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FP-2

Further Pure Maths-2

Differentiation Exercise

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Differentiation

1. It is given that $y = 2^x$

(a) By differentiating $\ln y$ with respect to x,
Show that, $\frac{dy}{dx} = 2^x \cdot \ln 2$ --[3]

(b) Write down $\frac{d^2y}{dx^2}$, --[1]

(c) Hence find the first three terms in the Maclaurin's series for 2^x. [S-20/21/Q2] -- [3]

2. A curve has equation $\cos y = x$ for $-\pi < x < \pi$

(i) Use implicit differentiation to show that:
$$\frac{d^2y}{dx^2} = -\cot y \left(\frac{dy}{dx}\right)^2 \quad ---[4]$$

(ii) Hence find the exact value of $\frac{d^2y}{dx^2}$ at the point $\left(\frac{1}{2}, \frac{\pi}{3}\right)$ on C. --[2]

[S-19/11/Q1]

3. The positive variables y and t are related by, $y = a^t$, where a is positive constant.

(i) (a) By differentiating $\ln y$ with respect to t,
Show that $\frac{dy}{dt} = a^t \ln a$ --[3]

(b) Write down $\frac{d^2y}{dt^2}$. --[1]

(ii) Determine the set of values of a for which the infinite series $y + \frac{dy}{dt} + \frac{d^2y}{dt^2} + \frac{d^3y}{dt^3} + \cdots$ is convergent. --[3]

A curve has parametric equations,
$$x = t^a, \quad y = a^t$$

(iii) Find $\frac{d^2y}{dx^2}$ in terms of a and t, and show that when $t = 2$, $\frac{d^2y}{dx^2} = 2^{1-2a}(1 - a + 2\ln a)\ln a$ ---[7]

[S-19/13/Q11(ii)]

4. It is given that $y = \ln(ax+1)$, where a is a positive constant, Prove by mathematical induction that, for every positive positive integer n. $\frac{d^ny}{dx^n} = (-1)^{n-1} \cdot \frac{(n-1)!\,a^n}{(ax+1)^n}$ [6]

[W-19/11/Q2]

5. It is given that $y = e^{x}u$, where u is a function of x. The rth derivatives $\frac{d^r y}{dx^r}$ and $\frac{d^r u}{dx^r}$ are denoted by $y^{(r)}$ and $u^{(r)}$ respectively. Prove by mathematical induction that, for all positive integers n,

$$y^{(n)} = e^{x}\left(\binom{n}{0}u + \binom{n}{1}u^{(1)} + \binom{n}{2}u^{(2)} + \dots + \binom{n}{r}u^{(r)} + \dots + \binom{n}{n}u^{(n)}\right)$$

--- [8]

[You may use $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$] |W-18|12|Q6|

6. The curve C is defined parametrically by

$$x = 18t - t^2 \text{ and } y = 8t^{3/2},$$

where $0 < t \leq 4$

(i) Show that at all points of C,

$$\frac{d^2y}{dx^2} = \frac{3(9+t)}{2t^{1/2}(9-t)^3}$$ --- [4]

(ii) Show that the mean value of $\frac{d^2y}{dx^2}$ with respect to x over the interval $0 < x \leq 56$ is $\frac{3}{70}$. --- [4]

(iii) Find the area of the surface when C is rotated through 2π radians about x-axis, showing full working. --- [6] |W-18|12|Q11(i)|

7. A curve has equation $\tan y = x$, for $x > 0$,

(i) Use implicit differentiation to show that,

$$\frac{d^2y}{dx^2} = -2x\left(\frac{dy}{dx}\right)^2$$ [3]

(ii) Hence find the value of $\frac{d^2y}{dx^2}$ at the point $(1, \frac{1}{4}\pi)$ on C. --- [2]

|S-17|11|Q3|

8. It is given that $x = t^{1/2}$, where $x > 0$ and $t > 0$, and y is a function of x.

(i) Show that $\frac{dy}{dx} = 2t^{1/2}\frac{dy}{dt}$ and $\frac{d^2y}{dx^2} = \frac{2dy}{dx} + 4t\frac{d^2y}{dt^2}$ --- [3]

(Continued →)

(continued →)

8(ii) Hence show that the differential equation

$$\frac{d^2y}{dx^2} - \left(8x + \frac{1}{x}\right)\frac{dy}{dx} + 12x^2y = 4x^2e^{-x^2} \quad (*)$$

reduces to the differential equation $\boxed{S-17 \mid 11 \mid Q10}$

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = e^{-t} \quad \text{---[1]}$$

(iii) Find the general solution of (*), giving y in terms of x. --[7]

9. A curve C has equation $x^3 - 3xy + y^2 = 4$. Find the value of $\frac{d^2y}{dx^2}$ at the point $(0,2)$ of C. $\boxed{S-17/13/Q4}$ ---[7]

10. (i) Show that $\frac{d^{n+1}}{dx^{n+1}}(x^{n+1}\ln x) = \frac{d^n}{dx^n}\left(x^n + (n+1)x^n \ln x\right)$ ---[2]

(ii) Prove by mathematical induction that, for all positive integers n, $\frac{d^n}{dx^n}(x^n \ln x) = n!\left(\ln x + 1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$ ---[5]

$\boxed{W-17/11/Q3}$

11. The curve C has equation $2x^3 + 3x^2y - 3y^3 - 16 = 0$

(i) Find the coordinates of the point A on C at which $\frac{dy}{dx} = 0$ and $x \neq 0$ ---[5]

(ii) Find the value of $\frac{d^2y}{dx^2}$ at A. ---- [3]

$\boxed{W-17/11/Q5}$

12. The curve C has equation $y = e^{-2x}$. Find, giving your answers correct to 3 significant figures,

(i) the mean value of $\frac{dy}{dx}$ over the interval $0 \leq x \leq 2$ ---[2]

(ii) the coordinates of the centroid of the region bounded by C, $x = 0$, $x = 2$ and $y = 0$. $\boxed{W-16/11/Q7}$ ---[9]

13. A curve C has equation $x^2 + 4xy - y^2 + 20 = 0$. Show that, at stationary points on C, $x = -2y$. ---[3]

Find the coordinates of the stationary points on C, and determine their nature by considering the value of $\frac{d^2y}{dx^2}$ at the stationary points. [W-16/11/Q8] ---[8]

14.(i) A curve has equation $x^2-6xy+25y^2=16$, show that $\frac{dy}{dx}=0$ at the point (3,1). ---[4]

(ii) By finding the value of $\frac{d^2y}{dx^2}$ at the point (3,1), determine the nature of this turning point. ---[5]
$\boxed{S-15/11/Q6}$

15 (i) The curve C has equation $x^2+2xy-4y^2+20=0$. Show that if the tangent to C at the point (x,y) is parallel to the x-axis then $x+y=0$ ---[3]

(ii) Hence find the coordinates of the stationary points on C, and determine their nature. $\boxed{S-15/13/Q7}$ ---[7]

16. The curve is defined parametrically by:
$$x=2\cos^3 t \text{ and } y=2\sin^3 t\text{, for } 0<t<\tfrac{1}{2}\pi$$
Show that, at the point with parameter t,
$$\frac{d^2y}{dx^2}=\frac{1}{6}\sec^4 t\operatorname{cosec} t. \quad \text{---[4]}$$
$\boxed{W-15/11/Q1}$

17. Given that a is a constant, prove by mathematical induction that, for every positive integer n,
$$\frac{d^n}{dx^n}(xe^{ax})=na^{n-1}e^{ax}+a^nxe^{ax} \quad \text{---[6]}$$
$\boxed{W-15/11/Q3}$

——— X ——— X ———

(Answers on the next page)

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Differentiation

Answers

1. (a) $y = 2^x \Rightarrow \ln y = x \ln 2$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln 2$

$\Rightarrow \frac{dy}{dx} = y \ln 2 = 2^x \ln 2$ ✓

(b) $\frac{d^2y}{dx^2} = 2^x \cdot (\ln 2)^2$ ✓

(c) $y(0) = 1, \quad y'(0) = \ln 2;$ and $y''(0) = (\ln 2)^2$

Now $2^x = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots$

$\therefore 2^x = 1 + (\ln 2)x + \frac{(\ln 2)^2}{2}x^2 + \cdots$ ✓

2. (i) $\cos y = x$ diff. w.r.t x

$-\sin y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = -(\sin y)^{-1}$ — ①

$\frac{d^2y}{dx^2} = +(\sin y)^{-2} \cdot \cos y \frac{dy}{dx}$

$= -(\sin y)^{-1} \cdot \left(\frac{-\cos y}{\sin y}\right)\frac{dy}{dx}$

from ①

$= -\cot y \cdot \left(\frac{dy}{dx}\right)^2$ — ②

(ii) at $\left(\frac{1}{2}, \frac{\pi}{3}\right)$ from ① $\frac{dy}{dx} = -\left(\sin\frac{\pi}{3}\right)^{-1} = \frac{-2}{\sqrt{3}}$

from ② $\frac{d^2y}{dx^2} = -\cot\frac{\pi}{3} \times \left(-\frac{2}{\sqrt{3}}\right)^2 = \frac{-4}{3\sqrt{3}} = \frac{-4}{9}\sqrt{3}$ ✓

3(i)(a) $y = a^t \Rightarrow \ln y = t \ln a \Rightarrow \frac{1}{y}\frac{dy}{dt} = \ln a$

$\Rightarrow \frac{dy}{dt} = y \ln a = a^t \ln a$ — ①

(i)(b) $\frac{d^2y}{dt} = a^t \cdot (\ln a)^2$ — ②

(ii) $r = \ln a$

$-1 < \ln a < 1 \Rightarrow e^{-1} < a < e$

uses convergence condition for a geometric series.

(continued →)

3(iii) $x = t^a, \quad y = a^t$

$\Rightarrow \frac{dx}{dt} = a t^{a-1}; \quad \frac{dy}{dt} = a^t \ln a$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{a^t \ln a}{a t^{a-1}}$

$\Rightarrow \frac{dy}{dx} = \left(\frac{\ln a}{a}\right) \cdot \left(\frac{a^t}{t^{a-1}}\right)$

diff. w.r.t t

$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{\ln a}{a}\left[\frac{t^{a-1} \cdot a^t \ln a - (a-1)t^{a-2} \cdot a^t}{t^{2(a-1)}}\right]$ — ①

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$

$= \frac{\ln a}{a}\left[\frac{t^{a-1} \cdot a^t \ln a - (a-1)t^{a-2} \cdot a^t}{t^{2(a-1)}}\right] \times \frac{1}{a t^{a-1}}$

$= \frac{\ln a}{a^2} \times a^t \times \frac{t^{a-1}\left[\ln a - (a-1)t^{-1}\right]}{t^{3(a-1)}}$

$= \frac{\ln a}{a^2} \times \left(\frac{\ln a - (a-1)t^{-1}}{t^{2(a-1)}}\right) a^t$ ✓

$\therefore \left(\frac{d^2y}{dx^2}\right)_{t=2} = \frac{\ln a}{a^2}\left[\frac{\ln a - \frac{(a-1)}{2}}{2^{2(a-1)}}\right] \times$

$= 2^{1-2a} \cdot \ln a\,[2\ln a - a + 1]$ ✓

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Answers

4. $y = \ln(ax+1)$

$\frac{dy}{dx} = \frac{a}{(ax+1)} = (-1)^0 \frac{0!\,a^1}{(ax+b)^1}$

To Prove: $\left[\frac{d^n y}{dx^n} = (-1)^{n-1} \cdot \frac{(n-1)!\,a^n}{(ax+1)^n}\right]$

$\therefore$ true for $n=1$

Now let the statement is true for $n=k$.

$\Rightarrow \frac{d^k y}{dx^k} = (-1)^{k-1} \cdot \frac{(k-1)!\,a^k}{(ax+1)^k}$ — (1)

$\Rightarrow \frac{d^{k+1} y}{dx^{k+1}} = -ka(-1)^{k-1} \cdot \frac{(k-1)!\,a^k}{(ax+1)^{k+1}}$

$= -(-1)^k \cdot \frac{k!\,a^{k+1}}{(ax+1)^{k+1}}$

So true for $n = k+1$

$\therefore$ True for every positive integer by induction.

6(i) $x = 18t - t^2 \Rightarrow \frac{dx}{dt} = (18-2t)$

$y = 8t^{3/2} \Rightarrow \frac{dy}{dt} = 12t^{\frac{1}{2}}$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{12t^{1/2}}{(18-2t)}$

or $\frac{dy}{dx} = \frac{6t^{1/2}}{(9-t)}$ — (1)

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$

$= \frac{(9-t)\cdot 3t^{-1/2} + 6t^{1/2}}{(9-t)^2\,(18-2t)}$

$= \frac{3t^{-1/2}[9-t+2t]}{2(9-t)^3} = \frac{3(9+t)}{2t^{1/2}(9-t)^3}$ ✓

(ii) Mean Value $= \frac{1}{56}\int_0^{56} \frac{d^2y}{dx^2}\,dx = \frac{1}{56}\left[\frac{6t^{1/2}}{9-t}\right]_{t=0}^{t=4}$

$= \frac{1}{56}\left[\frac{6\sqrt{4}}{9-4}\right] = \frac{3}{70}$ ✓

5. To Prove: Given $y = e^x \cdot u$, $\frac{d^2y}{dx^2} = y^{(2)}$, $\frac{du}{dx^2} = u^{(2)}$

$H_n: y^{(n)} = e^x\left[\binom{n}{0}u + \binom{n}{1}u^{(1)} + \binom{n}{2}u^{(2)} + \binom{n}{r}u^{(r)} + \cdots \binom{n}{n}u^{(n)}\right]$ — (1)

Proof: $y' = e^x u^{(1)} + ue^x = e^x\left[\binom{1}{0}u + \binom{1}{1}u^{(1)}\right] \Rightarrow H_1$ is True

Now assume that

$H_k: y^{(k)} = e^x\left[\binom{k}{0}u + \binom{k}{1}u^{(1)} + \cdots + \binom{k}{r}u^{(r)} + \cdots \binom{k}{k}u^{(k)}\right]$ is True ✓ — (2)

diff (2) $y^{(k+1)} = e^x\left[\binom{k}{0}u + \binom{k}{1}u^{(1)} + \cdots + \binom{k}{r}u^{(r)} + \cdots \binom{k}{k}u^{(k)}\right]$
$+ e^x\left[\binom{k}{0}u^{(1)} + \binom{k}{1}u^{(2)} + \cdots + \binom{k}{r}u^{(r+1)} + \cdots \binom{k}{k}u^{(k+1)}\right]$.

$= e^x\left[\binom{k}{0}u + \cdots \left(\binom{k}{r} + \binom{k}{r-1}\right)u^{(r)} + \cdots \binom{k}{k}u^{(k+1)}\right]$

$= e^x\left[\binom{k+1}{0}u + \cdots \binom{k+1}{r}u^{(r)} + \cdots \binom{k+1}{k+1}u^{(k+1)}\right]$

$\therefore H_k$ implies H_{k+1} is True. So by induction H_n is True for all $(n \geq 1)$

6(iii) $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (18-2t)^2 + 144t$
$= 4(t+9)^2$

$2\pi\int_0^4 8t^{3/2}(2(t+9))\,dt$

$= 32\pi\int_0^4 (t^{5/2} + 9t^{3/2})\,dt$

$= 32\pi\left[\frac{2}{7}t^{7/2} + \frac{18}{5}t^{5/2}\right]_0^4$

$= \frac{2^{11}\times 83}{35}\pi = \frac{169984}{35}\pi = \underline{15300}$ ✓

Answers

7. (i) $\tan y = x \Rightarrow \sec^2 y \frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \cos^2 y$

$\Rightarrow \frac{d^2y}{dx^2} = 2\cos y(-\sin y)\frac{dy}{dx}$

$= -2 \cdot \tan y \cdot \cos^2 y \frac{dy}{dx}$

or $\frac{d^2y}{dx^2} = -2x\left(\frac{dy}{dx}\right)^2$ ✓

(ii) $\left(\frac{d^2y}{dx^2}\right)_{(1,\frac{\pi}{4})} = -2\times1\times\left(\frac{1}{2}\right)^2$ $\left(\frac{dy}{dx}\right)_{(1,\frac{\pi}{4})} = \frac{1}{2}$

$= -\frac{1}{2}$ ✓

9. $x^3 - 3xy + y^2 = 4$

diff. w.r.t x

$3x^2 - \left(3y + 3x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$ — (1)

At $(0,2)$, $-6 + 4\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{3}{2}$ — (2)

Now diff (1)

$6x - 3\frac{dy}{dx} - \left(3\frac{dy}{dx} + 3x\frac{d^2y}{dx^2}\right) + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 0$

At $(0,2)$ $-\frac{9}{2} - \frac{9}{2} + \frac{9}{2} + 4\frac{d^2y}{dx^2} = 0$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{9}{8}$ ✓

8 (i) $x = t^{1/2}$ and $y = f(x)$

$\frac{dx}{dt} = \frac{1}{2}t^{-1/2} \Rightarrow \frac{dt}{dx} = 2t^{1/2}$

$\therefore \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = 2t^{1/2}\cdot\frac{dy}{dt}$ ✓ — (1)

Now $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\times\frac{dt}{dx}$

$= \left\{t^{-1/2}\frac{dy}{dt} + 2t^{1/2}\cdot\frac{d^2y}{dt^2}\right\}\times 2t^{1/2}$

or $\frac{d^2y}{dx^2} = 2\frac{dy}{dt} + 4\frac{d^2y}{dt^2}$ ✓ — (2)

(ii) $\frac{d^2y}{dx^2} - \left(8x + \frac{1}{x}\right)\frac{dy}{dx} + 12x^2y = 4x^2e^{-x^2}$ (*)

from (1) & (2) & (*)

$2\frac{dy}{dt} + 4\frac{d^2y}{dt^2} - \left(8t^{1/2} + t^{-\frac{1}{2}}\right)\times 2t^{1/2}\times\frac{dy}{dt} + 12ty = 4te^{-t}$

or $\cancel{2\frac{dy}{dt}} + 4\frac{d^2y}{dt} - 16t\frac{dy}{dt} - \cancel{2\frac{dy}{dt}} + 12ty = 4te^{-t}$

$\Rightarrow \frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = e^{-t}$ ✓

(iii) CF: $(m-1)(m-3) = 0$

$y = Ae^t + Be^{3t}$

PI: $y = ke^{-t} \Rightarrow y' = -ke^{-t}$

$\Rightarrow y'' = ke^{-t}$

$\Rightarrow ke^{-t} + 4ke^{-t} + 3ke^{-t} = e^{-t}$

$\Rightarrow k = 1/8$

$\therefore$ GS, $y = Ae^t + Be^{3t} + \frac{1}{8}e^{-t}$

$\Rightarrow y = Ae^{x^2} + Be^{3x^2} + \frac{1}{8}e^{-8}$ ✓

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Answers

10. $\dfrac{d^{n+1}}{dx^{n+1}}(x^{n+1}\ln x)$

(i) $= \dfrac{d^n}{dx^n}\left[x^{n+1}\cdot\dfrac{1}{x} + (n+1)x^n\ln x\right]$

$= \dfrac{d^n}{dx^n}\left[x^n + (n+1)x^n\ln x\right]$ ✓

(ii) Let H_k is True, i.e

$\dfrac{d^k}{dx^k}(x^k\ln x) = k!\left(\ln x + 1 + \dfrac{1}{2} \cdots + \dfrac{1}{n}\right)$

Now $\dfrac{d^{k+1}}{dx^{k+1}}(x^{k+1}\ln x) = \dfrac{d^k}{dx^k}\left(x^k + (n+1)x^k\ln x\right)$

$= k! + (k+1)\cdot k!\left(\ln x + 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n}\right)$

$= (k+1)!\left[\ln x + 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{k+1}\right]$

$\Rightarrow H_{k+1}$ is True ✓

for H_1: $\dfrac{d}{dx}(x\ln x) = 1\times\ln x + x\times\dfrac{1}{x}$

$= 1!(\ln x + 1)$ is True ✓

$\therefore$ H_1 is True and H_k True $\Rightarrow H_{k+1}$ is True.

$\therefore$ Using PMI, H_n is True for all positive integers n.

12(i) $M.V = \dfrac{\left[e^{-2x}\right]_0^2}{2-0} = \dfrac{e^{-4}-1}{2} = -0.491$ ✓

(ii) Coordinates of Centroid.

$\bar{x} = \dfrac{\int_0^2 xe^{-2x}}{\int_0^2 e^{-2x}dx}$ and $\bar{y} = \dfrac{1}{2}\dfrac{\int_0^2 e^{-4x}dx}{\int_0^2 e^{-2x}dx}$

Now $\int_0^2 e^{-2x}dx = \left[-\dfrac{1}{2}e^{-2x}\right]_0^2 = \dfrac{1}{2}(1-e^{-4})$

$= 0.4908$ ✓

$\int_0^2 xe^{-2x}dx = \left[-x\dfrac{e^{-2x}}{2}\right]_0^2 + \int_0^2 \dfrac{e^{-2x}}{2}dx$

$= \left[-\dfrac{xe^{-2x}}{2} - \dfrac{e^{-2x}}{4}\right]_0^4$

$= -\dfrac{5e^{-4}}{4} + \dfrac{1}{4} = 0.2271\ldots$ ✓

and

$\dfrac{1}{2}\int_0^2 e^{-4x}dx = \dfrac{1}{2}\left[\dfrac{-e^{-4x}}{4}\right]_0^2 = \dfrac{1}{2}\left[\dfrac{1}{4} - \dfrac{e^{-8}}{4}\right]$

$= 0.12495$ ✓

$\therefore \bar{x} = 0.463$ and $\bar{y} = 0.255$ ✓

11(i) $2x^3 + 3x^2y - 3y^3 - 16 = 0$ — ①

diff. w.r.t x

$6x^2 + 6xy + 3x^2y' - 9y^2\cdot y' = 0$ — ②

$\Rightarrow 2x(x+y) = (3y^2 - x^2)y'$

Now $y' = 0$ and $x \neq 0$

$\Rightarrow x + y = 0 \Rightarrow x = -y$

fr ① $2x^3 - 3x^3 + 3x^3 = 16 \Rightarrow x = 2$

$\therefore y = -2$

$\therefore$ Req. Point $A(2,-2)$ ✓

(ii) diff ②

$12x + 6xy' + 6y + 6xy' + 3x^2y'' - [18y(y')^2 + 9y^2y''] = 0$ — ③

Now at $A(2,-2)$, $x=2, y=-2, y'=0$

fr ③

$8 - 4 + 4y'' - 12y'' = 0$

$\Rightarrow y'' = \dfrac{1}{2}$ ✓

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Answers

13. $x^2+4xy-y^2+20=0$ — ①

Diff. w.r.t x

$2x+4(xy'+y)-2yy'=0$ — ②

$y'=0 \Rightarrow 2x+4y=0$

at stationary Pt, $\Rightarrow x=-2y$ — ○

for ① & ○ $4y^2-8y^2-y^2+20=0$

$\Rightarrow 5y^2=20 \Rightarrow y=\pm 2$

∴ Coordinates of stationary points $(4,-2)$ and $(-4,2)$.

Now Diff ②

$1+2(xy''+y'+y')-[yy''+(y')^2]=0$

at $(4,-2)$ with $y'=0$

$1+8y''+2y''=0$

$\Rightarrow y''=-\frac{1}{10} \Rightarrow$ Max.

Again at $(-4,2)$ and $y'=0$

$1-8y''-2y''=0 \Rightarrow y''=\frac{1}{10}$

$\Rightarrow$ Min.

14(i) $x^2-6xy+25y^2=16$

Diff w.r.t x

$2x-6[x\frac{dy}{dx}+y]+50y\frac{dy}{dx}=0$ — ①

Now at $(3,1) \Rightarrow 6-6[3\frac{dy}{dx}+1]+50\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=0$

Diff ①

$2-6(xy''+y'+y')+50[(y')^2+yy'']=0$

at $(3,1)$, $2-18y''+50y''=0$

$\Rightarrow y''=-\frac{1}{16}$

∴ Max at $(3,1)$

15 (i) $x^2+2xy-4y^2+20=0$ — ①

Diff w.r.t x

$2x+2[x\frac{dy}{dx}+y]-8y\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}(x-4y)=-(x+y)$ — ②

$\Rightarrow \frac{dy}{dx}=\frac{-(x+y)}{(x-4y)}=0$

$\Rightarrow (x+y=0)$

(ii) $\Rightarrow y=-x$

Put $y=-x$ in ①

$x^2-2x^2-4x^2+20=0$

$\Rightarrow 5x^2=20 \Rightarrow x=\pm 2$

∴ Coordinates are $(2,-2)$ and $(-2,2)$

Diff ②

$\frac{d^2y}{dx^2}(x-4y)+\frac{dy}{dx}\left(1-4\frac{dy}{dx}\right)=-1-\frac{dy}{dx}$ — ③

Now at $(2,-2)$ and $\frac{dy}{dx}=0$

for ③ $y''(2)=-\frac{1}{10} \Rightarrow$ Max

and at $(-2,2)$, $y''(-2)=\frac{1}{10} \Rightarrow$ Min

16. $x=2\cos^3 t$ and $y=2\sin^3 t$

$\frac{dx}{dt}=-6\cos^2 t\,\sin t$,

and $\frac{dy}{dt}=6\sin^2 t\cos t$

$\Rightarrow \frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=-\tan t$

Now $\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{dy}{dx}\right)\times\frac{dt}{dx}$

$=\frac{d}{dt}(-\tan t)\times\frac{-1}{6\cos^2 t\sin t}$

$=-\sec^2 t\times\frac{-1}{6\cos^2 t\sin t}$

$=\frac{1}{6}\cdot\sec^4 t\cdot\operatorname{cosec} t$

17. Let us denote $H_n : \frac{d^n}{dx^n}(xe^{ax}) = n a^{n-1} e^{ax} + a^n x e^{ax}$ — ①

from ①

for $n=1$, $\frac{d}{dx}(xe^{ax}) = a^0 e^{ax} + x a e^{ax} = e^{ax} + axe^{ax}$ — ②

Diff

Now $\frac{d}{dx}(xe^{ax}) = e^{ax} \times 1 + x \cdot a e^{ax} = e^{ax} + axe^{ax}$ — ③

from ② & ③ H_1 is True ✓

Now let H_k is True $\Rightarrow \frac{d^k}{dx^k}(xe^{ax}) = k a^{k-1} e^{ax} + a^k x e^{ax}$ — ④

diff ④ $\frac{d^{k+1}}{dx^{k+1}}(xe^{ax}) = k a^k e^{ax} + a^k e^{ax} + a^{k+1} x e^{ax}$

$= (k+1) a^k e^{ax} + a^{k+1} x e^{ax}$

$\Rightarrow$ H_{k+1} is True whenever H_k is True

$\therefore$ H_n is true using PMI for all positive integers n.

← × — × →