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Further Pure Maths_1

Summation Of Series Notes

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Summation of Series.
Notes

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- **Some Standard Results:** 1. $\sum_{r=1}^{n} 1 = n$

2. $\sum_{r=1}^{n} r = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

3. $\sum_{r=1}^{n} r^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{r=1}^{n} r^3 = 1^3+2^3+3^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{1}{4}\cdot n^2(n+1)^2$

- In General, $S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{r=1}^{n} a_r$

Partial Fractions:

Given an algebraic fraction, whose denominator is expressed as product of linear factors and $\deg N^r < \deg D^r$, can be expressed as algebraic sum of fractions with denominators as the factors of given expression in D^r.

- Example 1: $\frac{r+4}{r(r+1)(r+2)} = \frac{a}{r} + \frac{b}{r+1} + \frac{c}{r+2}$ —— (1)

multiply each term by $r(r+1)(r+2)$

$r+4 = a(r+1)(r+2) + b\,r(r+2) + c\cdot r(r+1)$ —— (2)

To get the value of 'a' put $r=0$ in (2)

$4 = a(0+1)(0+2) \rightarrow 2a = 4 \Rightarrow a = 2$ ✓

To get b, put $r=-1$ (zero of $(r+1)$, $r+1=0 \Rightarrow r=-1$) in (2)

$-1+4 = 0 + b(-1)(-1+2) \rightarrow -b = 3 \Rightarrow b = -3$ ✓

and for c put $r+2=0$ or $r=-2$ in (2)

$-2+4 = 0 + 0 + c\cdot(-2)(-2+1) \rightarrow 2c = 2 \Rightarrow c = 1$ ✓

Now put the value of a, b and c in (1)

The required partial fractions are.

$$\frac{2}{r} - \frac{3}{r+1} + \frac{1}{(r+2)}$$ ✓

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- **Example 2:** Use the method of differce to show that:

(i) $\sum_{r=1}^{N} \frac{1}{(3r+1)(3r-2)} = \frac{1}{3} - \frac{1}{3(3N+1)}$ ---[4]

(ii) Find the limit as $N \to \infty$, of ---[4]

$$\sum_{r=N+1}^{N^2} \frac{N}{(3r+1)(3r-2)}$$

$\boxed{S\text{-}19/13/Q4}$

Solution: Consider (for partial fractions)

(i) $\frac{1}{(3r+1)(3r-2)} = \frac{a}{(3r+1)} + \frac{b}{(3r-2)}$ —— (1)

multiply by $(3r+1)(3r-2)$ to each term of (1)

$\Rightarrow \quad 1 = a(3r-2) + b(3r+1)$ —— (2)

Now put (zero of $3r+1$) $r = -\frac{1}{3}$ in (2)

$\Rightarrow \quad 1 = a\left(3\times(-\frac{1}{3}) - 2\right) \Rightarrow -3a = 1 \Rightarrow -\frac{1}{3}$ ✓

Again put $r = \frac{2}{3}$ $(3r-2=0)$ in (2)

$\Rightarrow \quad 1 = 0 + b(3\times\frac{2}{3}+1) \Rightarrow 3b = 1 \Rightarrow b = \frac{1}{3}$ ✓

Put the values of a and b in (1)

we get $\frac{1}{(3r+1)(3r-2)} = \frac{1}{3}\left(\frac{1}{3r-2} - \frac{1}{3r+1}\right)$ —— (3)

$$\therefore \sum_{r=1}^{N} \frac{1}{(3r+1)(3r-2)} = \frac{1}{3}\left[\begin{array}{l} \frac{1}{1} - \cancel{\frac{1}{4}} \\ \cancel{\frac{1}{4}} - \frac{1}{7} \\ \vdots \qquad \vdots \\ \frac{1}{3N-2} - \frac{1}{3N+1} \end{array}\right]$$

$$= \frac{1}{3}\left[1 - \frac{1}{3N+1}\right] = \frac{1}{3} - \frac{1}{3(3N+1)}$$ ✓

Now

(ii) $\sum_{r=N+1}^{N^2} \frac{N}{(3r+1)(3r+2)}$

$$= \sum_{r=1}^{N^2} \frac{1}{(3r+1)(3r+2)} - \sum_{r=1}^{N} \frac{1}{(3r+1)(3r+2)}$$

$$= \frac{N}{3} - \frac{N}{3(3N^2+1)} - \left(\frac{N}{3} - \frac{N}{3(3N+1)}\right)$$

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2(ii)
$$= \frac{N}{3(3N+1)} - \frac{N}{3(3N^2+1)}$$

$$= \frac{N^3 - N^2}{(3N+1)(3N^2+1)}$$

$$= \frac{1 - \frac{1}{N}}{\left(3+\frac{1}{N}\right)\left(3+\frac{1}{N^2}\right)}$$ Divide N^r and D^r by N^3

as $n \to \infty$ $$= \frac{1-0}{(3+0)(3+0)}$$ $n \to \infty \Rightarrow \frac{1}{N} \Rightarrow 0$, $\frac{1}{N^2} \to 0$

$$= \frac{1}{9}$$ ✓

Example 3. Let $f(r) = r(r+1)(r+2)$ show that
$$f(r) - f(r-1) = 3r(r+1)$$
Hence show that $\sum_{r=1}^{n} r(r+1) = \frac{1}{3} n(n+1)(n+2)$

Using the standard results for $\sum_{r=1}^{n} r$, deduce that
$\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$

Find the sum of the series
$1^2 + 2\times 2^2 + 3^2 + 2\times 4^2 + 5^2 + 2\times 6^2 + \cdots + 2(n-1)^2 + n^2$
where n is odd. $\boxed{W\text{-}12 \mid 11 \mid Q4}$

Solution: $f(r) = r(r+1)(r+2)$ —— ①

$\therefore$ $f(r) - f(r-1) = r(r+1)(r+2) - (r-1)r(r+1)$
$$= r(r+1)[r+2-(r-1)] = 3r(r+1)$$ ✓ —— ②

Now $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}\sum_{r=1}^{n}\left(f(r) - f(r-1)\right)$ from ②

$$= \frac{1}{3}\left[\begin{array}{l} 6 - 0 \\ 24 - 6 \\ \vdots \\ n(n+1)(n+2) - (n-1)n(n+1) \end{array}\right]$$

$$= \frac{1}{3} n(n+1)(n+2)$$ ✓ —— ③

Now $\sum_{r=1}^{n} r^2 = \sum_{r=1}^{n}[r(r+1) - r] = \sum_{r=1}^{n} r(r+1) - \sum_{r=1}^{n} r$

from ③ $= \frac{1}{3} n(n+1)(n+2) - \frac{1}{2} n(n+1)$

$= \frac{1}{6} n(n+1)(2n+1)$ ✓

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3. Now to find the sum of the series.

$1^2 + 2\times2^2 + 3^2 + 2\times4^2 + 5^2 + 2\times6^2 + \cdots + 2(n-1)^2 + n^2$

$= (1^2 + 2^2 + 3^2 + 4^2 + \cdots n^2)$ (n is odd) (→ (n-1) is even)

$+ (2\times1)^2 + (2\times2^2) + \cdots + \left(2\times\frac{n-1}{2}\right)^2$

$= \sum_{r=1}^{n} r^2 + 2^2 . \sum_{r=1}^{\frac{n-1}{2}} r^2$

$= \frac{n(n+1)(2n+1)}{6} + 4\times\frac{\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\times n}{6}$

$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)(n-1)}{6}$

$= \frac{n(n+1)}{6}\left[(2n+1) + (n-1)\right]$

$= \frac{n(n+1)}{6}\times 3n = \frac{1}{2}n^2(n+1)$ ✓

Example 4. It is given that, $S_n = \sum_{r=1}^{n} u_r = 2n^2 + n$

Write down the values of S_1, S_2, S_3, S_4. Express u_r in terms of r, Justifying your answer.

Find $\sum_{r=n+1}^{2n} u_r$. $\boxed{W-13 / 11 / Q3}$

Solution: Given $S_n = \sum_{r=1}^{n} u_r = 2n^2 + n$ ——— (1)

from (1) $S_1 = 2\times1^2+1 = 3$ ✓

$S_2 = 2\times2^2+2 = 10$ ✓, $S_3 = 2\times3^2+3 = 21$ ✓

and $S_4 = 2\times4^2+4 = 36$ ✓

Now $u_r = \sum_{r=1}^{r} u_r - \sum_{r=1}^{r-1} u_r = S_r - S_{r-1}$

$= (2r^2 + r) - (2(r-1)^2 + r - 1)$

$= (4r-1)$ ✓

And $\sum_{r=n+1}^{2n} u_r = S_{2n} - S_n = (2(2n)^2 + 2n) - (2n^2 + n)$

$= 6n + n$ ✓

• **Example 5.** Given that $u_k = \frac{1}{\sqrt{(2k-1)}} - \frac{1}{\sqrt{(2k+1)}}$

Express $\sum_{k=13}^{n} u_k$ in terms of n. deduce the value of $\sum_{k=13}^{\infty} u_k$.

[W-14/11/Q1]

Solution: Given $u_k = \frac{1}{\sqrt{(2k-1)}} - \frac{1}{\sqrt{(2k+1)}}$

$$\sum_{k=13}^{n} u_k = \left[\frac{1}{\sqrt{25}} - \frac{1}{\sqrt{27}} \right.$$

$$+ \frac{1}{\sqrt{27}} - \frac{1}{\sqrt{29}}$$

$$\vdots \qquad \vdots$$

add: $$\left. \frac{1}{\sqrt{2n-1}} - \frac{1}{\sqrt{(2n+1)}} \right]$$

$$\therefore \sum_{k=13}^{n} u_k = \frac{1}{5} - \frac{1}{\sqrt{(2n+1)}}$$

$$\therefore \sum_{k=13}^{\infty} u_k = n\to\infty \left(\frac{1}{5} - \frac{1}{\sqrt{2n-1}} \right) = \underline{\frac{1}{5}}$$

• **Example 6:** Express $\frac{4}{r(r+1)(r+2)}$ into partial fractions and find $\sum_{r=1}^{n} \frac{4}{r(r+1)(r+2)}$ ---[5]

Deduce the value of $\sum_{r=1}^{\infty} \frac{4}{r(r+1)(r+2)}$ [S-16/11/Q2] ...[1]

Solution: $$\frac{4}{r(r+1)(r+2)} = \frac{a}{r} + \frac{b}{r+1} + \frac{c}{r+2} \quad \text{--- (1)}$$ for partial fraction

multiplying by $r(r+1)(r+2)$

we get, $$4 = a(r+1)(r+2) + br(r+2) + c\,r(r+1) \quad \text{--- (2)}$$

Put $r=0$ in (2) $\Rightarrow 4 = 2a \Rightarrow a = 2$

Now $r+1=0 \Rightarrow$ Put $r=-1$ in (2) $4 = 0 + b(-1)(+1) \Rightarrow b = -4$

and $r+2=0 \Rightarrow$ Put $r=-2$ in (2) $4 = 0 + 0 + c\times(-2)(-1) \Rightarrow c = 2$

$\therefore$ from (1) Req. partial fractions: $\frac{2}{r} - \frac{4}{r+1} + \frac{2}{r+2}$ --- (3)

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(Example6). ∴ $\sum_{r=1}^{n} \frac{4}{r(r+1)(r+2)} = \sum_{r=1}^{n} \left(\frac{2}{r} - \frac{4}{r+1} + \frac{2}{r+2} \right)$

$$= \Big[\frac{2}{1} - \frac{4}{2} + \frac{2}{3}$$
$$+ \frac{2}{2} - \frac{4}{3} + \frac{2}{4}$$
$$+ \frac{2}{3} - \frac{4}{4} + \frac{2}{5}$$
$$+ \frac{2}{n-1} - \frac{4}{n} + \frac{2}{n+1}$$
$$+ \frac{2}{n} - \frac{4}{n+1} + \frac{2}{n+2} \Big]$$

$$= 1 - \frac{2}{n+1} + \frac{2}{n+2}$$

Now $\sum_{r=1}^{\infty} \frac{4}{r(r+1)(r+2)}$

$$= \lim_{n\to\infty} \left(1 - \frac{2}{n+1} + \frac{2}{n+2} \right) = 1 - 0 + 0$$
$$= 1$$

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