S\text{-}17 / 11 / Q1}$

10. Verify that:

(i) $$\frac{2r+1}{r(r+1)(r+2)} = \frac{1}{2}\left\{ \frac{(2r+1)(2r+3)}{(r+1)(r+2)} - \frac{(2r-1)(2r+1)}{r(r+1)} \right\}$$ --[2]

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10(ii) Hence show that:

$$\sum_{r=1}^{n} \frac{2r+1}{r(r+1)(r+2)} = \frac{1}{2}\left\{\frac{(2n+1)(2n+3)}{(n+1)(n+2)} - \frac{3}{2}\right\}$$ --[2]

(iii) Deduce the value of $\sum_{r=1}^{\infty} \frac{2r+1}{r(r+1)(r+2)}$ --[2]

[S-17/13/Q2]

11. Find $\sum_{r=1}^{n} (4r-3)(4r+1)$, giving your answer in its simplest form. --[4]

[W-17/11/Q1]

12. Express $\frac{4}{r(r+1)(r+2)}$ into partial fractions and hence find $\sum_{r=1}^{n} \frac{4}{r(r+1)(r+2)}$ --[5]

Deduce the value of $\sum_{r=1}^{\infty} \frac{4}{r(r+1)(r+2)}$ --[1]

[S-16/11/Q2]

13. Verify: $\frac{1}{(3r+1)(3r+4)} = \frac{1}{3}\left(\frac{1}{3r+1} - \frac{1}{3r+4}\right)$ --[1]

Let S_N denotes $\sum_{r=1}^{N} \frac{1}{(3r+1)(3r+4)}$ and

Let S denote $\sum_{r=1}^{\infty} \frac{1}{(3r+1)(3r+4)}$, Find the least value of N, such that $S - S_N = \frac{1}{10\,000}$, --[5]

[S-16/13/Q1]

14. Use the method of difference to find, $\sum_{r=1}^{n} \frac{1}{(2r)^2-1}$, --[4]

[W-16/11/Q1]

Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2-1}$ --[1]

15. Using Formula (MF 10) to show that $\sum_{r=1}^{13} (3r^2-5r+1)$ and $\sum_{r=0}^{9} (r^3-1)$ have the same numerical value. --[4]

[S-15/11/Q1]

16. The sequence $a_1, a_2, a_3, \ldots$ is such that, for all positive integers n, $a_n = \frac{n+5}{\sqrt{(n^2-n+1)}} - \frac{n+6}{\sqrt{(n^2+n+1)}}$.

The sum $\sum_{n=1}^{N} a_n$ is denoted by S_N. Find

(i) the value of S_{30} correct to 3 decimal places. --[3]

(ii) the least value of N for which $S_N > 4.9$ --[4]

$\boxed{W\text{-}15/11/Q4}$

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Summation of Series

Answers:

1. (a) $f(r) = \frac{1}{(r+1)(r+2)}$

$f(r-1) = \frac{1}{r(r+1)}$

$f(r-1) - f(r)$

$= \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$

$= \frac{(r+2) - r}{r(r+1)(r+2)}$

$= \frac{2}{r(r+1)(r+2)}$ ✓

(b) $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$

$\sum_{r=1}^{n} \frac{1}{2}[f(r-1) - f(r)]$

$= \frac{1}{2}\left[f(0) - f(1) \right.$
$f(1) - f(2)$
$\left. f(n-1) - f(n) \right]$

$= \frac{1}{2}[f(0) - f(n)]$

$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right]$

$= \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$ ✓

(c) $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)} = \frac{1}{4}$ ✓

2. $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

(i) $u_n = \frac{2 \times 2\sin(n-\frac{1}{2})\sin\frac{1}{2}}{\cos(2n-1) + \cos 1}$

$= \frac{2[\cos(n-1) - \cos n]}{2\cdot\cos n \cdot \cos(n-1)}$

$= \frac{1}{\cos n} - \frac{1}{\cos(n-1)}$ ✓

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2(ii) $\sum_{n=1}^{N} \frac{4\sin(n-\frac{1}{2})\sin\frac{1}{2}}{\cos(2n-1) + \cos 1}$

$= \sum_{n=1}^{N} \frac{1}{\cos n} - \frac{1}{\cos(n-1)}$

$= \cancel{\frac{1}{\cos 1}} - \frac{1}{1}$
$\frac{1}{\cos 2} - \cancel{\frac{1}{\cos 1}}$
$+ \frac{1}{\cos N} - \cancel{\frac{1}{\cos(N-1)}}$

$= -1 + \frac{1}{\cos N}$ ✓

(iii) when $N \to \infty$, $\cos N$ oscillates
$\therefore u_1 + u_2 + u_3 + \cdots$ does not converge.

3.(i) $\frac{1}{(3r+1)(3r-2)} = \frac{1}{3}\left(\frac{1}{3r-2} - \frac{1}{3r+1}\right)$

$\sum_{r=1}^{N} \frac{1}{(3r+1)(3r-2)}$

$= \frac{1}{3}\left[\frac{1}{1} - \frac{1}{4} + \frac{1}{4} - \frac{1}{7} + \cdots + \frac{1}{3N-2} - \frac{1}{3N+1}\right]$

$= \frac{1}{3}\left[1 - \frac{1}{3N+1}\right]$ ✓

(ii) $\sum_{r=N+1}^{N^2} \frac{N}{(3r+1)(3r-2)}$

$= \sum_{r=1}^{N^2} \frac{N}{(3r+1)(3r-2)} - \sum_{r=1}^{N} \frac{N}{(3r+1)(3r-2)}$

$= \frac{N}{3} - \frac{N}{3(3N^2+1)} - \left(\frac{N}{3} - \frac{N}{3(3N+1)}\right)$

$= \frac{N}{3(3N+1)} - \frac{N}{3(3N^2+1)}$

$= \frac{N^3 - N^2}{(3N+1)(3N^2+1)}$

$\to \frac{1}{9}$ ✓ as $n \to \infty$

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Answers

4. (i) $\sum_{r=1}^{N}(5r+1)(5r+6)$

$= 25\sum_{r=1}^{N} r^2 + 35\sum_{r=1}^{N} r + 6N$

$= 25\left[\frac{1}{6}N(N+1)(2N+1)\right] + 35\left[\frac{1}{2}N(N+1)\right] + 6N$

$= N\left[\frac{25}{6}(2N^2+3N+1) + \frac{35}{2}N + \frac{35}{2} + 6\right]$

$S_N = \frac{1}{3}N(25N^2+90N+83)$ ✓

(ii) Now $\frac{1}{(5r+1)(5r+6)} = \frac{1}{5}\left[\frac{1}{5r+1} - \frac{1}{5r+6}\right]$

$\therefore T_N = \frac{1}{5}\left[\frac{1}{6} - \frac{1}{11} + \frac{1}{11} - \frac{1}{16} + \cdots - \frac{1}{5N+1} - \frac{1}{5N+6}\right]$

$= \frac{1}{5}\left[\frac{1}{6} - \frac{1}{5N+6}\right] = \frac{1}{30} - \frac{1}{5(5N+6)}$

(iii) $\frac{S_N}{N^3} \cdot T_N \to \frac{25}{3} \times \frac{1}{30} = \frac{5}{18}$ ✓

5. (i) $S_{2n} = 1^2 - 2^2 + 3^2 - 4^2 \cdots$

$= \sum_{r=1}^{2n} r^2 - 2\sum_{r=1}^{n}(2r)^2$

$= \sum_{r=1}^{2n} r^2 - 8\sum_{r=1}^{n} r^2$

$= \frac{1}{6}(2n)(2n+1)(4n+1) - \frac{8}{6}n(n+1)(2n+1)$

$= \frac{1}{3}n(2n+1)(4n+1-4n-4)$

$= -n(2n+1)$ ✓

(ii) $\lim_{n\to\infty} \frac{S_{2n}}{n^2} = -2$

$S_{2n+1} = S_{2n} + (-1)^{2n}\cdot(2n+1)^2$

$= -n(2n+1) + (2n+1)^2$

$= (2n+1)(n+1)$

Thus $\lim_{n\to\infty} \frac{S_{2n+1}}{n^2} = 2$ ✓

6(i) $\frac{1}{ne^n} - \frac{1}{(n+1)e^{n+1}} = \frac{(n+1)e - n}{n(n+1)e^{n+1}}$

$= \frac{n(e-1)+e}{n(n+1)e^{n+1}}$ ✓

(ii) $S_N = \sum_{n=1}^{N}\left[\frac{1}{ne^n} - \frac{1}{(n+1)e^{n+1}}\right]$

$= \left(\frac{1}{e} - \frac{1}{2e^2} + \frac{1}{2e^2} + \cdots - \frac{1}{(N+1)e^{N+1}}\right]$

$= \frac{1}{e} - \frac{1}{(N+1)e^{N+1}}$ ✓

(iii) $S = \frac{1}{e}$

$(N+1)(S - S_N) < 10^{-3}$

$\Rightarrow \frac{1}{e^{N+1}} < 10^{-3}$

$\Rightarrow e^{N+1} > 10^3$

$N+1 > 3\ln 10$

or $N+1 > 6.9$

$N > 5.9$

$\therefore$ least value of N is 6

7(i) $(2r+1)^2 - (2r-1)^2 = 8r$

$\Rightarrow 8\sum_{r=1}^{n} r = -1^2 + (2n+1)^2$

$= 4n(2n+2)$

$\Rightarrow \sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ ✓

(ii) $(2r+1)^4 - (2r-1)^4 = 16(4r^3+r)$

$\Rightarrow 4\sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r = -\left(1-\frac{1}{2}\right)^4 + \left(n+\frac{1}{2}\right)^4$

$4\sum_{r=1}^{n} r^3 = \left(n+\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^4 - \frac{1}{2}n(n+1)$

$= \left[\left(n+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right]\left[\left(n+\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right] - \frac{1}{2}n(n+1)$

$= n^2(n+1)^2$ ✓

$\therefore \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

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Answers

(continued →)

7(iii) $S = \frac{1}{4}(2N+1)^2(2N+2)^2 = (2N+1)^2(N+1)^2$ ✓

(iv) $T = S - \sum_{r=1}^{N}(2r)^3 = (2N+1)^2(N+1)^2 - 2N^2(N+1)^2$

$= (N+1)^2(2N^2+4N+1)$ ✓

(v) $\frac{S}{T} = \frac{(2N+1)^2}{2N^2+4N+1} = \frac{4N^2+4N+1}{2N^2+4N+1} \to 2$ as $N \to \infty$ ✓

8 (i) $\sum_{r=1}^{N}(3r+1)(3r+4) = 9\sum_{r=1}^{N}r^2 + 15\sum_{r=1}^{N}r + 4N$

$= 9\left[\frac{1}{6}N(N+1)(2N+1)\right] + 15\left(\frac{1}{2}N(N+1)\right) + 4N$

$= N\left[\frac{9}{6}(2N^2+3N+1) + \frac{15}{2}N + \frac{15}{2} + 4\right]$

$= N(3N^2+12N+13)$ ✓

(ii) $\frac{1}{(3r+1)(3r+4)} = \frac{1}{3}\left(\frac{1}{3r+1} - \frac{1}{3r+4}\right)$

$T_N = \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7} + \frac{1}{7} - \frac{1}{10} + \cdots + \frac{1}{3(N-1)+1} - \frac{1}{3N+4}\right)$

$= \frac{1}{3}\left(\frac{1}{4} - \frac{1}{3N+4}\right) = \frac{1}{12} - \frac{1}{3(3N+4)}$ ✓

(iii) $T_N = \frac{N}{4(3N+4)} \Rightarrow \frac{S_N}{T_N} = 4(3N+4)(3N^2+12N+13)$

So $\frac{S_N}{T_N}$ is an integer because all terms are integers.

9 (i) $\sum_{n+1}^{2n} u_r = (2n)^2(4n+3) - n^2(2n+3)$

$= 14n^3 + 9n^2$ ✓

(ii) $u_r = r^2(2r+3) - (r-1)^2(2r+1)$ $\left[u_r = \sum_{r=1}^{r} u_r - \sum_{r=1}^{(r-1)} u_r\right.$

$= 6r^2 - 1$ ✓

10 (i) $R.H.S. = \frac{1}{2}\left\{\frac{4r^3+8r^2+3r-(4r^2-1)(r+2)}{r(r+1)(r+2)}\right\}$

$= \frac{1}{2}\left\{\frac{4r^3+8r^2+3r-(4r^3+8r^2-r-2)}{r(r+1)(r+2)}\right\}$

$= \frac{1}{2}\left\{\frac{(4r+2)}{r(r+1)(r+2)}\right\} = \frac{(2r+1)}{r(r+1)(r+2)}$ ✓

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(Continued →) Answers

(ii) Sum to n terms:

$$\frac{1}{2}\left\{\left(\frac{3\cdot5}{2\cdot3}-\frac{1\cdot3}{1\cdot2}\right)+\left(\frac{5\cdot7}{3\cdot4}-\frac{3\cdot5}{2\cdot3}\right)+ \cdots +\left[\frac{(2n+1)(2n+3)}{(n+1)(n+2)}-\frac{(2n-1)(2n+1)}{n(n+1)}\right]\right.$$

$$=\frac{1}{2}\left\{\frac{(2n+1)(2n+3)}{(n+1)(n+2)}-\frac{3}{2}\right\} ✓$$

(iii) $S_\infty = \frac{1}{2}\times 4 - \frac{3}{4} = 1\frac{1}{4}$ ✓

11. $\sum_{r=1}^{n}(4r-3)(4r+1)$

$$=16\sum_{r=1}^{n} r^2 - 8\sum_{r=1}^{n} r - 3n$$

$$=16\times\frac{n(n+1)(2n+1)}{6} - 8\times\frac{n(n+1)}{2} - 3n$$

$$=\frac{n}{3}(16n^2+12n-13) ✓$$

12. $\frac{2}{r}-\frac{4}{r+1}+\frac{2}{r+2}$

$$=\left(2-2+\frac{2}{3}\right)+\left(1-\frac{4}{3}+\frac{1}{2}\right)+\cdots+\left(\frac{2}{n-1}-\frac{4}{n}+\frac{2}{n+1}\right)$$
$$+\left(\frac{2}{n}-\frac{4}{n+1}+\frac{2}{n+2}\right)$$

$$=1-\frac{2}{n+1}+\frac{2}{n+2} ✓$$

Sum to infinity = 1.

13. $\frac{1}{3}\left(\frac{1}{3r+1}-\frac{1}{3r+4}\right)=\frac{1}{3}\left(\frac{(3r+4)-(3r+1)}{(3r+1)(3r+4)}\right)=\frac{1}{(3r+1)(3r+4)}$

$$S_N=\frac{1}{3}\cdot\left[\left(\frac{1}{4}-\frac{1}{7}\right)\left(\frac{1}{7}-\frac{1}{10}\right)+\cdots\left(\frac{1}{3N+1}-\frac{1}{3N+4}\right)\right.$$

$$=\frac{1}{3}\left(\frac{1}{4}-\frac{1}{3N+4}\right) ✓$$

$S_\infty = \frac{1}{12}$ ✓

$$\Rightarrow S-S_N=\frac{1}{3(3N+4)}<\frac{1}{10000}$$

$$\Rightarrow 3N+4>\frac{10000}{3} \Rightarrow N>1109\frac{7}{9}$$

∴ Least Value of N = 1110.

Answers

14. $\dfrac{1}{(2r-1)(2r+1)} = \dfrac{1}{2}\left(\dfrac{1}{2r-1} - \dfrac{1}{2r+1}\right)$

$\sum_{r=1}^{n} \dfrac{1}{(2r)^2-1}$

$= \frac{1}{2}\left\{\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)\cdots\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right.$

$= \frac{1}{2}\left(1-\frac{1}{2n+1}\right) = \frac{n}{2n+1}$

$\therefore \sum_{1}^{\infty} \frac{1}{(2r)^2-1} = \lim_{n\to\infty} \frac{n}{2n+1}$

$= \frac{1}{2}$ ✓

15. $\sum_{r=1}^{13} (3r^2-5r+1)$

$= 3\times\frac{13\times14\times27}{6} - \frac{5\times13\times14}{2} + 13$

$= 2015$ ✓

and $\sum_{r=0}^{9} (r^3-1) = \left(\frac{9\times10}{2}\right)^2 - 10$

$= 2015$ ✓

16. (i) $\left(\frac{6}{\sqrt{1}}-\frac{7}{\sqrt{3}}\right)+\left(\frac{7}{\sqrt{3}}-\frac{8}{\sqrt{7}}\right)+ --- \left(\frac{35}{\sqrt{871}}-\frac{36}{\sqrt{931}}\right)$

$= 6 - \frac{36}{\sqrt{931}} = 4.820$ ✓

(ii) $6 - \frac{n+6}{\sqrt{n^2+n+1}} > 4.9 \Rightarrow 0.21n^2 - 10.79n - 34.79 > 0$

$\Rightarrow n > 54.42$

$\therefore$ Least value of $\underline{n=55}$. ✓