(P3)

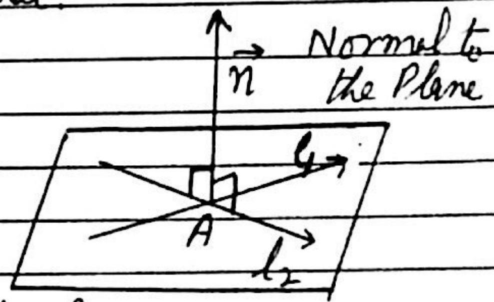
Vectors in 3D

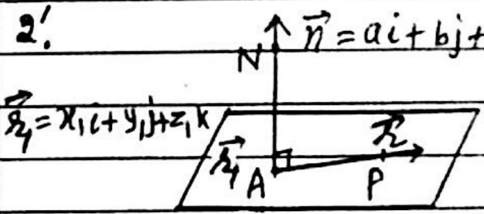
(Lines and Planes)

§ Plane in 3D

Direction of a Plane is expressed in terms of its Normal $\vec{n}$ to the plane, through the point of intersection of plane and normal.

$\vec{n} \perp l_1$ and $\vec{n} \perp l_2$



Cartesian form:	Vector form:
§ 1. General Equation of a Plane $ax + by + cz = d$ here a, b, c are components of Normal.	1'. $\vec{r} \cdot \vec{n} = d$ where $\vec{r} = xi + yj + zk$ and $\vec{n} = ai + bj + ck$
§ 2. Equation of Plane passing through a point $A(x_1, y_1, z_1)$ and components of normal are $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$	2'.  $(\vec{r} - \vec{r}_1) \cdot \vec{n} = 0$ $[\vec{AP} = (\vec{r} - \vec{r}_1)]$ $\vec{r} = xi + yj + zk$

§3. **Cross multiplication method** of solving two equations in three variables:

$$a_1 x + b_1 y + c_1 z = 0$$
$$a_2 x + b_2 y + c_2 z = 0$$

$$a_1 x + b_1 y + c_1 z = 0$$
$$a_2 x + b_2 y + c_2 z = 0$$

$$\frac{x}{(b_1 c_2 - b_2 c_1)} = \frac{-y}{(a_1 c_2 - a_2 c_1)} = \frac{z}{(a_1 b_2 - a_2 b_1)} = \lambda$$

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Vectors in 3D

(Lines and Planes)

Example. Solve for a, b, c using cross multiplication method.

$$a + 2b + 3c = 0$$
$$2a + b - 2c = 0$$

$$\frac{a}{2(-2)-(1\times3)} = \frac{-b}{[1\times(-2)-(2\times3)]} = \frac{c}{1\times1 - 2\times2}$$

or $$\frac{a}{-7} = \frac{+b}{+8} = \frac{c}{-3} = \lambda$$

$$\Rightarrow a = -7\lambda,\ b = 8\lambda,\ c = -3\lambda.$$

---

§4. To find the equation of a plane passing through three point $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$.

Solution: Equation of plane passing through a given point $A(x_1, y_1, z_1)$ and let the components of normal be a, b, c (To be determined)

$$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0 \quad \text{---(i)}$$

Point $B(x_2, y_2, z_2)$ lies on the plane (i), should satisfy (i)

$\therefore$ $$a(x_2-x_1) + b(y_2-y_1) + c(z_2-z_1) = 0 \quad \text{---(ii)}$$

Point $C(x_3, y_3, z_3)$ lies on (1)

$$a(x_3-x_1) + b(y_3-y_1) + c(z_3-z_1) = 0 \quad \text{---(iii)}$$

Solve (ii) and (iii) for a, b, c using cross-multiplication, and put these values of a, b, c in (i)

---

§5. Equation of a plane passing through the intersection of two given planes

P_1: $a_1x + b_1y + c_1z = d_1$

P_2: $a_2x + b_2y + c_2z = d_2$

Eqn of Required plane: $(a_1x + b_1y + c_1z - d_1) + \lambda(a_2x + b_2y + c_2z - d_2) = 0$ —— (1)

find the value λ, with one more given condition and replace in (1).

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P3

Vectors in 3D. (Lines and Planes)

Example 1. Points A, B, C are such that,

$$\vec{OA} = \begin{pmatrix}1\\2\\0\end{pmatrix} \quad \vec{OB} = \begin{pmatrix}3\\0\\1\end{pmatrix} \text{ and } \vec{OC} = \begin{pmatrix}1\\1\\4\end{pmatrix}$$

Find the equation of a plane m perpendicular to AB and contains point C. [SP-17/03/Q7(ii)] --- [2]

Solution: Plane passes through a point $C(1,1,4)$

Plane is perpendicular to (normal) $\vec{AB} = \begin{pmatrix}3-1\\0-2\\1-0\end{pmatrix} = \begin{pmatrix}2\\-2\\1\end{pmatrix}$

$\therefore$ components of normal to the plane $2, -2, 1$.

$\therefore$ Eqn of Plane m: $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

or $2(x-1)-2(y-1)+1(z-4)=0$

$\Rightarrow \underline{2x-2y+z=4}$ ✓

Example 2: Points A, B and C are such that, [S-16/31/Q9(i)]

$$\vec{OA} = \begin{pmatrix}1\\3\\2\end{pmatrix}; \vec{OB} = \begin{pmatrix}2\\1\\-1\end{pmatrix}; \vec{OC} = \begin{pmatrix}2\\4\\1\end{pmatrix}$$

Find the equation of plane containing points (Three) A, B and C, giving your answer in the form $ax+by+cz=d$ --- [6]

Solution: To find the equation of a Plane passing through three points $A(1,3,2)$, $B(2,1,-1)$ and $C(2,4,1)$.

Let the components of normal to the plane are a, b, c.

Equation of plane passing through a point $A(1,3,2)$,

$a(x-1)+b(y-3)+c(z-2)=0$ ——— (1)

$B(2,1,-1)$ lies on plane (1) $\Rightarrow a(2-1)+b(1-3)+c(-1-2)=0$

or $a-2b-3c=0$ ——— (2)

$C(2,4,1)$ also lies on plane (1) $\Rightarrow a(2-1)+b(4-3)+c(1-2)=0$

or $a+b-c=0$ ——— (3)

Solving (2) & (3) by Cross Multiplication method,

$$\frac{a}{(-2)(-1)-(1)(-3)} = \frac{-b}{1\times(-1)-(1)(-3)} = \frac{c}{1\times1-(1)(-2)} = \lambda$$

$\Rightarrow a = 5\lambda, b=-2\lambda, c=3\lambda$ Put in (1) gives the required.

$\underline{5x-2y+3z=5}$ ✓

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§6 General Results:

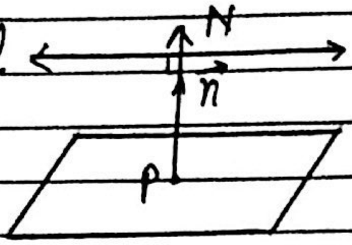
Given line 'l' $\vec{r} = \vec{a} + \lambda\vec{v}$ or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

and Plane 'P', $\vec{r}\cdot\vec{n} = d$ or $ax+by+cz=d$

(a) line l is parallel to plane P.

Then line l is perpendicular to the normal.

$l \perp PN$



$\begin{cases} \vec{v}\cdot\vec{n} = 0 \\ \text{or } av_1 + bv_2 + cv_3 = 0 \end{cases}$

(b) $l \perp$ Plane $\Rightarrow$ $l \parallel$ Normal $\Rightarrow$ $\vec{n} = k\vec{v}$

The direction of normal is same as direction of line.

§7 Given two planes.

$P_1: \vec{r}\cdot\vec{n}_1 = d_1$ and $P_2: \vec{r}\cdot\vec{n}_2 = d_2$

(a) The two planes are parallel if the normals are parallel $\rightarrow$ $\vec{n}_1 = \lambda\vec{n}_2$

(b) The two planes are perpendicular.

$\Leftrightarrow \vec{n}_1 \perp \vec{n}_2 \Leftrightarrow \vec{n}_1\cdot\vec{n}_2 = 0$

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(P3) Vectors in 3D.

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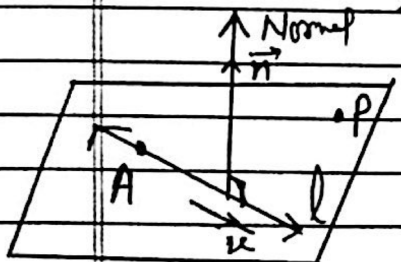
Page P-5

§ To find the equation of a Plane containing a line 'l' and a point P.



→ find the equation of plane through P
→ The point 'A' on line also lies on Plane
→ Normal to the plane is perp. to line ($\vec{v}\cdot\vec{n}=0$)

Example 3. The point 'P' has position vector $3i-2j+k$. The line 'l' has equation $r = 4i+2j+5k+\mu(i+2j+3k)$. --- [5]
Find the equation of plane containing 'l' and 'P' giving your answer in the form $ax+by+cz=d$. $\boxed{\text{S-18/31/Q10(ii)}}$

Solution: Required plane 'm' passes through a point $P(3,-2,1)$.
∴ Equation of plane 'm' is

$$a(x-3)+b(y+2)+c(z-1)=0 \quad \text{——— (1)}$$

where a, b, c are the component of normal to plane.

The given line 'l' contains a point $A(4,2,5)$ also lies on Plane (1), should satisfy (1)

$$a(4-3)+b(2+2)+c(5-1)=0$$

or $$a+4b+4c=0 \quad \text{——— (2)}$$

as line 'l' lies in plane (1) line 'l' is perpendicular to the normal to plane $\vec{n}=ai+bj+ck$ ——— (3)

and direction of line $\vec{v}=i+2j+3k$ ——— (4)

$$\Rightarrow \vec{v}\cdot\vec{n}=0 \Rightarrow a+2b+3c=0 \quad \text{(from (3) \& (4))} \quad (5)$$

Solving (2) and (5) by cross multiplication;

$$\frac{a}{4\times3-2\times4} = \frac{-b}{1\times3-1\times4} = \frac{c}{1\times2-1\times4} = \lambda$$

$\Rightarrow a=4\lambda,\ b=\lambda$ and $c=-2\lambda$

substitute the values a, b and c in (1) and simplify

The equation of plane is: $4x+y-2z=8$ ✓

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P3

Vectors in D.

§ Distance of point 'P' (x_1, y_1, z_1) from a plane $ax+by+cz=d$ is

$$= \left| \frac{ax_1+by_1+cz_1-d}{\sqrt{a^2+b^2+c^2}} \right|$$

§ Angle between two planes $\vec{r}\cdot\vec{n_1}=d_1$ & $\vec{r}\cdot\vec{n_2}=d_2$

is given by the angle 'θ' between their normals:

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}$$

Example 4: The planes 'm' and 'n' have equations, $3x+y-2z=10$ and $x-2y+2z=5$ respectively. The line 'l' has equation $r = 4i+2j+k+\lambda(i+j+2k)$.

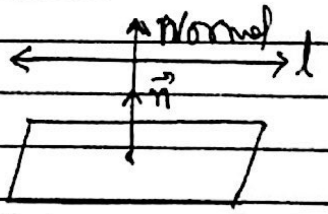
(i) Show that l is parallel to m. --[3]

(ii) Calculate the acute angle between the planes m and n. --[3]

(iii) A point P lies on 'l', the perpendicular distance of P from the plane 'n' is equal to 2. Find the position vector of the two possible positions of P. W-18/31/Q10 --[4]

Solution (i) l // Plane m $\Leftrightarrow$ $l \perp$ normal to Plane

$\Rightarrow \quad \vec{v}\cdot\vec{n} = 0$



line: $\vec{v} = i+j+2k$

normal to m, $\vec{n} = 3i+j-2k$ $\Big\} \Rightarrow \vec{v}\cdot\vec{n} = 1\times3+1\times1+2(-2) = 0$

$\therefore$ line // Plane ✓

(ii) Angle 'θ' between planes m & 'n' is angle between their normals:

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{(a_1^2+b_1^2+c_1^2)}\sqrt{(a_2^2+b_2^2+c_2^2)}}$$

$$= \left|\frac{3\times1+1\times(-2)+(-2)\times(2)}{\sqrt{3^2+1^2+(-2)^2}\sqrt{1^2+(-2)^2+2^2}}\right| = \frac{\cancel{3}}{\sqrt{14}\times\cancel{3}} = \frac{1}{\sqrt{14}}$$

$$\theta = \cos^{-1}\left(\frac{\sqrt{14}}{14}\right) = \underline{74.5^\circ} ✓$$

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Example 4(iii) $P(4+\lambda, 2+\lambda, 1+2\lambda)$ —— ①

the perp. distance of P ... from Plane 'π' $x-2y+2z=5$ is 2

or $\left|\frac{1(4+\lambda)-2(2+\lambda)+2(1+2\lambda)-5}{\sqrt{1^2+(-2)^2+(2)^2}}\right| = 2$

$\Rightarrow \frac{-3+3\lambda}{3} = \pm 2$

$\Rightarrow -3+3\lambda = \pm 6 \Rightarrow \lambda = 3$ or $\lambda = -1$

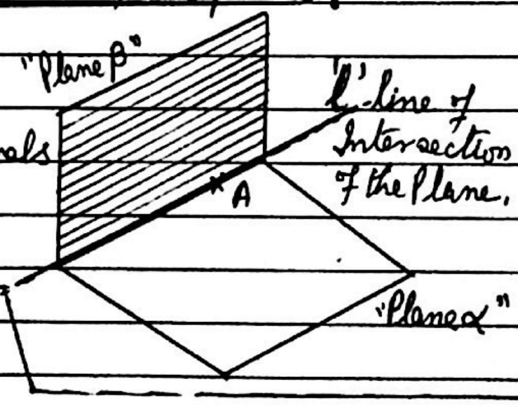
from ① $\lambda = 3$ $P(7i+5j+7k)$ ✓

for $\lambda = -1$ $P(3i+j-k)$ ✓

§ **To find the Equation of the line of intersection of two planes:**

(i) Find a point {P} common to both the plane, will lie on required line 'l'.

(ii) line 'l' is perpendicular to the normals of both the planes, separately.



Example 5: Two planes have equations, $3x+y-z=2$ and $x-y+2z=3$

Find the vector equation of line 'l' of intersection of the two planes. --[6]

Solution: Given equation of Plane① $3x+y-z=2$ —— ① [W-16/31/Q8(iii)]

and Plane② $x-y+2z=3$ —— ②

Put $x=0$ in ① and ② $\Rightarrow \begin{cases} y-z=2 \\ -y+2z=3 \end{cases}$ solving we get $y=7, z=5$

∴ Point $A(0,7,5)$ lies on line of intersection of ① & ②

Let the direction of required line l, $\vec{v} = ai+bj+ck$ —— ③

for Plane① direction of normal $\vec{n}_1 = 3i+j-k$ —— ④

for Plane② direction of normal $\vec{n}_2 = i-j+2k$ —— ⑤

$\vec{n}_1 \perp$ line $l \Rightarrow \vec{n}_1 \cdot \vec{v} = 0 \Rightarrow 3a+b-c=0$ —— ⑥

also $\vec{n}_2 \perp$ line $l \Rightarrow \vec{n}_2 \cdot \vec{v} = 0 \Rightarrow a-b+2c=0$ —— ⑦

Solving ⑥ & ⑦ by cross multiplication. $\frac{a}{2-1} = \frac{-b}{6-(-1)} = \frac{c}{-3-1} = \lambda$

(continued →)

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Example 5: Direction components $a:b:c = 1:-7:-4$

or $\vec{v} = i-7j-4k$

and line 'l' passes through a point $A(0,7,5) \Rightarrow \vec{OA} = 7j+5k$

∴ Equation of line 'l'; $r = a + \lambda\vec{v}$

or $r = 7j+5k+\lambda(i-7j-4k)$

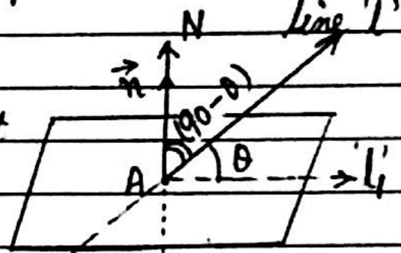
§ To find the angle between a line and a plane.

Let the line 'l' intersects the plane at A.

AN is normal to the plane.

Let the plane containing 'AN' and 'l' intersect the plane in line 'l_1'

The required angle between l & $l_1 = \theta$.



∴ Angle between the normal to plane and line 'l' is $(90-\theta)$.

$$\cos(90-\theta) = \frac{\vec{n}\cdot\vec{v}}{|\vec{n}||\vec{v}|}$$ (where $\vec{v}$ is the direction of line)

here $\hat{\vec{n}}$ is the normal to the plane,

Example 6: A plane has equation $4x-y+5z=39$. A line is parallel to the vector $(i-3j+4k)$ and passes through the point $A(0,2,-8)$. The line meets the plane at the point B. --[4]

Find the acute angle between the line and plane $\boxed{\text{W-15/33/Q8(iv)}}$

Solution: Direction of normal to plane $\vec{n} = 4i-j+5k$

direction of line $\vec{v} = i-3j+4k$.

Let the angle between the line and the plane is θ.

$$\cos(90-\theta) = \frac{\vec{n}\cdot\vec{v}}{|\vec{n}||\vec{v}|}$$

$$\text{or } \sin\theta = \frac{(4i-j+5k)\cdot(i-3j+4k)}{|4i-j+5k|\cdot|i-3j+4k|}$$

$$= \frac{4+3+20}{\sqrt{42}\sqrt{26}} = \frac{27}{\sqrt{42\times26}} = \frac{27}{33.05} = 0.817$$

$$\therefore \theta = \sin^{-1}(0.817) = 54.8^\circ$$

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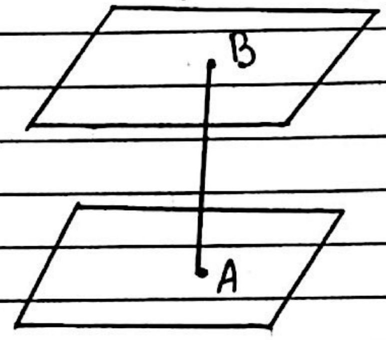
§ Distance between two Parallel Plane.

$P_1: ax+by+cz=d_1$ } Make the coefficients of x, y, z
$P_2: ax+by+cz=d_2$ } in both the equations equal.

Distance $AB = \dfrac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$



Example 7. Find the distance between the parallel planes: $2x-3y+6z=16$ — ①
$4x-6y+12z+4=0$ — ②
---[3]

Solution: Plane ① $2x-3y+6z=16$ — ①
To equate the coefficient of x, y and z in the two parallel planes
divide ② by 2, $\Rightarrow$ $2x-3y+6z=-2$ — ③

Required distance $= \dfrac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$

$= \dfrac{|16-(-2)|}{\sqrt{2^2+(-3)^2+6^2}} = \dfrac{18}{7}$ ✓

$= \dfrac{18}{7}$ ✓

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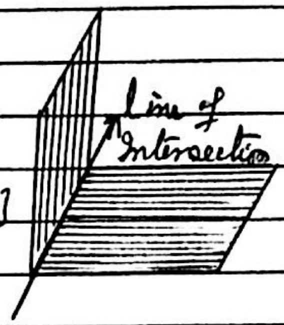
§ To find the equation of a plane passing through the intersection of two given planes $a_1x+b_1y+c_1z=d_1$ — ①
$a_2x+b_2y+c_2z=d_2$ — ②
is given by

$$(a_1x+b_1y+c_1z-d_1)+\lambda(a_2x+b_2y+c_2z-d_2)=0$$

Example 8. Find the equation of plane passing through the intersection of the planes $2x+3y-z+1=0$ and $x+y-2z+3=0$

(i) perpendicular to the plane $3x-y-2z-4=0$ --- [5]
(ii) passing through the point $(1,-2,3)$ --- [4]



Solution: Equation of any plane passing through the intersection of two given plane is $(2x+3y-z+1)+\lambda(x+y-2z+3)=0$ — ①

(i) Perpendicular to Plane $3x-y-2z-4=0$ — ②

Component of the normal to plane ② $\vec{n}_1 = \begin{pmatrix}3\\-1\\-2\end{pmatrix}$

and the component of the normal to plane ① $\vec{n}_2 = \begin{pmatrix}2+\lambda\\3+\lambda\\-1-2\lambda\end{pmatrix}$

as Planes ① & ② are perpendicular

$$\vec{n}_1\cdot\vec{n}_2=0 \to \begin{pmatrix}3\\-1\\-2\end{pmatrix}\cdot\begin{pmatrix}2+\lambda\\3+\lambda\\-1-2\lambda\end{pmatrix}=0$$

$$\Rightarrow 3(2+\lambda)-1(3+\lambda)-2(-1-2\lambda)=0 \Rightarrow \lambda=-5/6$$

Put $\lambda=-5/6$ in ① $\Rightarrow$ Eqn of Plane $\underline{7x+13y+4z=9}$ ✓

(ii) The required plane passes through the point $(1,-2,3)$ satisfy ① $(2\times1+3(-2)-3+1)+\lambda(1-2-2(3)+3)=0$

$$\Rightarrow \lambda=-3/2$$

Put $\lambda=-\frac{3}{2}$ in ① $\Rightarrow$ $\underline{x+3y+4z=7}$ ✓

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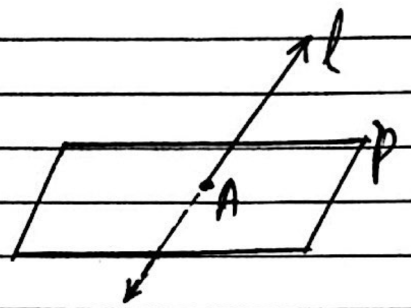
§ To find the point of intersection of a line and a plane:



Example 9. The line 'l' has equation $r = i + 2j - k + \lambda(3i - 2j + 2k)$ and the plane 'p' has equation $2x + 3y - 5z = 18$. --[3]
find the position vector of the point of intersection of l and p.
[S-14/33/Q10]

Solution: Equation of line 'l' $r = i + 2j - k + \lambda(3i - 2j + 2k)$

$$\text{or} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+3\lambda \\ 2-2\lambda \\ -1+2\lambda \end{pmatrix} \quad \text{--- (1)}$$

Eqn of Plane $2x + 3y - 5z = 18$ --- (2)
for the point of intersection of (1) & (2)
Put the values of x, y, z from (1) in (2) to find $\lambda = ?$

$$2(1+3\lambda) + 3(2-2\lambda) - 5(-1+2\lambda) = 18$$

$$\Rightarrow \lambda = -\tfrac{1}{2}$$

$$\text{from (1)} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+3(-\frac{1}{2}) \\ 2-2(-\frac{1}{2}) \\ -1+2(-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 3 \\ -2 \end{pmatrix}$$

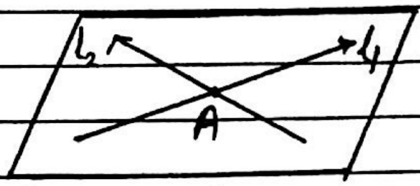
$\therefore$ Position Vector of the point of intersection

$$\left(-\tfrac{1}{2}i + 3j - 2k\right) \checkmark$$

(P3)

Vectors in 3D

§ Equation of Plane containing two intersecting lines:



Example 10. Two lines have equations: $r = \begin{pmatrix}5\\1\\-4\end{pmatrix} + s\begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $r = \begin{pmatrix}p\\4\\-2\end{pmatrix} + t\begin{pmatrix}2\\5\\-4\end{pmatrix}$ where p is a constant. It is given that the lines intersect.

(i) Find the value of p and the coordinates of the point of intersection. --[5]

(ii) Find the equation of plane containing the two lines, giving your answer in the form $ax+by+cz=d$ [W-12/33/Q8] --[5]

Solution: Given line 'l_1' $r = \begin{pmatrix}5\\1\\-4\end{pmatrix} + s\begin{pmatrix}1\\-1\\3\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}5+s\\1-s\\-4+3s\end{pmatrix}$ —— ①

and line 'l_2' $r = \begin{pmatrix}p\\4\\-2\end{pmatrix} + t\begin{pmatrix}2\\5\\-4\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}p+2t\\4+5t\\-2-4t\end{pmatrix}$ —— ②

(i) as lines l_1 and l_2 intersect

$\Rightarrow \begin{pmatrix}5+s\\1-s\\-4+3s\end{pmatrix} = \begin{pmatrix}p+2t\\4+5t\\-2-4t\end{pmatrix} \Rightarrow \begin{matrix}5+s = p+2t\\1-s = 4+5t\\-4+3s = -2-4t\end{matrix} \Rightarrow \begin{matrix}s-2t = p-5 & —③\\-s-5t = 3 & —④\\3s+4t = 2 & —⑤\end{matrix}$

Solving ④ & ⑤ we get $s = 2$ and $t = -1$, Put in ③ $p = 9$ ✓

∴ from ① Point of intersection is $A(7,-1,2)$ ✓

(ii) Now the plane contain the l_1 and l_2 also contains their point of intersection $A(7,-1,2)$, Let a, b, c are component of normal, Equation of Plane through A; $a(x-7)+b(y+1)+c(z-2)=0$ —⑥

as l_1 lies in Plane, normal is perp to $l_1 \Rightarrow \vec{v}\cdot\vec{n} = 0$

direction of l_1, $\vec{v} = \begin{pmatrix}1\\-1\\3\end{pmatrix} \Rightarrow a - b + 3c = 0$ —⑦

Similarly for l_2, $\Rightarrow 2a+5b-4c=0$ —⑧

Solving ⑦ & ⑧ by Cross multiplication,

we get $a = 11\lambda$, $b = -10\lambda$, $c = -7\lambda$

Put the values of a, b and c in ⑥

The required equation of Plane: $11x - 10y - 7z = 73$ ✓

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