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FP2

Further Pure Maths-2

Complex Numers Exercise.

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Complex numbers

1 (a) Using de Moivre's theorm, show that

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta} \quad \text{--[5]}$$

(b) Hence show that the equation $x^2 - 10x + 5 = 0$ has roots $\tan^2\left(\frac{1}{5}\pi\right)$ and $\tan^2\left(\frac{2}{5}\pi\right)$ --[5]

[SP-20/02/Q6]

2. (a) Find the roots of the equation $z^3 = -1-i$, giving your answer in the form $re^{i\theta}$, where $r > 0$ and $0 \leq \theta < 2\pi$. --[5]

Let $W = z_1^{3k} + z_2^{3k} + z_3^{3k}$, where k is a positive integer and z_1, z_2 and z_3 are the roots of $z^3 = -1-i$

(b) Express W in the form $Re^{i\alpha}$, where $R > 0$, giving R and α in terms of k. [S-20/21/Q3] --[3]

3. (a) Use de Moivre's theorem to show that:

$$\sin^6\theta = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10) \quad \text{--[6]}$$

It is given that $\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$.

(b) Find the exact value of $\int_0^{\frac{1}{3}\pi}\left(\cos^6\left(\frac{1}{4}\right) + \sin^6\left(\frac{1}{4}\right)\right)dx$ --[4]

(c) Express each root of the equation $16c^6 + 16(1-c^2)^3 - 13 = 0$ in the form $\cos k\pi$, where k is a rational number. --[5]

[S-20/23/Q8]

4. By letting $z = \frac{1}{2}(\cos\theta + i\sin\theta)$, use de Moivre's theorem to deduce that

$$\sum_{m=1}^{\infty}\left(\frac{1}{2}\right)^m \sin m\theta = \frac{2\sin\theta}{5 - 4\cos\theta} \quad \text{---[5]}$$

[S-19/11/Q8(ii)]

5. (i) Write down the fifth roots of unity. --[2]

(ii) Find all the roots of the equation

$$z^{10} + z^5 + 1 = 0$$

giving each root in the form $e^{i\theta}$. ---[5]

[S-19/13/Q3]

6.(i) Use de Moivre's theorem to show that:

$$\sec 6\theta = \frac{\sec^6\theta}{32-48\sec^2\theta+18\sec^4\theta-\sec^6\theta} \quad \text{--[6]}$$

(ii) Hence obtain the roots of the equation:

$$3x^6-36x^4+96x^2-64=0$$

in the form $\sec q\pi$, where q is rational. [W-19| 11| Q9] --[5]

7 (i) Show that if $z=e^{i\theta}$ and $z\neq-1$ then

$$\frac{z-1}{z+1}=i\tan\tfrac{1}{2}\theta \quad \text{---[3]}$$

(ii) Hence, or otherwise, show that if z is a cube root of unity then, $\dfrac{z^3-1}{z^3+1}+\dfrac{z^2-1}{z^2+1}+\dfrac{z-1}{z+1}=0$ ---[5]

(iii) Hence write down three roots of the equation,

$$(z^3-1)(z^2+1)(z+1)+(z^2-1)(z^3+1)(z+1)+(z-1)(z^3+1)(z^2+1)=0$$

and find the other three roots. Give your answer in an exact form. [S-18| 11| Q11(i)] ---[6]

8 (i) Use de Moivre's theorem to show that,

$$\cos 4\theta = \cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta \quad \text{--[3]}$$

(ii) Hence find all the roots of the equation

$$x^4-6x^2+1=0 \quad \text{--[5]}$$

in the form $\tan q\pi$, where q is a positive rational number

[S-18| 13| Q3]

9 (i) Use de Moivre's theorem to show that

$\sin 8\theta = 8\sin\theta\cos\theta(1-10\sin^2\theta+24\sin^4\theta-16\sin^6\theta)$. ---[6]

(ii) Use the equation $\dfrac{\sin 8\theta}{\sin 2\theta}=0$ to find the roots of

$$16x^6-24x^4+10x^2-1=0$$

in the form $\sin k\pi$, where k is rational. ---[4]

[W-18| 11| Q7]

10 (i) By considering the binomial expansion of $\left(z+\frac{1}{z}\right)^6$, where $z=\cos\theta+i\sin\theta$, express $\cos^6\theta$ in the form

$$\frac{1}{32}\left(p+q\cos 2\theta+r\cos 4\theta+s\cos 6\theta\right) \quad \text{---[6]}$$

where p, q, r and s are the integers to be determined. [W-18/12/Q8]

(ii) Hence find the exact value of

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^6\left(\tfrac{1}{2}x\right)dx \quad \text{---[4]}$$

11 (i) Let $z=\cos\theta+i\sin\theta$; show that $z-\frac{1}{z}=2i\sin\theta$ and hence express $16\sin^5\theta$ in the form $\sin 5\theta+p\sin 3\theta+q\sin\theta$, where p and q are integers to be determined. ---[6]

(ii) Hence find the exact value of $\int_0^{\frac{1}{3}\pi} 16\sin^5\theta\, d\theta$ ---[3]

[S-17/11/Q8]

12 (i) Use de Moivre's theorem to prove that

$$\tan 4\theta=\frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta} \quad \text{---[5]}$$

(ii) Hence find the solution of the equation

$$t^4-4t^3-6t^2+4t+1=0 \quad \text{---[5]}$$

giving your answer in the form $\tan k\pi$, where k is a rational number.

[S-17/13/Q7]

13 (i) Use de Moivre's theorem to show that

$$\sin 5\theta=5\sin\theta-20\sin^3\theta+16\sin^5\theta \quad \text{---[5]}$$

(ii) Hence explain why the roots of the equation ---[3]

$16x^4-20x^2+5=0$ are $x=\pm\sin\frac{1}{5}\pi$ and $x=\pm\sin\frac{2}{5}\pi$

(iii) Without using a calculator, find the exact value of

$\sin\frac{1}{5}\pi\,\sin\frac{2}{5}\pi\,\sin\frac{3}{5}\pi\,\sin\frac{4}{5}\pi$ and $\sin^2\frac{1}{5}\pi+\sin^2\left(\frac{2}{5}\pi\right)$ ---[4]

[W-17/11/Q10]

14 (i) Use de Moivre's theorem to express $\cot 7\theta$, in term of $\cot\theta$. ---[4]

(ii) Use the equation $\cot 7\theta=0$ to show that the roots of the equation $x^6-21x^4+35x^2-7=0$

(continued →)

(Continued →)

14(ii) are $\cot\left(\frac{1}{14}k\pi\right)$ for $k = 1, 3, 5, 9, 11, 13$ and deduce that,

$$\cot^2\left(\frac{1}{14}\pi\right)\cot^2\left(\frac{3}{14}\pi\right)\cot^2\left(\frac{5}{14}\pi\right) = 7 \quad \text{---[5]}$$

[S-16/11/Q6]

15.(i) Use de-Moivre's theorem to show that,

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3) \quad \text{---[4]}$$

(ii) Find the corresponding expression for $\sin^4\theta$ in terms of $\cos 4\theta$ and $\cos 2\theta$ ---[4]

(iii) Hence find the exact value of $\int_0^{\frac{1}{8}\pi}(\cos^4\theta + \sin^4\theta)\,d\theta$ ---[3]

[S-16/13/Q9]

16(i) Let $z = \cos\theta + i\sin\theta$, Show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta \text{ and } z^n - \frac{1}{z^n} = 2i\sin n\theta \quad \text{---[2]}$$

(ii) By considering $\left(z - \frac{1}{z}\right)^4\left(z + \frac{1}{z}\right)^2$, show that

$$\sin^4\theta\cos^2\theta = \frac{1}{32}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2) \quad \text{---[7]}$$

(iii) Hence find the exact value of. $\int_0^{\frac{1}{4}\pi}\sin^4\theta\cos^2\theta\,d\theta$ ---[3]

[W-16/11/Q10]

17(i) By considering $\sum_{r=1}^{n} z^{2r-1}$, where $z = \cos\theta + i\sin\theta$, show that, if $\sin\theta \neq 0$

$$\sum_{r=1}^{n}\sin(2r-1)\theta = \frac{\sin^2 n\theta}{\sin\theta} \quad \text{---[7]}$$

(ii) Deduce that,

$$\sum_{r=1}^{n}(2r-1)\cos\left[\frac{(2r-1)\pi}{2n}\right] = -\operatorname{cosec}\left(\frac{\pi}{2n}\right)\cot\left(\frac{\pi}{2n}\right) \quad \text{---[4]}$$

[S-15/11/Q8]

18(i) Let $z = \cos\theta + i\sin\theta$, Use the binomial expansion of $(1+z)^n$, where n is a positive integer, to show that.

$$\binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + \cdots + \binom{n}{n}\cos n\theta = 2^n\cos^n\left(\tfrac{1}{2}\theta\right)\cos\left(\tfrac{1}{2}n\theta\right) - 1 \quad \text{---[7]}$$

(ii) Find $\binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta + \cdots \binom{n}{n}\sin n\theta$ ---[2]

[S-15/13/Q6]

19 (i) Using de Moivre's theorem, show that

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$ --[5]

(ii) Hence show that the equation $x^2 - 10x + 5 = 0$ has roots $\tan^2\left(\frac{1}{5}\pi\right)$ and $\tan^2\left(\frac{2}{5}\pi\right)$ --[4]

(iii) Deduce a quadratic equation, with integer coefficients, having roots $\sec^2\left(\frac{1}{5}\pi\right)$ and $\sec^2\left(\frac{2}{5}\pi\right)$ --[3]

$\boxed{W-15/11/Q10}$

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(Answers on the next page)

Answers

Using de Moivre's theo.

1. (a) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$ — ①

also $(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$ — ②

Comparing real & imaginary parts ① & ②

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

and $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

Now $\tan 5\theta = \sin 5\theta / \cos 5\theta$

$$= \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$$

Dividing N^r & D^r by $\cos^5\theta$

$$\Rightarrow \tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta} \checkmark \quad ③$$

(b) from ③ $\tan 5\theta = 0$

(let $\tan\theta = t$) $\Rightarrow 5t - 10t^3 + t^5 = 0$

$\Rightarrow t[t^4 - 10t^2 + 5] = 0$

$\Rightarrow t^4 - 10t^2 + 5 = 0$ — ④

Now $\tan 5\theta = 0$ roots are

$\Rightarrow \tan 5\theta = n\pi \Rightarrow \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$ — ⑤

$n = \cancel{0}, 1, 2, 3, 4,$

$\therefore$ roots of ④ are shown in ⑤

$\Rightarrow \left(t - \tan\frac{\pi}{5}\right)\left(t - \tan\frac{2\pi}{5}\right)\left(t - \tan\frac{3\pi}{5}\right)\left(t - \tan\frac{4\pi}{5}\right) = 0$

as $\tan 4\pi/5 = -\tan\pi/5$ and $\tan\frac{3\pi}{5} = -\tan\frac{2\pi}{5}$

$\left(t^2 - \tan^2\left(\frac{\pi}{5}\right)\right)\left(t^2 - \tan^2\left(\frac{2\pi}{5}\right)\right) = 0$

So the roots of $x^2 - 10x + 5 = 0$

are $\tan^2\left(\frac{\pi}{5}\right)$ ✓ and $\tan^2\left(\frac{2\pi}{5}\right)$ ✓

2. $(-1 - i) = r(\cos\theta + i\sin\theta) = re^{i\theta}$

(a) $r\cos\theta = -1$, $r\sin\theta = -1$ $\Rightarrow r = \sqrt{2} = 2^{1/2}$, $\theta = \frac{5}{4}\pi$

$\therefore z^3 = 2^{\frac{1}{2}} e^{i\frac{5}{4}\pi}$ — ①

Roots: $z_1 = 2^{\frac{1}{6}} e^{i\frac{5}{12}\pi}$

roots $z = 2^{\frac{1}{6}} e^{i\left(\frac{2n\pi + \frac{5\pi}{4}}{3}\right)}$, $n = 0, 1, 2$

$z_2 = 2^{\frac{1}{6}} \cdot e^{i\frac{13}{12}\pi}$, $z_3 = 2^{\frac{1}{6}} \cdot e^{i\cdot\frac{21}{12}\pi}$

(b) $z_1^{3k} + z_2^{3k} + z_3^{3k} = 3\left(2^{\frac{1}{2}k} e^{i\frac{5}{4}\pi}\right)$

$R = |W| = 3\left(2^{\frac{1}{2}k}\right)$ ✓

$\alpha = \frac{5}{4}k\pi$ ✓

3. (a) $z - z^{-1} = 2i\sin\theta$

$(z - z^{-1})^6 = (z^6 + z^{-6}) - 6(z^4 + z^{-4}) + 15(z^2 + z^{-2}) - 20$

$(2i\sin\theta)^6 = 2\cos 6\theta - 6(2\cos 4\theta) + 15 \times 2\cos 2\theta - 20$

$\Rightarrow$

$\sin^6\theta = \frac{1}{32}\left[-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10\right]$ ✓

Given

(b) $\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$

$\int_0^{\frac{1}{3}\pi} \left(\cos^6\tfrac{1}{4}x + \sin^6\tfrac{1}{4}x\right)dx = \frac{1}{8}\int_0^{\frac{1}{3}\pi}(3\cos x + 5)\,dx$

$= \frac{1}{8}\left[3\sin x + 5x\right]_0^{\frac{1}{3}\pi} = \frac{1}{8}\left[\frac{5}{3}\pi + \frac{3}{2}\sqrt{3}\right]$ ✓

(c) $c = \cos\theta$, $1 - c^2 = \sin^2\theta$

Now $16c^6 + 16(1 - c^2)^3 - 13 = 0$

$\Rightarrow 6\cos 4\theta - 3 = 0$

$4\theta = \frac{1}{3}\pi, \frac{5}{3}\pi, \frac{7}{3}\pi, \frac{11}{3}\pi$

$\Rightarrow c = \cos\left(\frac{1}{12}\pi\right), \cos\left(\frac{5}{12}\pi\right), \cos\left(\frac{7}{12}\pi\right), \cos\left(\frac{11}{12}\pi\right)$ ✓

Answers

$z = \frac{1}{2}(\cos\theta + i\sin\theta)$

4. Since $|z|<1$, $\sum_{z=0}^{\infty} z^m = \frac{-1}{z-1}$

$$\sum_{m=1}^{\infty} 2^{-m}\sin m\theta = \text{Im}\left(\sum_{m=0}^{\infty} z^m\right)$$

$$= \text{Im}\left(\frac{-1}{\frac12\cos\theta + i\frac12\sin\theta - 1}\right)$$

$$= \text{Im}\left(\frac{-(\frac12\cos\theta - 1 - i\frac12\sin\theta)}{\frac14\cos^2\theta - \cos\theta + 1 + \frac14\sin^2\theta}\right)$$

$$= \frac{\frac12\sin\theta}{\frac54 - \cos\theta} = \frac{2\sin\theta}{(5-4\cos\theta)} \checkmark$$

5. (i) $1 = (\cos 0 + i\sin 0) = e^{i0}$

∴ Fifth roots $= e^{i(\frac{2\pi k}{5})}$, $k = 0, \pm1, \pm2$

$1, e^{i(2\pi/5)}, e^{i(-\frac{2\pi}{5})}, e^{i(4\pi/5)}, e^{i(-\frac{4\pi}{5})}$

(ii) $z^5 = -\frac12 \pm i\frac{\sqrt3}{2}$

$= e^{i2\pi(\frac13 + k)}$ or $e^{i2\pi(-\frac13+k)}$

∴ Required roots $e^{i(\pm\frac{2\pi}{15})}, e^{i(\pm\frac{4\pi}{15})}, e^{i(\pm\frac{8\pi}{15})}, e^{i(\pm\frac{10\pi}{15})}, e^{i(\pm\frac{14\pi}{15})}$

6. (i) Write $\cos\theta = c$ and $\sin\theta = s$

$(\cos6\theta + i\sin6\theta) = (c+is)^6$

$\Rightarrow \cos6\theta = c^6 - 15c^4s^2 + 15c^2s^4 - s^6$

$= c^6 - 15c^4(1-c^2) + 15c^2(1-c^2)^2 - (1-c^2)^3$

$= c^6 - 15c^4(1-c^2) + 15c^2(1-2c^2+c^4) - (1-3c^2+3c^4-c^6)$

$= 32c^6 - 48c^4 + 18c^2 - 1$

$$\Rightarrow \sec6\theta = \frac{1}{32c^6-48c^4+18c^2-1}$$

$$= \frac{\sec^6\theta}{32 - 48\sec^2\theta + 18\sec^4\theta - \sec^6\theta} \checkmark$$

6(ii) $x^6 = 2(32 - 48x^2 + 18x^4 - x^6)$

$$\Rightarrow \frac{x^6}{32-48x^2+18x^4-x^6} = 2$$

$\Rightarrow \sec6\theta = 2 \Rightarrow \cos6\theta = \frac12$

$\Rightarrow x = \sec\frac{\pi}{18}$

$x = \sec q\pi$, $q = \frac{5}{18}, \frac{7}{18}, \frac{11}{18}, \frac{13}{18}, \frac{17}{18}$ ✓

7(i) $\frac{z-1}{z+1} = \frac{z^{1/2} - z^{-1/2}}{z^{1/2}+z^{-1/2}}$ —— (1)

$z = \cos\theta + i\sin\theta/2$

$z^{1/2} = (\cos\theta + i\sin\theta)^{1/2} = (\cos\frac\theta2 + i\sin\frac\theta2)$

and $z^{-1/2} =$ —— $= (\cos\theta/2 - i\sin\theta/2)$

$z^{1/2}+z^{-1/2} = 2\cos\theta/2$

$z^{1/2}-z^{-1/2} = 2i\sin\theta/2$

from (1) $\frac{z-1}{z+1} = \frac{2i\sin\theta/2}{2\cos\theta/2} = i\tan\theta/2$ ✓

(ii) If $z=1 \Rightarrow z^2-1=0$ ∴ the result.

If $z\ne1$, $z = e^{i2\pi/3}$ or $e^{-i2\pi/3}$

$$\frac{z^3-1}{z^3+1} + \frac{z^2-1}{z^2+1} + \frac{z-1}{z+1} = 0 + \frac{e^{\pm i4\pi/3}-1}{e^{\pm i4\pi/3}+1} + \frac{e^{\pm i2\pi/3}-1}{e^{\pm i2\pi/3}+1}$$

$$= i\tan\frac12\left(\frac{4\pi}{3}\right) + i\tan\frac12\left(\frac{2\pi}{3}\right) = 0 \checkmark$$

(iii) $z = 1, e^{\pm i2\pi/3}$ ✓

Now $(z^3-1)(z^2+1)(z+1) + \cdots + \cdots = 0$

$= 3z^6 + z^5 + z^4 - z^2 - z - 3 = 0$

$\Rightarrow (z+1)(z^3-1)(3z^2-2z+3) = 0$

$\Rightarrow z = -1, \frac13 \pm i\frac23\sqrt2$ ✓

Answers

8(i) $(c+is)^4 = c^4+4c^3(is)+6c^2(is)^2+4c(is)^3+(is)^4$

$\Rightarrow \cos 4\theta = c^2-6c^2s^2+s^4$ ✓

(ii) $\frac{\cos 4\theta}{\cos^4\theta} = \tan^4\theta - 6\tan^2\theta + 1$

Let $x=\tan\theta$, then $x^4-6x^2+1=0$

$\Rightarrow \cos 4\theta = 0$

$\Rightarrow 4\theta = \pm\frac{\pi}{2}+2m\pi,\ m\in\mathbb{Z}$

Roots are $\tan q\pi$, $q=\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$

$q>0$ ✓

9(i) $\cos 8\theta + i\sin 8\theta = (c+is)^8$

$\Rightarrow \sin 8\theta = 8c^7s-56c^5s^3+56c^3s^5-8cs^7$

$= 8cs[c^6-7c^4s^2+7c^2s^4-s^6]$

$= 8cs[(1-s^2)^3-7(1-s^2)^2s^2+7(1-s^2)s^4-s^6]$

$\therefore \sin 8\theta = 8cs(1-10s^2+24s^4-16s^6)$ ✓ —(1)

(ii) $\sin 2\theta = 2cs$ —(2)

from (1),(2) $\Rightarrow \frac{\sin 8\theta}{\sin 2\theta} = 4[1-10s^2+24s^4-16s^6]$

$\sin 8\theta = 0 \Rightarrow \theta = \frac{n\pi}{8}$

$x=\sin\pi/8$

Req. roots are

$\sin\frac{k\pi}{8}$, $k=\pm1,\pm2,\pm3$

or 1, 2, 3, 9, 10, 11 ✓

10(i) $z+z^{-1} = 2\cos\theta$

$(z+z^{-1})^6 = (z^6+z^{-6})+6(z^4+z^{-4})+15(z^2+z^{-2})+20$

$\Rightarrow 64\cos^6\theta = 2\cos6\theta+12\cos4\theta+30\cos2\theta+20$

$\Rightarrow \cos^6\theta = \frac{1}{32}[10+15\cos2\theta+6\cos4\theta+\cos6\theta]$

(ii) $\int_{-\pi/2}^{\pi/2}\cos^6\frac{1}{2}x\,dx$

$=\frac{1}{32}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 10+15\cos x+6\cos2x+\cos3x)\,dx$

$=\frac{1}{32}\left[10x+15\sin x+3\sin2x+\frac{1}{3}\sin3x\right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}$

$=\frac{1}{32}\left[5\pi+\frac{44}{3}\right]$ ✓

11(i) $z=(\cos\theta+i\sin\theta)$ —(1)

$z^{-1}=(\cos(-\theta)+i\sin(-\theta))$

or $z^{-1}=(\cos\theta-i\sin\theta)$ —(2)

from (1) & (2) $z-z^{-1}=2i\sin\theta$

$(z-z^{-1})^5=(z^5-\frac{1}{z^5})-5(z^3-\frac{1}{z^3})+10(z-\frac{1}{z})$

$\Rightarrow 32i\sin^5\theta = 2i\sin5\theta-10i\sin3\theta+20i\sin\theta$

$\Rightarrow 16\sin^5\theta = \sin5\theta-5\sin3\theta+10\sin\theta$ ✓

(ii) $\int_0^{\pi/3}16\sin^5\theta = \left[-\frac{\cos5\theta}{5}+5\frac{\cos3\theta}{3}-10\cos\theta\right]_0^{\frac{1}{3}\pi}$

$=\left[-\frac{1}{10}-\frac{5}{3}-5\right]-\left[-\frac{1}{5}+\frac{5}{3}-10\right]=\frac{53}{30}$ ✓

12. $(c+is)^4 = c^4+4c^3(is)+6c^2(is)^2+4c(is)^3+(is)^4$

$\Rightarrow \cos4\theta = c^4-6c^2s^2+s^4$

$\sin4\theta = 4c^3s-4cs^3$

$\tan4\theta = \frac{(4c^3s-4cs^3)\div c^4}{(c^4-6c^2s^2+s^4)\div c^4}$

$=\frac{(4\tan\theta-4\tan^3\theta)}{1-6\tan^2\theta+\tan^4\theta}$ ✓

(ii) $\tan4\theta=-1 \Rightarrow t^4-4t^3-6t^2+4t+1=0$

$\Rightarrow 4\theta = \frac{3\pi}{4},\frac{7\pi}{4},\frac{11\pi}{4},\frac{15\pi}{4}$

$t=\tan\frac{3\pi}{16},\tan\frac{7\pi}{16},\tan\frac{11\pi}{16},\tan\frac{15\pi}{16}$ ✓

[or $(\frac{k}{4}-\frac{1}{16})\pi$, $k=0,1,2,3$] ✓

13(i) $\sin5\theta = Im(c+is)^5$

$=Im[c^5+5c^4(is)+10c^3(is)^2+10c^2(is)^3+5c(is)^4+(is)^5]$

$\Rightarrow \sin5\theta = 5c^4s-10c^2s^3+s^5$

$=s[5(1-s^2)^2-10s^2(1-s^2)+s^4]$

$=5\sin\theta-20\sin^3\theta+16\sin^5\theta$ ✓

(continued →)

Answers

13(ii) If $\theta = 0, \pm \frac{1}{5}\pi, \pm \frac{2}{5}\pi$

Then $\sin 5\theta = 0$

$\Rightarrow 16s^5 - 20s^3 + 5s = 0$ where $s = \sin\theta$.

$\Rightarrow s(16s^4 - 20s^2 + 5) = 0$

$s = 0 \Rightarrow \theta = 0$

Hence the roots of $16s^4 - 20s^2 + 5 = 0$ are $\pm \sin\frac{1}{5}\pi\ ;\ \pm \sin\frac{2}{5}\pi$ ✓

(iii) as $\sin\frac{4}{5}\pi = -\sin\left(-\frac{1}{5}\pi\right)$ and $\sin\frac{3}{5}\pi = -\sin\left(-\frac{2\pi}{5}\right)$

$\sin\left(\frac{4}{5}\pi\right)\sin\left(\frac{3}{5}\pi\right)\cdot\sin\left(\frac{2}{5}\pi\right)\cdot\sin\left(\frac{1}{5}\pi\right)$

$= \sin\left(-\frac{1}{5}\pi\right)\left(\sin -\frac{2}{5}\pi\right)\sin\left(\frac{1}{5}\pi\right)\left(\sin\frac{2}{5}\pi\right)$

$= 5/16$ ✓

and $\sin^2\frac{1}{5}\pi + \sin^2\left(\frac{2}{5}\pi\right) = \frac{-(-20)}{16} = \frac{5}{4}$ ✓

14(i) $(c+is)^7 = c^7 + 7c^6(is) + \cdots + (is)^7$

$\frac{\cos 7\theta}{\sin 7\theta} = \frac{c^7 - 21c^5s^2 + 35c^3s^4 - 7cs^6}{7c^6s - 35c^4s^3 + 21c^2s^5 - s^7}$

$\cot 7\theta = \frac{\cot^7\theta - 21\cot^5\theta + 35\cot^3\theta - 7\cot\theta}{7\cot^6\theta - 35\cot^4\theta + 21\cot^2\theta - 1}$ ✓

(ii) $\cot 7\theta = 0$; and $\cot\theta \neq 0$

$\Rightarrow x^6 - 21x^4 + 35x^2 - 7 = 0$ where $x = \cot\theta$

and $\theta = \frac{k\pi}{14}$ where $k = 1, 3, 5, 9, 11, 13$

Product of roots $\Rightarrow$

$\boxed{\cot\frac{\pi}{14}\cot\frac{3\pi}{14}\cot\frac{5\pi}{14}\cot\frac{9\pi}{14}\cot\frac{11\pi}{14}\cot\frac{13\pi}{14} = -7}$

But $\cot\frac{\pi}{14} = -\cot\frac{13\pi}{14}$, $\cot\frac{3\pi}{14} = -\cot\frac{11\pi}{14}$, $\cot\frac{5\pi}{14} = -\cot\frac{9\pi}{14}$

$\therefore \cot^2\frac{\pi}{14}\cdot\cot^2\frac{3\pi}{14}\cdot\cot^2\frac{5\pi}{14} = 7.$ ✓

15(i) $\left(z+\frac{1}{z}\right)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$

Taking $z = \cos\theta + i\sin\theta$,

$(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$

$\Rightarrow \cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ ✓

And (ii)

$\left(z - \frac{1}{z}\right)^4 = \left(z^4 + \frac{1}{z^4}\right) - 4\left(z^2 + \frac{1}{z^2}\right) + 6$

$(2i\sin\theta)^4 = 2\cos 4\theta - 8\cos 2\theta + 6$

$\Rightarrow \sin^4\theta = \frac{1}{8}[\cos 4\theta - 4\cos 2\theta + 3]$ ✓

(iii) $\int_0^{\frac{1}{8}\pi}(\cos^4\theta + \sin^4\theta)\,d\theta$

$= \frac{1}{4}\int_0^{\frac{1}{8}\pi}(\cos 4\theta + 3)\,d\theta$

$= \frac{1}{4}\left[\frac{\sin 4\theta}{4} + 3\theta\right]_0^{\frac{1}{8}\pi} = \frac{1}{16} + \frac{3\pi}{32}$ ✓

16(i) $z^n = \cos n\theta + i\sin n\theta$ and

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$

$\Rightarrow z^n + \frac{1}{z^n} = 2\cos n\theta$ and

$z^n - \frac{1}{z^n} = 2i\sin n\theta.$

(ii) $\left(z - \frac{1}{z}\right)^4\left(z + \frac{1}{z}\right)^2 = \left(z - \frac{1}{z}\right)^2\cdot\left(z^2 - \frac{1}{z^2}\right)^2$

$= \left(z^2 - 2 + \frac{1}{z^2}\right)\left(z^4 - 2 + \frac{1}{z^4}\right)$

$= \left(z^6 + \frac{1}{z^6}\right) - 2\left(z^4 + \frac{1}{z^4}\right) - 2\left(z^2 + \frac{1}{z^2}\right) + 4$

$\Rightarrow 64\sin^4\theta\cos^2\theta = 2\cos 6\theta - 4\cos 2\theta + 4$

$\Rightarrow \sin^4\theta\cos^2\theta = \frac{1}{32}[\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2]$

(iii) $\int_0^{\frac{1}{4}\pi}\sin^4\theta\cos^2\theta\,d\theta$

$= \frac{1}{32}\left[\frac{\sin 6\theta}{6} - \frac{\sin 4\theta}{2} - \frac{\sin 2\theta}{2} + 2\theta\right]_0^{\frac{1}{4}\pi}$

$= \frac{1}{32}\left[-\frac{1}{6} - 0 - \frac{1}{2} + \frac{\pi}{2}\right]$

$= \frac{(3\pi - 4)}{192}$ ✓

Answers

17.(i) $\sum_{r=1}^{n} z^{2r-1} = \dfrac{z\left[1-(z^2)^n\right]}{1-z^2} = \dfrac{z - z^{2n+1}}{1-z^2}$

$= \dfrac{1-z^{2n}}{z^{-1}-z} = \dfrac{1-(\cos 2n\theta + i\sin 2n\theta)}{(\cos\theta - i\sin\theta)-(\cos\theta+i\sin\theta)}$

$= \dfrac{1-\cos 2n\theta + i\sin 2n\theta}{-2i\sin\theta}$

(ii) Equating imaginary part

$\sum_{r=1}^{n} \sin(2r-1)\theta = \dfrac{2\sin^2 n\theta}{2\sin\theta}$

$= \dfrac{\sin^2 n\theta}{\sin\theta}$ ✓

differentiating

$\sum_{r=1}^{n} (2r-1)\cdot\cos(2r-1)\theta$

$= 2n\sin n\theta\cos n\theta\,\mathrm{cosec}\,\theta$

$- \sin^2 n\theta\,\mathrm{cosec}\,\theta\cot\theta$

Putting $\theta = \dfrac{\pi}{2n}$

$\sum_{r=1}^{n} (2r-1)\cos(2r-1)\dfrac{\pi}{2n} = n\sin\dfrac{\pi}{2}\cos\dfrac{\pi}{2}\,\mathrm{cosec}\,\dfrac{\pi}{2n}$

$- \sin^2\dfrac{\pi}{2}\,\mathrm{cosec}\,\dfrac{\pi}{2n}\cdot\cot\dfrac{\pi}{2n}$

$\Rightarrow \sum_{r=1}^{n}(2r-1)\cos\left(\dfrac{(2r-1)\pi}{2n}\right) = -\mathrm{cosec}\left(\dfrac{\pi}{2n}\right)\cot\left(\dfrac{\pi}{2n}\right)$

19 (i)
(ii) } same as Q1.

19(iii).

$\sec^2\alpha = 1+\tan^2\alpha$

$y = 1+x \Rightarrow x = y-1$

$\Rightarrow (y-1)^2 - 10(y-1) + 5 = 0$

$\Rightarrow y^2 - 12y + 16 = 0$ ✓

⟵ X —— X ⟶

18 (i) $(1+z)^n = 1 + \binom{n}{1}z + \binom{n}{2}z^2 + \cdots \binom{n}{n}z^n$

$\mathrm{Re}\,(1+z)^n = 1 + \binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta \cdots + \binom{n}{n}\cos n\theta$ — ①

$\mathrm{Re}\left[(1+\cos\theta)+i\sin\theta\right]^n = \mathrm{Re}\left(2\cos^2\theta/2 + 2i\sin\theta/2\cos\theta/2\right)^n$

$= 2^n\cos^n\theta/2\cdot\mathrm{Re}\left(\cos\theta/2 + i\sin\theta/2\right)^n$ ②

from ① & ②

$\binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + \cdots + \binom{n}{n}\cos n\theta$

$= 2^n\cos^n\theta/2\cdot\cos\dfrac{n\theta}{2} - 1$ ✓

(ii) $\mathrm{Im}(1+z)^n = \binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta + \cdots + \binom{n}{n}\sin n\theta$

$\Rightarrow \binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta + \cdots + \binom{n}{n}\sin n\theta = 2^n\cos^n\theta/2\cdot\sin\left(\dfrac{n\theta}{2}\right)$. ✓