FP_1

Further
Pure Maths_1

Vectors_3D
Exercise

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FP$_1$

Vectors (3D)

Date

1. The position vectors of the points A, B, C and D are $2i + 4j - 3k$; $-2i+5j-4k$; $i+4j+k$; $i+5j+mk$ respectively, where m is an integer. It is given that the shortest distance between the line through A and B, and the line through C and D is 3.

(a) Show that the only possible value of m is 2. --[7]

(b) Find the shortest distance of D from the line through A and C. --[3]

(c) Show that the acute angle between the planes ACD and BCD is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$. SP-20/07/Q6 --[4]

2. The lines l_1 and l_2 have equations,

$$r = 6i+2j+7k+\lambda(i+j) \text{ and}$$

$$r = 4i+4j+\mu(-6j+k) \text{ respectively.}$$

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vectors of P and Q. --[8]

S-19/11/Q3

3. The line l_1 passes through the points A(−3, 1, 4) and B(−1, 5, 9). The line l_2 passes through the points C(−2, 6, 5) and D(−1, 7, 5). --[5]

(i) Find the shortest distance between the lines l_1 and l_2

(ii) Find the acute angle between the line l_2 and the plane containing A, B, and D. S-19/13/Q7 --[5]

4. With O as the origin, the points A, B, and C have position vectors $i-j$, $2i+j+7k$, $i-j+k$ respectively:

(i) Find the shortest distance between the lines OC and AB. ---[5]

continued→

4(ii) Find the cartesian equation of the plane containing the line OC and the common perpendicular of lines OC and AB. W-19/11/Q6 --[4].

5. The line l_1 is parallel to the vector $ai-j+k$, where a is a constant, and passes through the point whose position vector is $9j+2k$. The line l_2 is parallel to the vector $-ai+2j+4k$ and passes through the point whose position vector is $-6i-5j+10k$.

(i) It is given that l_1 and l_2 intersect.

(a) Show that $a=-6/13$ --[3]

(b) Find a cartesian equation of the plane containing l_1 and l_2. --[4]

(ii) Given instead that the perpendicular distance between l_1 and l_2 is $3\sqrt{30}$, find the value of a. --[5]

S-18/11/Q 10

6. The lines l_1 and l_2 have vector equations:

$$r = ai+9j+13k + \lambda(i+2j+3k)$$ and

$$r = -3i+7j-2k + \mu(-i+2j-3k)$$

respectively. It is given that l_1 and l_2 intersect.

(i) Find the value of the constant a --[3]

The point P has position vector $3i+j+6k$.

(ii) Find the perpendicular distance from P to the plane containing l_1 and l_2. --[4]

(iii) Find the perpendicular distance from P to l_2. --[4]

S-18/13/Q7

7. The Π_1 has equation:

$$r = \begin{pmatrix}5\\1\\0\end{pmatrix} + s\begin{pmatrix}-4\\1\\3\end{pmatrix} + t\begin{pmatrix}0\\1\\2\end{pmatrix}$$

(i) Find a cartesian equation of Π_1. --[3]

The plane Π_2 has equation, $3x+y-z=3$

(ii) Find the acute angle between Π_1 and Π_2, giving your answer in degrees. --[2]

(Continued→)

Vectors

(Continued→)

7(iii) Find an equation of line of intersection of Π_1 and Π_2, giving your answer in the form, $r = a + \lambda b$. [W-18/11/Q8] --[5]

8. The position vectors of the points A, B, C, D are $i+j+3k$, $3i+4j+5k$, $-i+3k$, $mj+4k$ respectively, where m is a constant. ---[1]

(i) Show that the lines AB and CD are parallel when $m = 3/2$

(ii) Given that $m \neq 3/2$, find the shortest distance between the lines AB and CD. ---[5]

(iii) When $m = 2$, Find the acute angle between the planes ABC and ABD, giving your answer in degrees. [W-18/12/Q10] ---[6]

9. The position vectors of the points A, B, C, D are $i+j+3k$, $3i-j+5k$, $3i-j+k$, $5i-5j+\alpha k$ respectively, where α is a positive integer. It is given that the shortest distance between the line AB and the line CD is equal to $2\sqrt{2}$

(i) Show that the possible values of α are 3 and 5. --[7]

(ii) Using $\alpha = 3$, find the shortest distance of the point of the point D from the line AC, giving your answer correct to 3 significant figures. ---[3]

(iii) Using $\alpha = 3$, find the acute angle between the planes ABC and ABD, giving your answer in degrees. --[4]

[S-17/11/Q12or]

10. The plane Π_1 passes through the points (1, 2, 1) and (5, −2, 9) and is parallel to the vector $i+2j+3k$.

(i) Find the cartesian equation of Π_1 --[4]

The plane Π_2 contains the lines:

$r = 2i-3j+k+\lambda(i-2j-k)$ and $r = 2i-3j+k+\mu(2i+3j-k)$

(ii) Find the cartesian equation of Π_2. [S-17/13/Q9] [4]

(iii) Find the acute angle between Π_1 and Π_2 --[3]

(FP$_1$)

Vectors

P-4

Date ________

11. The points A, B and C have position vectors, $2i - j + k$, $3i + 4j - k$ and $-i + 2j + 4k$ respectively.

(i) Find the area of triangle ABC. --[4]

(ii) Find the perpendicular distance of the point A from the line BC. --[3]

(iii) Find the cartesian equation of the plane through A, B and C. [W-17|11|Q6] --[2]

12. Find a cartesian equation of the plane Π_1 passing through the points with coordinates $(2,-1,3)$, $(4,2,-5)$ and $(-1,3,-2)$. --[4]

The plane Π_2 has cartesian equation $3x - y + 2z = 5$. Find the acute angle between Π_1 and Π_2. --[3]

Find a vector equation of the line of intersection of the planes Π_1 and Π_2. [S-16|11|Q8] --[4]

13. The position vectors of the points A, B, C, D are, $a = 2i + \lambda j - 3k$, $b = 6i + 3j - 2k$, $c = i + 2j - k$, $d = i + 7j + 4k$ respectively. It is given that the shortest distance between the lines AB and CD is 3.

(i) Show that $\lambda^2 + \lambda - 20 = 0$ --[7]

(ii) The planes p_1 and p_2 are the planes through A, B and D corresponding to the two values of λ, satisfying the equation in part (i). Find the acute angle between p_1 and p_2. [S-16|13|Q11] --[7]

14. The lines l_1 and l_2 have equations:

$r = 6i - 3j + s(3i - 4j - 2k)$ and $r = 2i - j - 4k + t(i - 3j - k)$

The point P on l_1 and point Q on l_2 are such that PQ is perp. to both l_1 and l_2. Show that the position vector of P is $3i + j + 2k$ and find the position vector of Q. --[7]

Find the eqn of plane Π, $r = a + \lambda b + \mu c$ which passes through P and is perp. to l_1. [W-16|11|Q11] --[3]

The plane Π meets plane $r = bi + 2j$ in line l_3. Find the vector equation of l_3. --[4]

15. The lines l_1 and l_2 have equations, $r = 8i+2j+3k+\lambda(i-2j)$ and $r = 5i+3j-14k+\mu(2j-3k)$ respectively. The point P on l_1 and Q on l_2 are such that PQ is perp. to both, l_1 and l_2. Find the position vector of the point P and the position vector of the point Q. --[8]

The points with position vectors $8i+2j+3k$ and $5i+3j-14k$ are denoted by A and B respectively. Find

(i) $\overrightarrow{AP} \times \overrightarrow{AQ}$ and hence the area of the triangle APQ.

(ii) the volume of the tetrahedron APQB.

(Volume of tetrahedron $= \frac{1}{3} \times$ area of base $\times$ perp. height)

S-15/11/Q11 --[6]

16. A line passing through the point A(3,0,2), has vector equation $r = 3i+2k+\lambda(2i+j-2k)$. It meets the plane Π, which has equation $r \cdot (i+2j+k) = 3$, at point P. Find the coordinates of P. --[3]

Write down a vector n which is perp. to Π, and calculate the vector w to determine

$$w = n \times (2i+j-2k). \quad --[3]$$

The point Q lies in Π, and is the foot of perp from A to Π. Use the vector w to determine an equation of line PQ in the form:

$$r = u + \lambda v.$$ --[4]

S-15/13/Q8

(FP1)

Vectors

17. The point A, B and C have position vectors i, $2j$ and $4k$ respectively, relative to an origin O. The point N is the foot of perp. from O to the plane ABC. The point P on the line-segment ON is such that $OP = \frac{3}{4} ON$. The line AP meets the plane OBC at Q. Find a vector perp. to the plane ABC, and show that the length of ON is $\frac{4}{\sqrt{(21)}}$. --[4]
Find the position vector of Q. --[5]
Show that the acute angle between the planes ABC and ABQ is $\cos^{-1}(2/3)$ $\boxed{W-15/11/Q11}$ --[5]

Vectors(3D)

Answers

1. $\vec{a} = 2i+4j-3k \rightarrow \vec{a}_1$
$\vec{b} = -2i+5j-4k \downarrow$
$\vec{c} = i+4j+k \rightarrow \vec{a}_2$
$\vec{d} = i+5j+mk.$

dirn l_1: $\overrightarrow{BA} = \vec{a}-\vec{b} = 4i-j+k$: d_1
dirn l_2 $\overrightarrow{CD} = \vec{d}-\vec{c} = j+(m-1)k$: d_2

Shortest distance l_1 and l_2 is

$$\boxed{\frac{(a_1-a_2)\cdot(d_1\times d_2)}{|d_1\times d_2|} \text{ — ①}}$$

ⓐ $\vec{n} = \vec{d}_1 \times \vec{d}_2 = \overrightarrow{BA}\times\overrightarrow{CD}$

$$= \begin{vmatrix} i & j & k \\ 4 & -1 & 1 \\ 0 & 1 & m-1 \end{vmatrix}$$

$\vec{d}_1 \times \vec{d}_2 = -mi - 4(m-1)j + 4k$ —②

{ now a_1 is point on line $\overrightarrow{BA} = (2i+4j-3k)$
and a_2 is point on line $\overrightarrow{CD} = (i+4j+k)$

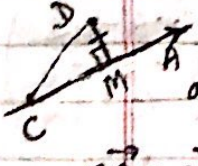
$\Rightarrow \vec{a}_1 - \vec{a}_2 = (i-4k)$ —③

for ② and ③ in ①, The shortest distance.

$$\frac{\begin{pmatrix}1\\0\\-4\end{pmatrix}\cdot\begin{pmatrix}-m\\-4(m-1)\\4\end{pmatrix}}{\sqrt{[m^2+16(1-2m+m^2)+16]}} = 3$$

$\Rightarrow 19m^2 - 40m + 4 = 0$
$\Rightarrow (19m-2)(m-2) = 0$
$\Rightarrow m = 2$ ($\because$ m is integer)

ⓑ $\overrightarrow{CA} = \begin{pmatrix}1\\0\\-4\end{pmatrix}$; $\overrightarrow{CD} = \begin{pmatrix}0\\1\\1\end{pmatrix}$



$$\text{dis} = \frac{|\overrightarrow{CA}\times\overrightarrow{CD}|}{|\overrightarrow{CA}|} \text{ — ④}$$

$$\overrightarrow{CA}\times\overrightarrow{CD} = \begin{vmatrix} i & j & k \\ 1 & 0 & -4 \\ 0 & 1 & 1 \end{vmatrix} = \begin{pmatrix}4\\-1\\1\end{pmatrix} \text{ — ⑤}$$

(continued→)

1(b) from ④ distance

$$= \frac{\sqrt{4^2+1^2+1^2}}{\sqrt{17}} = \sqrt{\frac{18}{17}} = 1.03 ✓$$

(c) $\overrightarrow{CD} = \begin{pmatrix}0\\1\\1\end{pmatrix}$; $\overrightarrow{BC} = \begin{pmatrix}3\\-1\\5\end{pmatrix}$

$\therefore$ normal to Plane BCD

$$\vec{n}_1 = \begin{vmatrix} i & j & k \\ 3 & -1 & 5 \\ 0 & 1 & 1 \end{vmatrix} = (-6i-3j+3k) = -2(2i+j-k) ✓ \text{ — ⑥}$$

and normal vector to Plane ACD
$\vec{n}_2 = 4i-j+k$ from ⑤

~~∴ ⑤ and ⑥ Required angle~~

$$\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

$$= \frac{(4i-j+k)\cdot(2i+j-k)}{\sqrt{16+1+1}\,\sqrt{4+1+1}}$$

$$\cos\theta = \frac{6}{\sqrt{18}\sqrt{6}} = \frac{1}{\sqrt{3}}$$

$\therefore$ Req. angle $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

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2. l_1: $\vec{r} = 6i+2j+7k+\lambda(i+j)$

$\overrightarrow{OP} = \begin{pmatrix}6+\lambda\\2+\lambda\\7\end{pmatrix}$ on l_1 ; $\overrightarrow{OQ} = \begin{pmatrix}4\\4-6\mu\\\mu\end{pmatrix}$ on l_2.

$\Rightarrow \overrightarrow{PQ} = \begin{pmatrix}-2-\lambda\\2-\lambda-6\mu\\\mu-\lambda\end{pmatrix}$ ✓

$\overrightarrow{PQ} \perp l_1 \Rightarrow$
$\overrightarrow{PQ}\cdot\vec{V}_1 = \begin{pmatrix}-2-\lambda\\2-\lambda-6\mu\\\mu-\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix} = 0$

$\Rightarrow -2\lambda - 6\mu = 0$ —— (i)

(Here V_1 is the direction of l_1)

(continued→)

(continued)

Answers continued →

2. $\vec{PQ} \perp l_2$ (V_2 is the direction of l_2)

$$\Rightarrow \vec{PQ}\cdot V_2 = \begin{pmatrix}-2-\lambda\\ 2-\lambda-6\mu\\ \mu-7\end{pmatrix}\begin{pmatrix}0\\ -6\\ 1\end{pmatrix}=0$$

$$\Rightarrow 6\lambda+37\mu=19 \quad —②$$

Solving ① and ② $\lambda=-3,\ \mu=1$

$$\Rightarrow \vec{OP}=\begin{pmatrix}3\\ -1\\ 7\end{pmatrix} ✓ ;\ \vec{OQ}=\begin{pmatrix}4\\ -2\\ 1\end{pmatrix} ✓$$

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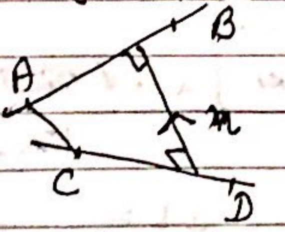
3. $\vec{AB}=\begin{pmatrix}2\\ 4\\ 5\end{pmatrix};\ \vec{CD}=\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}$

(i)

$$\vec{n}=\begin{pmatrix}i & j & k\\ 2 & 4 & 5\\ 1 & 1 & 0\end{pmatrix}=\begin{pmatrix}-5\\ 5\\ -2\end{pmatrix}$$



$$\vec{AC}=\begin{pmatrix}1\\ 5\\ 1\end{pmatrix}$$

$$S.D = \frac{\vec{AC}\cdot\vec{n}}{|\vec{n}|}=\frac{\begin{pmatrix}1\\ 5\\ 1\end{pmatrix}\cdot\begin{pmatrix}-5\\ 5\\ -2\end{pmatrix}}{\sqrt{5^2+5^2+2^2}}$$

$$=\frac{18}{\sqrt{54}}=\sqrt{6}=\underline{2.45} ✓$$

(ii) $\vec{AB}=\begin{pmatrix}2\\ 4\\ 5\end{pmatrix};\ \vec{AD}=\begin{pmatrix}2\\ 6\\ 1\end{pmatrix}$

Normal Vector to Plane ABD $=\vec{AB}\times\vec{AD}$

$$=\begin{vmatrix}i & j & k\\ 2 & 4 & 5\\ 2 & 6 & 1\end{vmatrix}=\begin{pmatrix}-26\\ 8\\ 4\end{pmatrix}\sim\begin{pmatrix}-13\\ 4\\ 2\end{pmatrix}$$

θ; angle between the normal and line $\vec{CD}$

$$\cos\theta=\frac{\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}\begin{pmatrix}-13\\ 4\\ 2\end{pmatrix}}{\sqrt{1^2+1^2+0}\cdot\sqrt{13^2+4^2+2^2}}=\frac{-9}{\sqrt{378}}$$

3(ii) $\cos\theta=-\sqrt{42}/14$

$$\text{Required angle}=\cos^{-1}\left(\frac{-\sqrt{42}}{14}\right)-90$$

$$=\underline{27.6^{\circ}} ✓$$

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4. $\vec{AB}=\begin{pmatrix}1\\ 2\\ 7\end{pmatrix};\ \vec{OC}=\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}$

(i)

$$\vec{OC}\times\vec{AB}=\begin{vmatrix}i & j & k\\ 1 & -1 & 1\\ 1 & 2 & 7\end{vmatrix}=\begin{pmatrix}-9\\ -6\\ 3\end{pmatrix}\sim\begin{pmatrix}3\\ 2\\ -1\end{pmatrix}$$

$$S.D=\frac{\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}\cdot\begin{pmatrix}3\\ 2\\ -1\end{pmatrix}}{\sqrt{3^2+2^2+1^2}}=\frac{1}{\sqrt{14}}=\underline{0.267} ✓$$

(ii) Normal to the plane.

$$\hat{n}=\begin{vmatrix}i & j & k\\ 1 & -1 & 1\\ 3 & 2 & -1\end{vmatrix}=t\begin{vmatrix}-1\\ 4\\ 5\end{vmatrix}$$

∴ Eqn of Plane

$$-1\cdot x+4y+5z=d \quad —①$$

Passes through origin $O(0,0,0)$

for ① $-1\times0+4\times0+5\times0=d \Rightarrow d=0$

from ① Req eqn of Plane

$$-x+4y+5z=0 ✓$$

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5. (i) a Point on l_1 $\begin{pmatrix}\lambda a\\ 9-\lambda\\ 2+\lambda\end{pmatrix}$; Point on l_2 $\begin{pmatrix}-6-\mu a\\ -5+2\mu\\ 10+4\mu\end{pmatrix}$

For Point of intersection

$$\begin{pmatrix}\lambda a\\ 9-\lambda\\ 2+\lambda\end{pmatrix}=\begin{pmatrix}-6-\mu a\\ -5+2\mu\\ 10+4\mu\end{pmatrix} \Rightarrow \mu=1 ✓\ \lambda=12,\ \therefore a=\frac{-6}{13} ✓$$

(FP₁)

(continued→)

Answers

5(i)(b) Normal to the Plane ,

$$\vec{n} = \begin{vmatrix} i & j & k \\ \frac{6}{13} & -1 & -1 \\ \frac{6}{13} & 2 & 4 \end{vmatrix} = -6i + \frac{30}{13}j - \frac{6}{13}k \sim (-13i + 5j - k)$$

Eqⁿ of Plane $-13x + 5y - z = d$ — ①

Passes through point $(0, 9, 2)$

In ① $5 \times 9 - 2 = d = 43$

In ① $\underline{-13x + 5y - z = 43}$ ✓

(ii)
$$\begin{vmatrix} i & j & k \\ a & -1 & 1 \\ -a & 2 & 4 \end{vmatrix} = -6i - 5aj + ak$$

$$\left| \begin{pmatrix} -6 \\ -5a \\ a \end{pmatrix} \right| = \sqrt{36 + 26a^2}$$

$$\begin{pmatrix} -6 \\ -5-9 \\ 10-2 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ -5a \\ a \end{pmatrix} = 36 + 78a$$

$$\Rightarrow 3\sqrt{30}\sqrt{30 + 26a^2} = |36 + 78a|$$

$$\Rightarrow 15(18 + 13a^2) = (6 + 13a)^2$$

$$\Rightarrow 26(a-3)^2 = 0 \Rightarrow a = 3$$ ✓

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6. (i) $9 + 2\lambda = 7 + 2\mu$ — ①

$13 + 3\lambda = -3 - 3\mu$ — ②

Solving ① & ② $\lambda = -3, \mu = -2$

Using the third component

$\therefore a + 3\lambda = -3 - \mu$

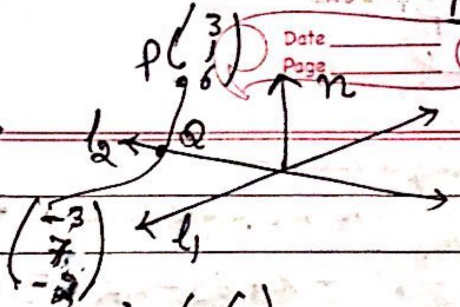
Using the values of λ & $\mu \Rightarrow a = 2$ ✓

(ii) Normal to the Plane $\vec{n} = \vec{V_1} \times \vec{V_2}$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -1 & 2 & -3 \end{vmatrix} = -12i + 4k$$ ✓

(Continued

6 (ii)



$$\vec{PQ} = \begin{pmatrix} -6 \\ 6 \\ -8 \end{pmatrix}$$

Req perp dis $= \frac{\vec{PQ} \cdot \vec{n}}{|\vec{n}|}$

$$= \frac{1}{\sqrt{160}} \begin{pmatrix} -6 \\ 6 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} -12 \\ 0 \\ 4 \end{pmatrix} = \sqrt{10} = \underline{3.16}$$ ✓

(iii) $\vec{PQ} \times$ direction of l_2

$$= \begin{pmatrix} -6 \\ 6 \\ -8 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ -10 \\ -6 \end{pmatrix}$$

perp. distance from P to l_2

$$= \frac{|2i + 10j + 6k|}{|-i + 2j - 3k|} = \sqrt{10}$$

$= \underline{3.16}$ ✓

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7 Normal to the plane

(i)
$$\vec{n} = \begin{vmatrix} i & j & k \\ -4 & 1 & 3 \\ 0 & 1 & 2 \end{vmatrix} = -i + 8j - 4k$$

$\therefore$ Eqⁿ of Plane

$-x + 8y - 4z = d$ — ①

Passes through a point $(5, 1, 0)$

for ① $-5 + 8 - 0 = d = 3$

In ① Eqⁿ of the plane,

$-x + 8y - 4z = 3$ — ②

(ii) Angle between two planes is the angle between their normals

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{\begin{pmatrix} -1 \\ 8 \\ -4 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}}{\sqrt{81}\sqrt{11}}$$

$$\cos\theta = \frac{9}{9\sqrt{11}}$$

$\theta = 72.5°$ ✓

$\Rightarrow$ Kontinued)

(continued →)

Answers.

7(iii) The two planes are

$\Pi_1: -x+8y-4z=3$ — ②

$\Pi_2: 3x+y-z=3$ — ③

l is line of int of Π_1 & Π_2

$l \perp \vec{n_1}$ $\vec{n_1}, \vec{n_2}$ are

$l \perp \vec{n_2}$ normals to Π_1 & Π_2

l is along $\vec{n_1} \times \vec{n_2}$

$$\begin{vmatrix} i & j & k \\ 3 & 1 & -1 \\ -1 & 8 & -4 \end{vmatrix} = 4i+13j+25k$$

Put $x=1$ in ② & ③ $\begin{cases} 8y-4z=4 \\ y-z=0 \end{cases}$

solving we get $y=1, z=1$

∴ $(1,1,1)$ is common point of Π_1 & Π_2 lies on l.

∴ Eqn of the line of Intersection is

$\vec{r} = i+j+k + \lambda(4i+13j+25k)$

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8 $\vec{AB} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$, $\vec{CD} = \begin{pmatrix} 1 \\ m \\ 1 \end{pmatrix}$

and $\vec{AC} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$, $\vec{AD} = \begin{pmatrix} -1 \\ m-1 \\ 1 \end{pmatrix}$, $\vec{BC} = \begin{pmatrix} -4 \\ -4 \\ -2 \end{pmatrix}$

(i) for $m = 3/2$, $\vec{CD} = \begin{pmatrix} 1 \\ 3/2 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$

$\Rightarrow \vec{CD} = \frac{1}{2}\vec{AB}$

∴ lines $\vec{AB}$ and $\vec{CD}$ are parallel.

8(ii) $m \neq 3/2$

$$\vec{AB} \times \vec{CD} = \vec{n} = \begin{vmatrix} i & j & k \\ 2 & 3 & 2 \\ 1 & m & 1 \end{vmatrix} = \begin{pmatrix} 3-2m \\ 0 \\ 2m-3 \end{pmatrix}$$

parallel to $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Now $\dfrac{|\vec{AC} \cdot \vec{n}|}{|\vec{n}|} = \dfrac{|-2(3-2m)+0+0|}{\sqrt{(3-2m)^2+(2m-3)^2}}$

$= \frac{2}{\sqrt{2}} = \underline{\sqrt{2}}$ ✓

(iii) for $m=2$

Angle between planes is the angle between their normal.

Normal to Plane ABC,

$$\begin{array}{c} \\ \vec{AB} \rightarrow \\ \vec{AC} \rightarrow \end{array} \vec{n_1} = \begin{vmatrix} i & j & k \\ 2 & 3 & 2 \\ - & -1 & 0 \end{vmatrix} = \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

Normal to Plane ABD

$$\begin{array}{c} \\ \vec{AB} \rightarrow \\ \vec{AD} \rightarrow \end{array} \vec{n_2} = \begin{vmatrix} i & j & k \\ 2 & 3 & 2 \\ -1 & 1 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$$

for m = 2

∴ $\cos\theta = \dfrac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|} =$

$= \dfrac{1+8+10}{\sqrt{1^2+2^2+2^2}\sqrt{1^2+4^2+5^2}}$

$= \dfrac{19}{\sqrt{378}}$

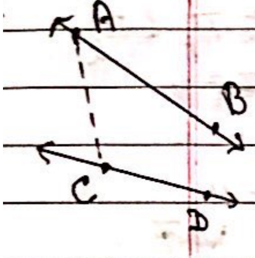
$\theta = \underline{12.2°}$ ✓

Answers (continued →)

9. $\vec{AC} = \begin{pmatrix}2\\-2\\-2\end{pmatrix}$, $\vec{AB}\begin{pmatrix}2\\-2\\2\end{pmatrix}$, $\vec{CD} = \begin{pmatrix}2\\-4\\\alpha-1\end{pmatrix}$

(i)

$$\vec{AB}\times\vec{CD} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 2 & -4 & \alpha-1 \end{vmatrix}$$



$$= \begin{pmatrix}5-\alpha\\3-\alpha\\-2\end{pmatrix} \sim \begin{pmatrix}\alpha-5\\\alpha-3\\2\end{pmatrix}$$

Shortest distance between $\vec{AB}$ and $\vec{CD}$

$$= \frac{|\vec{AC}\cdot(\vec{AB}\times\vec{CD})|}{|\vec{AB}\times\vec{CD}|}$$

$$= \frac{\left|\begin{pmatrix}2\\-2\\-2\end{pmatrix}\cdot\begin{pmatrix}\alpha-5\\\alpha-3\\2\end{pmatrix}\right|}{\sqrt{(\alpha-5)^2+(\alpha-3)^2+24}} = 2\sqrt{2}$$

$\Rightarrow 2\sqrt{2}\sqrt{2\alpha^2-16\alpha+38} = 8$

$\Rightarrow \alpha^2-8\alpha+15=0 \Rightarrow \underline{\alpha=3 \text{ or } 5}$ ✓

(ii) $\vec{AD} = \begin{pmatrix}4\\-6\\0\end{pmatrix}$, $\vec{AC} = \begin{pmatrix}2\\-2\\-2\end{pmatrix} \sim \begin{pmatrix}1\\-1\\-1\end{pmatrix}$

Distance of D from AC

$$= \frac{|\vec{AD}\times\vec{AC}|}{|\vec{AC}|} \quad \text{——} ①$$

$$\vec{AD}\times\vec{AC} = \begin{vmatrix} i & j & k \\ 4 & -6 & 0 \\ 1 & -1 & -1 \end{vmatrix} = 6i+4j+2k$$

fr ① Req Distance $= \dfrac{\sqrt{6^2+4^2+2^2}}{\sqrt{1^2+1^2+1^2}} = \sqrt{\dfrac{56}{3}}$

$= \underline{4.32}$ ✓

(iii) ABC, $\vec{n}_1 = \begin{vmatrix} i & j & k \\ 1 & -1 & -1 \\ 1 & - & 1 \end{vmatrix} = \begin{pmatrix}-1\\-1\\0\end{pmatrix}$

(continue →)

9(iii) ABD, $\vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & & \\ 2 & & \end{vmatrix} = \begin{pmatrix}3\\2\\-1\end{pmatrix}$

$$\cos\theta = \frac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|} = \frac{|-3-2+0|}{\sqrt{2}\sqrt{14}} = \frac{5}{\sqrt{28}}$$

$\underline{\theta = 19.1^\circ}$ ✓

10. $A(1,2,1)$, $B(5,-2,9)$, $\vec{v} = i+2j+3k$

(i) $\vec{AB} = \begin{pmatrix}4\\-4\\8\end{pmatrix} \sim \begin{pmatrix}1\\-1\\2\end{pmatrix}$

Π_1, $\vec{n}_1 = \begin{pmatrix} i & j & k \\ 1 & -1 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix}-7\\-1\\3\end{pmatrix}$

∴ Equⁿ of Π_1 is

$7x+y-3z=d$ —— ①

Passes through $(1,2,1) \Rightarrow d=6$

fr ① Equⁿ of Π_1; $\underline{7x+y-3z=6}$ ✓

(ii) Π_2, $\vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & -2 & -1 \\ 2 & 3 & -1 \end{vmatrix} = \begin{pmatrix}5\\-1\\7\end{pmatrix}$

∴ Equⁿ of Π_2, $5x-y+7z=d$ —— ②

Passes through $(2,-3,1) \Rightarrow d=20$.

from ② Equⁿ of Π_2, $\underline{5x-y+7z=20}$ ✓

(iii) $\cos\theta = \dfrac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|} = \dfrac{\begin{pmatrix}7\\1\\-3\end{pmatrix}\begin{pmatrix}5\\-1\\7\end{pmatrix}}{\sqrt{49+1+9}\cdot\sqrt{25+1+49}}$

$\Rightarrow \cos\theta = \dfrac{13}{\sqrt{59}\sqrt{75}}$

$\Rightarrow \underline{\theta = 78.7^\circ}$ or $\underline{1.37 \text{ rad}}$

(FP1)

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Answers

11. (i) $\vec{AB} = (i + 5j - 2k)$

$\vec{BC} = -4i - 2j + 5k$, $\vec{AC} = -3i + 3j + 3k$

$$\vec{AB} \times \vec{BC} = \begin{vmatrix} i & j & k \\ 1 & 5 & -2 \\ -4 & -2 & 5 \end{vmatrix} = 21i + 3j + 18k \text{ — ①}$$

Area of triangle $= \frac{1}{2}|\vec{AB} \times \vec{BC}|$

$$= \frac{1}{2}\sqrt{21^2 + 3^2 + 18^2} = \frac{3}{2}\sqrt{86} \text{ — ②}$$

$$= \underline{13.9} \checkmark$$

(ii) $d = \dfrac{|\vec{AB} \times \vec{BC}|}{|\vec{BC}|} = \dfrac{3\sqrt{86}}{\sqrt{(4^2 + 2^2 + 5^2)}}$ from ②

$$= \underline{4.15} \checkmark$$

from ①

(iii) ∵ ABC, $\vec{n} = \vec{AB} \times \vec{BC} = 21i + 3j + 18k$

∴ Eqn of ABC, $\vec{n} = \begin{pmatrix} 21 \\ 3 \\ 18 \end{pmatrix} \sim \begin{pmatrix} 7 \\ 1 \\ 6 \end{pmatrix}$

$7x + y + 6z = d$ — ③

Passes through $A(2, -1, 1) \Rightarrow d = 19$

from ③ $\underline{7x + y + 6z = 19} \checkmark$

12, (i) $\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 2 & 3 & -8 \\ -3 & 4 & -5 \end{vmatrix} = \begin{pmatrix} 17 \\ 34 \\ 17 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

ABC, $\vec{n}$ =

Eqn of Plane Π_1: $x + 2y + z = d$ — ①

Passes through $(2, -1, 3)$ from ① $d = 3$

(ii) ∴ Eqn of Plane Π_1, $\underline{x + 2y + z = 3}\checkmark$ — ②

Given Π_2 $3x - y + 2z = 5$ — ③

$$\vec{n}_2 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

∴ Acute angle θ, $\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}}{|\vec{n}_1||\vec{n}_2|}$

$$\cos\theta = \frac{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}}{\sqrt{1^2 + 2^2 + 1^2}\sqrt{3^2 + 1^2 + 2^2}} = \frac{3}{\sqrt{14 \times 6}}$$

$$\theta = 70.9^\circ \text{ or } 1.24 \text{ rad}$$

12.(iii) Direction of line $\vec{v} \perp n_1$, $\vec{v} \perp n_2$

$$\vec{v} = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 3 & -1 & 2 \end{vmatrix} = \begin{pmatrix} 5 \\ 1 \\ -7 \end{pmatrix}$$

Now a point common to Π_1 & Π_2

Put $x = -1 \Rightarrow \begin{cases} 2y + z = 4 \\ -y + 2z = 8 \end{cases}$

Solving $y = 0$, $z = 4$

Point $= (-1, 0, 4) \leftarrow \vec{a}$

∴ Eqn of line of intersection of Π_1 & Π_2 $\vec{r} = \vec{a} + \lambda\vec{v}$

$$\vec{r} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 5 \\ 1 \\ -7 \end{pmatrix} \checkmark$$

13. (i) $\vec{AB} = 4i + (3 - \lambda)j + k$

$\vec{CD} = 0 + 5j + 5k \rightarrow \vec{v} = \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\vec{DB} = 5i - 4j - 6k.$

$$\vec{n} = \vec{AB} \times \vec{CD} = \begin{pmatrix} i & j & k \\ 0 & 1 & 1 \\ 4 & 3-\lambda & 1 \end{pmatrix} = \begin{pmatrix} \lambda - 2 \\ 4 \\ -4 \end{pmatrix}$$

Shortest distance between $\vec{AB}$ & $\vec{CD}$

$$= \frac{|\vec{DB} \cdot \vec{n}|}{|\vec{n}|} = \begin{pmatrix} 5 \\ -4 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} \lambda - 2 \\ 4 \\ -4 \end{pmatrix}$$

Given $3 = \dfrac{}{\sqrt{(\lambda - 2)^2 + 16 + 16}}$

$$= (5\lambda - 2)^2 = 9(\lambda^2 - 4\lambda + 36)$$

$$\Rightarrow \lambda^2 + \lambda - 20 = 0 \checkmark$$

$$\Rightarrow \underline{\lambda = -5 \text{ or } 4.} \checkmark$$

(continued →)

Answers

(Continued →)

13(ii) $\lambda = 4 \Rightarrow a = \begin{pmatrix}2\\4\\-3\end{pmatrix} \Rightarrow \vec{AB} = \begin{pmatrix}4\\-1\\1\end{pmatrix}$

$\vec{AD} = -i+3j+7k$

Normal to ABD $= \vec{n_1}$ $\begin{vmatrix} i & j & k \\ 4 & -1 & 1 \\ -1 & 3 & 7 \end{vmatrix} = \begin{pmatrix}-10\\-29\\11\end{pmatrix}$

If $\lambda = -5 \Rightarrow \vec{a} = \begin{pmatrix}2\\-5\\-3\end{pmatrix} \Rightarrow \vec{AB} = 4i+8j+k$

$\vec{AD} = -i+12j+7k$

Normal to ABD, $\vec{n_2} = \begin{pmatrix} i & j & k \\ 4 & 8 & 1 \\ -1 & 12 & 7 \end{pmatrix} = \begin{pmatrix}44\\-29\\56\end{pmatrix}$

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{-440+841+616}{\sqrt{10^2+29^2+11^2}\sqrt{44^2+29^2+56^2}}$$

$$\cos\theta = \frac{1017}{\sqrt{1062}\sqrt{5913}} \Rightarrow \underline{\theta = 66.1^\circ}$$

---

14. $l_1: \vec{r} = (6i-3j) + s(3i-4j-2k)$

P is a point of $l_1 \Rightarrow$

$$\vec{OP} = \vec{r} = \begin{pmatrix}6+3s\\-3-4s\\-2s\end{pmatrix} \quad \text{—①}$$

$l_2: \vec{r} = 2i-j-4k+t(i-3j-k)$

Q is any point on l_2

$$\vec{OQ} = \vec{r} = \begin{pmatrix}2+t\\-1-3t\\-4-t\end{pmatrix} \quad \text{—②}$$

fr ① & ② $\vec{PQ} = \vec{OQ} - \vec{OP}$

$$= \begin{pmatrix}t-3s-4\\-3t+4s+2\\-t+2s-4\end{pmatrix} \quad \text{—③}$$

as $\vec{PQ}$ is perp to both l_1 & l_2.

direction of $\vec{PQ} = V_1 \times V_2$

$$= \begin{vmatrix} i & j & k \\ 1 & -3 & -1 \\ 3 & -4 & -2 \end{vmatrix} = \begin{pmatrix}2\\-1\\5\end{pmatrix}$$

$$\vec{PQ} = \alpha\begin{pmatrix}2\\-1\\5\end{pmatrix} \quad \text{—④}$$

(Continued →)

(Continued →)

14. from ③ & ④ Equating the resp. components.

$$\begin{cases} t-3s-2\alpha = 4 \\ -3t+4s+\alpha = -2 \\ -t+2s-5\alpha = 4 \end{cases}$$

Solving the three equations

$$s=-1, \; t=-1, \; \alpha=-1$$

Put $s=-1$ in ① & $t=-1$ in ②

we get $\underline{OP} = \begin{pmatrix}3\\1\\2\end{pmatrix} = \underline{3i+j+2k}$ ✓

and

$\underline{\vec{OQ}} = \begin{pmatrix}1\\2\\-3\end{pmatrix} = \underline{i+2j-3k}$ ✓

Now Plane π through P and perp to l_1.

$$\begin{vmatrix} i & j & k \\ 3 & -4 & -2 \\ 2 & -1 & 5 \end{vmatrix} = \begin{pmatrix}-22\\-19\\5\end{pmatrix}$$

∴ Eqn of plane π is

$$\vec{r} = \begin{pmatrix}3\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}-22\\-19\\5\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\5\end{pmatrix} \checkmark$$

Direction of l_3.

$$\begin{vmatrix} i & j & k \\ 3 & -4 & -2 \\ 0 & 0 & 1 \end{vmatrix} = \begin{pmatrix}-4\\-3\\0\end{pmatrix} \sim \begin{pmatrix}4\\3\\0\end{pmatrix}$$

and normal to π,

$$\begin{vmatrix} i & j & k \\ 22 & 19 & -5 \\ 2 & -1 & 5 \end{vmatrix} = \begin{pmatrix}90\\-120\\-60\end{pmatrix} = \begin{pmatrix}3\\-4\\-2\end{pmatrix}$$

Cartesian Eqn of π, $3x-4y-2z=1$
(Since P lies on π)

Vector Eqn of l_3.

$$r = \begin{pmatrix}-1\\-1\\0\end{pmatrix} + \lambda\begin{pmatrix}4\\3\\0\end{pmatrix} \checkmark$$

(Since $z=0$)

FP1

Answers

15. $p=\begin{pmatrix}8+\lambda\\2-2\lambda\\3\end{pmatrix}, q=\begin{pmatrix}5\\3+2\mu\\-14-3\mu\end{pmatrix}$

$\Rightarrow \vec{QP}=\begin{pmatrix}3+\lambda\\-1-2\lambda-2\mu\\17+2\mu\end{pmatrix}$

$\begin{pmatrix}1\\-2\\0\end{pmatrix}\cdot\begin{pmatrix}3+\lambda\\-1-2\lambda-2\mu\\17+2\mu\end{pmatrix}=0\Rightarrow 5\lambda+4\mu=-5$

and $\begin{pmatrix}0\\2\\-3\end{pmatrix}\begin{pmatrix}3+\lambda\\-1-2\lambda-2\mu\\17+2\mu\end{pmatrix}=0\Rightarrow -4\lambda-13\mu=53$

Solving $\lambda=3, \mu=-5$

hence $p=\begin{pmatrix}11\\-4\\3\end{pmatrix}, q=\begin{pmatrix}5\\-7\\1\end{pmatrix}$ ✓

(i) $\vec{AP}=\begin{pmatrix}3\\-6\\0\end{pmatrix}, \vec{AQ}=\begin{pmatrix}-3\\-9\\-2\end{pmatrix}$

$\vec{AP}\times\vec{AQ}=\begin{vmatrix}i & j & k\\3 & -6 & 0\\-3 & -9 & -2\end{vmatrix}=\begin{pmatrix}12\\6\\-45\end{pmatrix}$

Area of triangle APQ $=\frac{1}{2}|\vec{AP}\times\vec{AQ}|$

$=\frac{1}{2}\sqrt{12^2+6^2+45^2}=\frac{1}{2}\sqrt{2205}$

$=23.5$ ✓

(ii) $\vec{AB}=\begin{pmatrix}-3\\1\\-17\end{pmatrix}$

$V=\frac{1}{3}\times\frac{1}{2}\sqrt{2205}\cdot\left|\frac{\begin{pmatrix}-3\\1\\45\end{pmatrix}\cdot\begin{pmatrix}12\\6\\-45\end{pmatrix}}{\sqrt{2205}}\right|$

$=\frac{735}{6}$ $=122.5$ ✓

16. $\begin{pmatrix}3+2\lambda\\\lambda\\2-2\lambda\end{pmatrix}\cdot\begin{pmatrix}k\\2\\1\end{pmatrix}=3\Rightarrow\lambda=-1$

$P(1,-1,4)$ ✓

$n=\begin{pmatrix}1\\2\\1\end{pmatrix}; w=\begin{vmatrix}i & j & k\\1 & 2 & 1\\2 & 1 & -2\end{vmatrix}=\begin{pmatrix}5\\4\\-3\end{pmatrix}$

direction $PQ=\begin{vmatrix}i & j & k\\5 & 4 & -3\\1 & 2 & 1\end{vmatrix}=\begin{pmatrix}10\\2\\-14\end{pmatrix}$

$\sim\begin{pmatrix}5\\1\\-7\end{pmatrix}$

Equ^n of PQ is

$r=\begin{pmatrix}1\\-1\\4\end{pmatrix}+\mu\begin{pmatrix}5\\1\\-7\end{pmatrix}$ ✓

17. $a=i, b=2j, c=4k$

$\Rightarrow \vec{AB}=-i+2j; \vec{AC}=-i+4k$

Plane ABC, $\vec{n}=\begin{vmatrix}i & j & k\\-1 & 2 & 0\\-1 & 0 & 4\end{vmatrix}=\begin{pmatrix}8\\4\\2\end{pmatrix}\sim\begin{pmatrix}4\\2\\1\end{pmatrix}$

$\frac{\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}4\\2\\1\end{pmatrix}}{\sqrt{4^2+2^2+1^2}}=\frac{4}{\sqrt{21}}$

$p=\frac{3}{\sqrt{21}}\cdot\left(\frac{4i+2j+k}{\sqrt{21}}\right)$

$=\frac{1}{7}(4i+2j+k)$

Line AP: $r=\begin{pmatrix}1\\0\\0\end{pmatrix}+t\begin{pmatrix}-3\\2\\1\end{pmatrix}$

for Q, $1-3t=0\Rightarrow t=\frac{1}{3}$

$q=\frac{1}{3}\begin{pmatrix}0\\-4\\1\end{pmatrix}$

Now $AB=\begin{pmatrix}-1\\2\\0\end{pmatrix}, BQ=\frac{1}{3}\begin{pmatrix}0\\-4\\2\end{pmatrix}$

$\begin{vmatrix}i & j & k\\-1 & 2 & 0\\0 & - & 1\end{vmatrix}=\begin{pmatrix}2\\1\\4\end{pmatrix}$

$\Rightarrow$

$\Rightarrow$

$\cos^{-1}\left|\frac{\begin{pmatrix}4\\2\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\4\end{pmatrix}}{\sqrt{21}\sqrt{21}}\right|$

$=\cos^{-1}\frac{14}{21}$

$=\cos^{-1}\left(\frac{2}{3}\right)$ ✓