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FP2

Further Pure Maths 2

Integration Exercise

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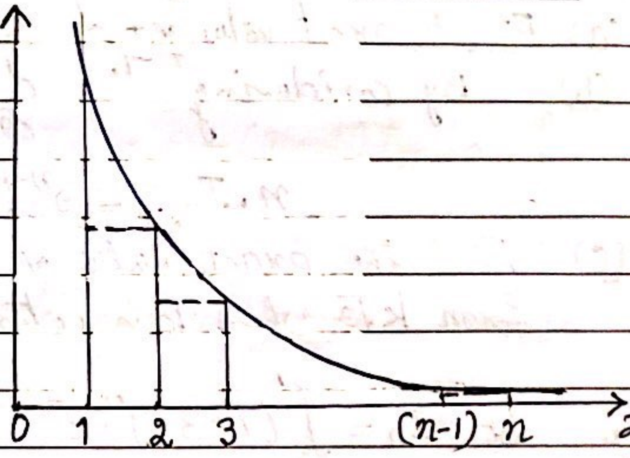
Integration

1. Find the exact value of $\int_0^1 \frac{1}{\sqrt{3+4x-4x^2}}\,dx$ [SP-20/02/Q2] --[6]

2. The diagram shows the curve with equation $y=\frac{1}{x^2}$ for $x>0$, together with a set of $(n-1)$ rectangles of unit width.



(a) By considering the sum of the areas of these rectangles, show that,
$$\sum_{r=1}^{n}\frac{1}{r^2}<\frac{2n-1}{n}.$$ --[5]

(b) Use a similar method to find, in terms of n, a lower bound for $\sum_{r=1}^{n}\frac{1}{r^2}$ [SP-20/02/Q4] --[3]

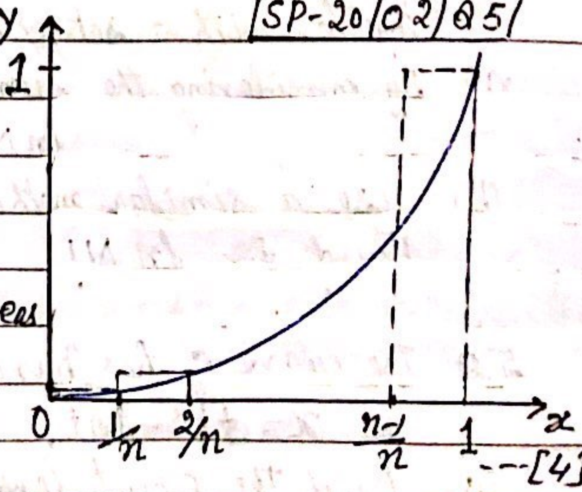
3. The curve C has parametric equations.
$$x=e^t-4t+3,\quad y=8e^{\frac{1}{2}t},\quad \text{for } 0\leq t\leq 2$$

(a) Find in terms of e, the length of C. --[5]

(b) Find, in terms of π and e, the area of the surface generated when C is rotated through 2π radians about x-axis. --[5]

[SP-20/02/Q5]

4. The diagram shows the curve with equation $y=x^2$ for $0\leq x\leq 1$, together with a set of n rectangles of width $\frac{1}{n}$.



(a) By considering the sum of the areas of these rectangles, show that
$$\int_0^1 x^2\,dx<\frac{2n^2+3x+1}{6n^2}$$ ---[4]

(b) Use a similar method to find, in terms of n, a lower bound for $\int_0^1 x^2\,dx$. ---[4]

[S-20/21/Q4]

5. The integral I_n, where n is an integer, is defined by,
$$I_n = \int_0^{\frac{1}{2}} (1-x^2)^{-\frac{1}{2}n}\,dx$$
(a) Find the exact value of I_1. --[2]
(b) By considering $\frac{d}{dx}\left(x(1-x^2)^{-\frac{1}{2}n}\right)$, or otherwise, show that
$$n.I_{n+2} = 2^{n-1}.3^{-\frac{1}{2}n} + (n-1)I_n$$ --[5]
(c) Find the exact value of I_5 giving your answer in the form $k\sqrt{3}$, where k is rational number to be determined. --[3]

$\boxed{S\text{-}20/21/Q6}$

6. Let $I_n = \int_0^1 (1+3x)^n \cdot e^{-3x}\,dx$, where n is an integer.
(a) Show that $3I_n = 1 - 4^n e^{-3} + 3nI_{n-1}$ ---[3]
(b) Find the exact value of I_2 --[3]

$\boxed{S\text{-}20/23/Q2}$

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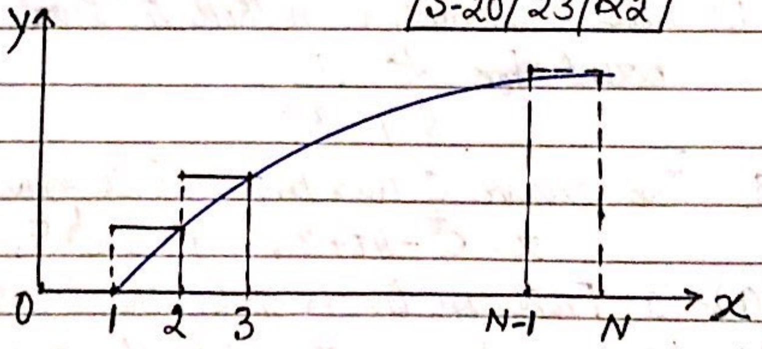


The diagram shows the curve with equation $y = \ln x$ for $x \geq 1$, together with a set of $(N-1)$ rectangles of unit width.
(a) By considering the sum of areas of these rectangles, show that
$$\ln N! > N\ln N - N + 1,$$ ---[5]
(b) Use a similar method to find, in terms of N, an upper bound for $\ln N!$ $\boxed{S\text{-}20/23/Q4}$ --[3]

8. The curve C has parametric equations,
$$x = \tfrac{1}{2}t^2 - \ln t, \quad y = 2t+1, \quad \text{for } \tfrac{1}{2} \leq t \leq 2$$
(a) Find the exact length of C. ---[5]
(b) Find $\frac{d^2y}{dx^2}$ in terms of t, simplify your answer. --[4]

$\boxed{S\text{-}20/23/Q5}$

9. It is given that, for $n \geq 0$, $I_n = \int_0^1 x^n e^{x^3} dx$

(i) Show that $I_2 = \frac{1}{3}(e-1)$ ---[2]

(ii) Show that, for $n \geq 3$, $3I_n = e-(n-2)I_{n-3}$ ---[3]

(iii) Hence find the exact value of I_8 ---[3]

[S-19/11/Q4]

10. A curve C is defined parametrically by
$$x = \frac{2}{e^t + e^{-t}} \text{ and } y = \frac{e^t - e^{-t}}{e^t + e^{-t}},$$
for $0 \leq t \leq 1$. The area of the surface generated when C is rotated through 2π radians about the x-axis is denoted by S.

(i) Show that $S = 4\pi \int_0^1 \frac{e^t - e^{-t}}{(e^t + e^{-t})^2} dt$ ---[5]

(ii) Using the substitution $u = e^t + e^{-t}$, or otherwise, find S in terms of π and e. [S-19/11/Q5] ---[3]

11. Let $I_n = \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot^n x \, dx$, where $n \geq 0$

(i) By considering $\frac{d}{dx}(\cot^{n+1} x)$, or otherwise, show that
$$I_{n+2} = \frac{1}{n+1} - I_n$$ ---[5]

The curve C has equation $y = \cot x$, for $\frac{1}{4}\pi \leq x \leq \frac{1}{2}\pi$

(ii) Find, in exact form, the y-coordinate of the centroid of the region enclosed by C, the line $x = \frac{1}{4}\pi$ and the x-axis. --[6]

[S-19/13/Q10]

12. The integral I_n, where n is a positive integer, is defined by
$$I_n = \int_{\frac{1}{2}}^1 x^{-n} \cdot \sin \pi x \, dx$$

(i) Show that $n(n+1) I_{n+2} = 2^{n+1} n + \pi - \pi^2 I_n$ ---[5]

(ii) Find I_5 in terms of π and I_1. [W-19/11/Q3] ---[2]

13. The curve C is defined parametrically by:

$$x = e^t - t \;; \quad y = 4e^{\frac{1}{2}t}$$

Find the length of the arc of C from the point where $t=0$ to the point where $t=3$. [S-18/11/Q1] --[5]

14. (i) Using the substitution $u = \tan x$, or otherwise, find:

$$\int \sec^2 x \tan^2 x \, dx$$ --[2]

It is given that, for $n \geq 0$, $I_n = \int_0^{\frac{1}{4}\pi} \sec^n x \tan^2 x \, dx$

(ii) Using the result that $\frac{d}{dx}(\sec x) = \tan x \cdot \sec x$, show that for $n \geq 2$,

$$(n+1) I_n = (\sqrt{2})^{n-2} + (n-2) I_{n-2}$$ ---[5]

(iii) Hence find the mean value of $\sec^4 x \tan^2 x$ with respect to x over the interval $0 \leq x \leq \frac{1}{4}\pi$, giving your answer in exact form. [S-18/11/Q9] ---[3]

15(i) Show that: $\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^x \cos x \, dx = \frac{1}{2}\left(e^{\frac{1}{2}\pi} + e^{-\frac{1}{2}\pi}\right)$ --[4]

(ii) It is given that, for $n \geq 0$, $I_n = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{2x} \cos^n x \, dx$

Show that, for $n \geq 2$,

$$4I_n = n(n-1)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{2x} \sin^2 x \cos^{n-2} x \, dx - nI_n$$

and deduce the reduction formula,

$$(n^2+4) I_n = n(n-1) I_{n-2}$$ ---[6]

(iii) Using the result in part (i) and the reduction formula in part (ii), find the y-coordinate of the centroid of the region bounded by the x-axis and the arc of the curve $y = e^x \cos x$ from $x = -\frac{1}{2}\pi$ to $x = \frac{1}{2}\pi$. Give your answer correct to 3 s.f. --[4]

[S-18/13/Q11(i)]

Integration

16. A curve is defined parametrically by

$$x = t - \frac{1}{2}\sin 2t \text{ and } y = \sin^2 t$$

The arc of the curve joining the point where $t=0$ to the point where $t=\pi$ is rotated through one complete revolution about the x-axis. The area of the surface generated is denoted by S.

(i) Show that $S = a\pi\int_0^{\pi} \sin^3 t\,dt$ where the constant a is to be found. --[5]

(ii) Using the result $\sin 3t = 3\sin t - 4\sin^3 t$, find the exact value of S. --[3]

W-18/11/Q4

17. The curve C has equation $x^2 + 2xy = y^3 - 2$

(i) Show that $A(-1, 1)$ is the only point on C with x-coordinate equal to -1. --[2]

for $n \geq 1$, let A_n denote the value of $\frac{d^n y}{dx^n}$ at the point $A(-1,1)$

(ii) Show that $A_1 = 0$ --[3]

(iii) Show that $A_2 = \frac{2}{5}$ --[3]

Let $I_n = \int_{-1}^{0} x^n \frac{d^n y}{dx^n}\,dx$

(iv) Show that for $n \geq 2$,

$$I_n = (-1)^{n+1} A_{n+1} - n I_{n-1}$$ --[3]

(v) Deduce the value of I_3 in terms of I_1 --[2]

W-18/11/Q11(ii)

18. Let $I_n = \int_1^{\sqrt{2}} (x^2-1)^n\,dx$

(i) Show that, for $n \geq 1$, $(2n+1)I_n = \sqrt{2} - 2n I_{n-1}$ --[5]

(ii) Using the substitution $x = \sec\theta$, show that

$$I_n = \int_0^{\frac{1}{4}\pi} \tan^{2n+1}\theta \sec\theta\,d\theta$$ --[4]

(iii) Deduce the exact value of: $\int_0^{\frac{1}{4}\pi} \frac{\sin^7\theta}{\cos^8\theta}\,d\theta$ --[5]

W-18/12/11(ii)

(FP-2)

19. Let $I_n = \int_0^{\frac{1}{2}\pi} x^n \sin x \, dx$.

(i) Prove that, for $n \geq 2$, $I_n + n(n-1)I_{n-2} = n\left(\frac{1}{2}\pi\right)^{n-1}$ --- [4]

(ii) Calculate the exact value of I_1 and deduce the exact value of I_3. [S-17/11/Q6] --- [3]

20. The curve C has polar equation $r = a(1+\sin\theta)$ for $-\pi < \theta \leq \pi$, where a is a positive constant.

(i) Sketch C. --- [2]

(ii) Find the area of the region enclosed by C. --- [4]

(iii) Show that the length of the arc of C from the pole to the point furthest from the pole is given by --- [3]

$$s = (\sqrt{2})a \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sqrt{(1+\sin\theta)}\, d\theta$$

(iv) Show that the substitution $u = 1+\sin\theta$ reduces this integral for s to $(\sqrt{2})a \int_0^2 \frac{1}{\sqrt{2-u}}\, du$

Hence evaluate s. [S-17/11/Q11] --- [4]

21. The curve C has equation $y = \frac{1}{2}(e^x + e^{-x})$ for $0 \leq x \leq 4$

(i) The region R is bounded by C, the x-axis, the y-axis and the line $x=4$. Find, in terms of e, the coordinates of the centroid of the region R. --- [10]

(ii) Show that $\frac{ds}{dx} = \frac{1}{2}(e^x + e^{-x})$, where s denotes the arc length of C, and find surface area generated when C is rotated through 2π radians about the x-axis. --- [4]

[S-17/11/Q12]

22. A curve C has parametric equations

$$x = \frac{2}{5}t^{5/2} - 2t^{1/2}, \quad y = \frac{4}{3}t^{3/2}, \quad \text{for } 1 \leq t \leq 4.$$

(i) Find the exact value of the arc length of C. --- [5]

(ii) Find also the exact value of the surface area generated when C is rotated through 2π radians about the x-axis. --- [3]

[S-17/13/Q5]

23. Let I_n denote $\int_0^2 (4+x^2)^{-n}\,dx$

(i) Find $\frac{d}{dx}\left(x(4+x^2)^{-n}\right)$ and hence show that.

$$8n I_{n+1} = (2n-1)I_n + 2\times 8^{-n} \quad \text{---[5]}$$

(ii) Use the result for integrating $\frac{1}{x^2+a^2}$ with respect to x, to find the value of I_1 and deduce that $I_3 = \frac{3}{1024}\pi + \frac{1}{128}$

[S-17/13/Q6]

24. A curve C has polar equation $r = 2a\cos(2\theta + \frac{1}{2}\pi)$ for $0 \le \theta \le 2\pi$, where a is a positive constant.

(i) Show that $r = -2a\sin 2\theta$ and sketch C. ---[4]

(ii) Deduce that the cartesian equation of C is,

$$(x^2+y^2)^{3/2} = -4axy \quad \text{---[2]}$$

(iii) Find the area of one loop of C. ---[5]

(iv) Show that at the points (other than the pole) at which a tangent to C is parallel to the initial line,

$$2\tan\theta = -\tan 2\theta \quad \text{---[3]}$$

[S-17/13/Q11]

25. Let $I_n = \int_0^{\frac{1}{4}\pi} \sec^n x\,dx$ for $n > 0$

(i) Find the value of I_2 ---[2]

(ii) Show that, for $n > 2$,

$$(n-1)I_n = 2^{\frac{1}{2}n-1} + (n-2)I_{n-2} \quad \text{---[5]}$$

(iii) The curve C has equation $y = \sec^3 x$ for $0 \le x \le \frac{1}{4}\pi$. The region R is bounded by C, the x-axis, the y-axis and the line $x = \frac{1}{4}\pi$. Find the volume of revolution generated when R is rotated through 2π radians about the x-axis. ---[4]

[W-17/11/Q8]

26. The polar equation of a curve C is $r = a(1+\cos\theta)$ for $0 \le \theta < 2\pi$, where a is positive constant.

(i) Sketch C. ---[2]

(continued→)

26(ii) Show that the cartesian equation of C is
$$x^2+y^2 = a\left(x+\sqrt{(x^2+y^2)}\right) \quad \text{---[2]}$$
(iii) Find the area of the sector of C between $\theta=0$ and $\theta=\frac{1}{3}\pi$ ---[4]
(iv) Find the arc length of C between the point where $\theta=0$ and the point where $\theta=\frac{1}{3}\pi$ [W-17/11/Q11] --[5]

27. A curve C has polar equation $r^2 = 8\operatorname{cosec}2\theta$ for $0<\theta<\frac{1}{2}\pi$.
(i) Find a cartesian equation of C. --[3]
(ii) Sketch C --[2]
(iii) Determine the exact area of the sector bounded by the arc of C between $\theta=\frac{1}{6}\pi$ and $\theta=\frac{1}{3}\pi$, the half-line $\theta=\frac{1}{6}\pi$ and the half-line $\theta=\frac{1}{3}\pi$ --[3]
[It is given that $\int \operatorname{cosec} x\,dx = \ln|\tan\frac{1}{2}x|+c$] [S-16/11/Q4]

28. Let $I_n = \int_0^{\frac{1}{2}\pi} \cos^n x \sin^2 x\,dx$, for $n\geq 0$. By differentiating $\cos^{n-1}x \cdot \sin^3 x$ with respect to x, prove that
(i) $(n+2)I_n = (n-1)I_{n-2}$ for $n\geq 2$ ---[5]
(ii) Hence find the exact value of I_4. [S-16/11/Q5] ---[4]

29. A curve C has parametric equations
$$x = e^{2t}\cos 2t,\quad y = e^{2t}\sin 2t,\quad \text{for } -\tfrac{1}{2}\pi \leq t \leq \tfrac{1}{2}\pi$$
(i) Find the arc length of C. --[6]
(ii) Find the area of the surface generated when C is rotated through 2π radians about the x-axis. ---[8]
[S-16/11/Q11]

30. (i) The curve C has equation $y=-\ln(1-x^2)$ for $-\frac{1}{2}\leq x\leq \frac{1}{2}$.
Show that $1+\left(\frac{dy}{dx}\right)^2 = \left(\frac{1+x^2}{1-x^2}\right)^2$ --[2]
(ii) Show further that $\frac{1+x^2}{1-x^2}$ may be expressed in the form $\frac{P}{1+x}+\frac{Q}{1-x}+R$, where P, Q and R are the constants to be determined. --[2]
(iii) Find the exact arc length of C. [S-16/13/Q4] --[4]

31. Let $I_n = \int_0^2 x^n (4-x^2)^{\frac{1}{2}} dx$, for $n \geq 1$, By Considering $\frac{d}{dx}\{x^n(4-x^2)^{3/2}\}$

(i) Show that,

$$(n+3) I_{n+1} = 4n I_{n-1} \text{ where } n \geq 2$$ --[4]

(ii) Find the value of I_1 and deduce the exact value of I_3 --[4]

[S-16/13/Q6]

32. A curve has polar equation $r = \frac{1}{1-\cos\theta}$, for $0 < \theta < 2\pi$.

(i) Find in the form $y^2 = f(x)$, the cartesian equation of the curve. --[3]

(ii) Hence sketch the curve, and shade the region whose area is given by $\frac{1}{2}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{1}{(1-\cos\theta)^2} d\theta$ ---[3]

(iii) Using the cartesian equation of the curve, find the area of this region. [S-16/13/Q7] --[3]

33(i) Given that y is a function of x and that $x = e^u$, show that

$$x\frac{dy}{dx} = \frac{dy}{du} \text{ and } x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$$ --- [3]

(ii) Given also that, $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 17y = 34\ln x + 21$,

deduce that $\frac{d^2y}{du^2} + 2\frac{dy}{du} + 17y = 34u + 21$ ---[1]

(iii) Find y in terms of x given that $y = 0$ and $\frac{dy}{dx} = -1$ when $x = 1$. ---[9]

[S-16/13/Q10]

34(i) Evaluate $\int_0^{\frac{1}{2}\pi} x \sin x \, dx$ --- [2]

(ii) Given that $I_n = \int_0^{\frac{1}{2}\pi} x^n \sin x \, dx$, prove that, for $n > 1$,

$$I_n = n\left(\tfrac{1}{2}\pi\right)^{n-1} - n(n-1) I_{n-2}$$ ---[4]

(iii) By first using the substitution $x = \cos^{-1} u$, find the value of

$$\int_0^1 (\cos^{-1} u)^3 \, du$$ ---[5]

giving your answer in an exact form.

[W-16/11/Q9]

35. A curve has parametric equations.

(a) $x = 1 - 3t^2$, $y = t(1-3t^2)$, for $0 \leq t \leq \frac{1}{\sqrt{3}}$

Show that $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1+9t^2)^2$ --[2]

(b) Hence find: (i) the arc length of C. --[2]

(ii) the surface area generated when C is rotated through 2π radians about the x-axis. --[3]

(c) Use the fact that $t = y/x$ to find a cartesian equation of C. Hence show that the polar equation of C is $r = \sec\theta(1 - 3\tan^2\theta)$, and state the domain of θ. --[4]

Find the area of the region enclosed between C and the initial line. [W-16/11/Q11] --[3]

36. The curves C_1 and C_2 have polar equations.

$C_1 : r = \frac{1}{\sqrt{2}}$, for $0 \leq \theta < 2\pi$

$C_2 : r = \sqrt{(\sin\frac{1}{2}\theta)}$, for $0 \leq \theta \leq \pi$

(i) Find the polar coordinates of the point of intersection of C_1 and C_2. ---[2]

(ii) Sketch C_1 and C_2 on the same diagram. --[3]

(iii) Find the exact value of the area of the region enclosed by C_1, C_2 and the half-line $\theta = 0$. [S-15/11/Q5] --[4]

37 (i) Let $I_n = \int_0^{\frac{1}{2}\pi} x^n \sin x \, dx$, where n is a non-negative integer. Show that, $I_n = n(\frac{1}{2}\pi)^{n-1} - n(n-1)I_{n-2}$, for $n \geq 2$ -- [5]

(ii) Find the exact value of I_4. [S-15/11/Q7] --[4]

38. The curve C has parametric equations,

$x = 4t + 2t^{3/2}$, $y = 4t - 2t^{3/2}$, for $0 \leq t \leq 4$,

(i) Find the arc length of C, giving your answer correct to 3 s.f. --[6]

(ii) Find the mean value of y with respect to x over the interval $0 \leq x \leq 32$, --[5]

[S-15/11/Q9]

39. The curve C has polar equation $r = e^{4\theta}$ for $0 \leq \theta \leq \alpha$, where α is measured in radians. The length of C is 2015. Find the value of α. [S-15/13/Q2] --- [6]

40. Let $I_n = \int_0^{\frac{1}{2}\pi} \frac{\sin 2n\theta}{\cos\theta}\, d\theta$, where n is a non-negative integer.

(i) Use the identity $\sin P + \sin Q = 2\sin\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$ to show that $I_n + I_{n-1} = \frac{2}{2n-1}$, for all positive integer n. --- [5]

(ii) Find the exact value of $\int_0^{\frac{1}{2}\pi} \frac{\sin 8\theta}{\cos\theta}\, d\theta$ --- [4] [S-15/13/Q5]

41. (i) It is given that $I_n = \int_1^e (\ln x)^n\, dx$ for $n \geq 0$ show that $I_n = (n-1)[I_{n-2} - I_{n-1}]$ for $n \geq 2$ --- [6]

(ii) Hence find, in a exact form, the mean value of $(\ln x)^3$ with respect to x over the interval $1 \leq x \leq e$ [W-15/11/Q9] --- [6]

42. (i) The curve C has polar equation, $r = a(1-\cos\theta)$ for $0 \leq \theta < 2\pi$. Sketch C. --- [2]

(ii) Find the area of the region enclosed by the arc of C for which $\frac{1}{2}\pi \leq \theta \leq \frac{3}{2}\pi$, the half line $\theta = \frac{1}{2}\pi$ and the half line $\theta = \frac{3}{2}\pi$. --- [5]

(iii) Show that, $\left(\frac{ds}{d\theta}\right)^2 = 4a^2 \sin^2\left(\frac{1}{2}\theta\right)$

where s denotes arc length, and find the length of the arc of C for which $\frac{1}{2}\pi \leq \theta \leq \frac{3}{2}\pi$ --- [7] [W-15/11/Q11]

← × — × →

Answers

1. $\int_0^1 \frac{1}{\sqrt{(3+4x-4x^2)}}dx$

$= \frac{1}{2}\int_0^1 \frac{1}{\sqrt{(1-(x-\frac{1}{2})^2}}dx$

$= \frac{1}{2}\left[\sin^{-1}\left(x-\frac{1}{2}\right)\right]_0^1$ $\quad \left[\because \int \frac{1}{\sqrt{a^2-x^2}}dx = \frac{1}{a}\sin^{-1}\frac{x}{a}\right.$

$= \frac{1}{2}\left[\sin^{-1}\frac{1}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$

$= \frac{1}{2}\left(\frac{\pi}{6}+\frac{\pi}{6}\right] = \frac{\pi}{6}$ ✓

2. (a) Sum of Areas of rectangles

$= \frac{1}{2^2}+\frac{1}{3^2}+\cdots\cdots+\frac{1}{n^2}$

$< \int_1^n \frac{1}{x^2}dx = 1-\frac{1}{n}$

$\sum_{r=1}^n \frac{1}{r^2} < 2-\frac{1}{n} = \frac{2n-1}{n}$ ✓

(b) Sum of areas of appropriate rectangles

$= \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(n-1)^2} > \int_1^n \frac{1}{x^2}dx = 1-\frac{1}{n}$

Hence $\sum_{r=1}^n \frac{1}{r^2} > 1-\frac{1}{n}+\frac{1}{n^2} = \frac{n^2-n+1}{n^2}$ ✓

3 (a) $x = e^t - 4t + 3,\ y = 8e^{\frac{1}{2}t}$

$\frac{dx}{dt} = e^t - 4\ ;\ \frac{dy}{dt} = 4e^{\frac{1}{2}t}$

$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$

$= \sqrt{e^{2t}+16+8e^t} = e^t+4$.

$\therefore$ Arc length $= \int_0^2 (e^t+4)dt = \left[e^t+4t\right]_0^2$

$= e^2+7$ ✓

(b) $S = 2\pi\int_0^2 8e^{\frac{1}{2}t}(e^t+4)dt$ $\quad \boxed{S = 2\pi\int_0^2 y\,\frac{ds}{dt}\times dt}$

$= 16\pi\int_0^2 (e^{3/2t}+4e^{\frac{1}{2}t})dt$

$= 16\pi\left(\frac{2}{3}e^3+8e-\frac{26}{3}\right)$ ✓

4(a) $\int_0^1 x^2dx < \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^2+\frac{1}{n}\left(\frac{2}{n}\right)^2+\cdots+\frac{1}{n}\left(\frac{n}{n}\right)^2$

$= \frac{1}{n^3}\sum_{r=1}^n r^2 = \frac{n(n+1)(2n+1)}{6n^3} = \frac{2n^2+3n+1}{6n^2}$

(b) $\int_0^1 x^2dx > \frac{1}{n}\left(\frac{1}{n}\right)^2+\frac{1}{n}\left(\frac{2}{n}\right)^2+\cdots+\frac{1}{n}\left(\frac{n-1}{n}\right)^2$

$= \frac{1}{n^3}\sum_{r=1}^{n-1} r^2 = \frac{(n-1)n(2n-2+1)}{6n^3}$

$= \frac{2n^2-3n+1}{6n^2}$ ✓

5(a) $I_1 = \int_0^{\frac{1}{2}} (1-x^2)^{-\frac{1}{2}}dx = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}dx$

$= \left[\sin^{-1}x\right]_0^{\frac{1}{2}} = \frac{1}{6}\pi$ ✓

(b) $\frac{d}{dx}\left[x(1-x^2)^{-\frac{1}{2}n}\right] = nx^2(1-x^2)^{-\frac{1}{2}n-1} + (1-x^2)^{-\frac{1}{2}n}$

$= n\left(1-(1-x^2)\right)(1-x^2)^{-\frac{1}{2}-1}+(1-x^2)^{-\frac{1}{2}n}$

$\left[x(1-x^2)^{-\frac{1}{2}n}\right]_0^{\frac{1}{2}} = nI_{n+2} - nI_n + I_n$

$\frac{1}{2}\left(\frac{3}{4}\right)^{-\frac{1}{2}n} = nI_{n+2} - (n-1)I_n$

$\Rightarrow n.I_{n+2} = 2^{n-1}3^{-\frac{1}{2}n} + (n-1)I_n$ ✓

(c) $I_3 = 3^{-\frac{1}{2}}$

$3I_5 = 2^2.3^{-3/2} + 2I_3 \Rightarrow I_5 = \frac{10}{27}\sqrt{3}$ ✓

6(a) $I_n = \left[-\frac{1}{3}e^{-3x}.(1+3x)^n\right]_0^1$

$+ n\int_0^1 e^{-3x}.(1+3x)^{n-1}dx$

$= -\frac{1}{3}e^{-3}.4^n + \frac{1}{3} + nI_{n-1}$

$\Rightarrow 3I_n = 1 - 4^ne^{-3} + 3nI_{n-1}$ ✓

(b) $I_0 = \int_0^1 e^{-3x}dx = -\frac{1}{3}\left[e^{-3x}\right]_0^1 = \frac{1}{3}(1-e^{-3})$

$I_1 = \frac{1}{3}(1-4e^{-3}+1-e^{-3}) = \frac{1}{3}(2-5e^{-3})$

$I_2 = \frac{1}{3}(1-16e^{-3}+2(2-5e^{-3})$

$I_2 = \frac{1}{3}(5-26e^{-3})$ ✓

FP.2

Answers

7.(a) $\ln N! = \ln 2 + \ln 3 + \cdots + \ln N$

$> \int_1^N \ln x \, dx$

$= [x \ln x]_1^N - \int_1^N 1 \, dx$

$= N\ln N - N + 1$

$\therefore \ln N! > N\ln N - N + 1$ ✓

(b) $\ln 1 + \ln 2 + \cdots + \ln (N-1)$

$< \int_1^N \ln x \, dx$ [or $< \int_1^N \ln x \, dx$

$\Rightarrow \ln N! < N\ln N - N + 1 + \ln N$ ✓

$\Rightarrow \ln N! < (N+1)\ln N - N + 2(1 - \ln 2)$

8 (a) $x = \frac{1}{2}t^2 - \ln t, \; y = 2t + 1$

$\frac{dx}{dt} = t - t^{-1}, \quad \frac{dy}{dt} = 2$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = t^2 - 2 + t^{-2} + 4 = (t + t^{-1})^2$

$C = \int_{\frac{1}{2}}^{2} (t + t^{-1}) \, dt = \left[\frac{1}{2}t^2 + \ln t\right]_{\frac{1}{2}}^{2}$

$= \frac{15}{8} + 2\ln 2$ ✓

(b) $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{2}{t - t^{-1}}$

$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{-2(1 - t^{-2})}{(t - t^{-1})^2}$

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$

$= \frac{-2(1 - t^{-2})}{(t - t^{-1})^2} \times \frac{1}{(t - t^{-1})^2}$

$= -\frac{2(1 - t^{-2})}{(t - t^{-1})^3}$ ✓

9 (i) $I_2 = \int_0^1 x^2 e^{x^3} dx = \frac{1}{3}\left[e^{x^3}\right]_0^1$

$= \frac{1}{3}(e - 1)$ ✓

(ii) $I_n = \int_0^1 x^{n-2} \cdot x^2 e^{x^3} dx$

$= \left[\frac{1}{3} x^{n-2} e^{x^3}\right]_0^1 - \frac{n-2}{3} \int_0^1 x^{n-3} e^{x^3} dx$

$= \frac{e}{3} - \frac{n-2}{3} I_{n-3}$

$\Rightarrow 3I_n = e - (n-2) I_{n-3}$ ✓

(iii) $I_5 = \frac{1}{3}(e - 3I_2) = \frac{1}{3}(e - e + 1) = \frac{1}{3}$

$I_8 = \frac{1}{3}(e - 6I_5) = \frac{1}{3}(e - 2)$ ✓

10 (i) $\frac{dx}{dt} = \frac{-2(e^t - e^{-t})}{(e^t + e^{-t})^2}$

$\frac{dy}{dt} = \frac{(e^t + e^{-t})^2 - (e^t - e^{-t})^2}{(e^t + e^{-t})^2}$

$= \frac{(2e^t)(2e^{-t})}{(e^t + e^{-t})^2} = \frac{4}{(e^t + e^{-t})^2}$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{4(e^t - e^{-t})^2 + 16}{(e^t + e^{-t})^4}$

$\left(\frac{ds}{dt}\right)^2 = \frac{4(e^t + e^{-t})^2}{(e^t + e^{-t})^4}$

$S = 2\pi \int_0^1 y \cdot \left(\frac{ds}{dt}\right) dt$

$= 2\pi \int_0^1 \frac{(e^t - e^{-t})}{(e^t + e^{-t})} \times \left(\frac{2}{e^t + e^{-t}}\right) dt$

$= 4\pi \int_0^1 \frac{e^t - e^{-t}}{(e^t + e^{-t})^2} \, dt$

(ii) $S = 4\pi \int_2^{e + e^{-1}} u^{-2} \, du = 4\pi \left[-u^{-1}\right]_2^{e + e^{-1}}$ $\quad u = e^t + e^{-t}$

$= 4\pi \left(\frac{1}{2} - \frac{1}{e + e^{-1}}\right)$ ✓

Answers

11(i) $\frac{d}{dx}(\cot^{n+1}x) = -(n+1)\cot^n x \csc^2$

$= -(n+1)\cot^n x(\cot^2 x+1)$

$= -(n+1)\cot^{n+2}x - (n+1)\cot^n x$

$[\cot^{n+1}x]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} = -(n+1)(I_{n+2}+I_n)$

$\Rightarrow 0-1 = -(n+1)(I_{n+2}+I_n)$

$\Rightarrow I_{n+2} = \frac{-1}{n+1} - I_n$ ✓

(ii) $\bar{y} = \frac{1}{2A}\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot^2 x\,dx = \frac{I_2}{2A}$

$A = \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot x\,dx = [\ln \sin x]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}$

$= \ln\sqrt{2}$

$I_2 = 1 - I_0 = 1-\frac{1}{4}\pi$

$\bar{y} = \frac{1}{\ln 2}\left(1-\frac{1}{4}\pi\right)$ ✓

12 (i) $I_{n+2} = \left[\frac{x^{-n-1}}{-n-1}\sin\pi x\right]_{\frac{1}{2}}^{1} - \pi\int_{\frac{1}{2}}^{1}\frac{x^{-n-1}}{-n-1}\cos\pi x\,dx$

$= \frac{2^{n+1}}{n+1} + \frac{\pi}{n+1}\left(\left[\frac{x^{-n}}{-n}\cos\pi x\right]_{\frac{1}{2}}^{1} + \pi\int_{\frac{1}{2}}^{1}\frac{x^{-n}}{-n}\sin\pi x\,dx\right)$

$= \frac{2^{n+1}}{n+1} + \frac{\pi}{n+1}\left(\frac{1}{n} - \frac{\pi}{n}I_n\right)$

$\Rightarrow (n+1)I_{n+2} = 2^{n+1} + \pi\left(\frac{1}{n} - \frac{\pi}{n}I_n\right)$

$\Rightarrow n(n+1)I_{n+2} = n\cdot 2^{n+1} + \pi - \pi^2 I_n$ ✓

(ii) $2I_3 = 4 + \pi - \pi^2 I_1$

$12 I_5 = 48 + \pi - \frac{\pi^2}{2}(4+\pi-\pi^2 I_1)$

$\Rightarrow I_5 = 4 + \frac{1}{24}(2\pi - 4\pi^2 - \pi^3 + \pi^4 I_1)$

13. $\frac{dx}{dt} = e^t - 1$ and $\frac{dy}{dt} = 2e^{\frac{1}{2}t}$

$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^t+1)^2$

$\therefore$ Arc length $= \int_0^3 (e^t+1)\,dt = [e^t + t]_0^1 = 2 + e^3$ ✓

14(i) $\int \sec^2 x \tan^2 x\,dx$

$= \int u^2\,du \quad \begin{cases} u = \tan x \\ du = \sec^2 x\,dx \end{cases}$

$= \frac{1}{3}u^3 = \frac{1}{3}\tan^3 x + c$ ✓

(ii) $I_n = \int_0^{\frac{\pi}{4}} \sec^{n-2}x \sec^2 x \tan^2 x\,dx$

$= \frac{1}{3}\left[\sec^{n-2}x\cdot\tan^3 x\right]_0^{\frac{\pi}{4}} - \frac{n-2}{3}\int_0^{\frac{\pi}{4}}\sec^{n-2}x\tan^4 x\,dx$

$= \frac{2^{\frac{n-2}{2}}}{3} - \frac{n-2}{3}\int_0^{\frac{\pi}{4}}\sec^{n-2}x\tan^2 x(\sec^2 x - 1)\,dx$

$= \frac{2^{\frac{n-2}{2}}}{3} - \frac{n-2}{3}I_n + \frac{(n-2)}{3}I_{n-2}$

$\Rightarrow (3+n-2)I_n = (\sqrt{2})^{n-2} + (n-2)I_{n-2}$

$\Rightarrow (n+1)I_n = (\sqrt{2})^{n-2} + (n-2)I_{n-2}$ ✓

(iii) $I_2 = \left[\frac{\tan^3 x}{3}\right]_0^{\frac{\pi}{4}} = \frac{1}{3}$

$5I_4 = 2 + 2I_2$

Mean value $= \frac{4}{\pi}I_4$

$= \frac{82}{15\pi}$ ✓

15(i) $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \cos x\,dx$

$= [e^x\cos x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x(-\sin x)\,dx$

$= [e^x \sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - I$

$\Rightarrow 2I = e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}$

$I = \frac{1}{2}\left[e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}\right]$ ✓

(Continued→)

Answers

15(ii) $I_n = \left[\frac{e^{2x}}{2}\cos^n x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{n}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}\cos^{n-1}x\sin x\,dx$

$= 0 - \frac{n}{4}\int_{-\pi/2}^{\pi/2} e^{2x}\left[\cos^n x - (n-1)\sin^2 x\cos^{n-2}x\right]dx$

$\Rightarrow 4I_n = n(n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}\sin^2 x\cos^{n-2}dx - nI_n$

$\Rightarrow (n+4)I_n = n(n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}(1-\cos^2 x)\cos^{n-2}x\,dx$

$(n+4)I_n = n(n-1)\cdot I_{n-2} - n(n-1)I_n$

$\Rightarrow (n^2+4)I_n = n(n-1)\,I_{n-2}$ ✓

(iii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y^2\,dx = I_2 = \frac{1}{4}I_0 = \frac{1}{4}\left[\frac{e^{2x}}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{8}(e^{\pi}-e^{-\pi})$

$\Rightarrow \bar{y} = \frac{I_2}{2I} = \frac{1}{8}\left[\frac{e^{\pi}-e^{-\pi}}{e^{\pi/2}+e^{-\frac{\pi}{2}}}\right] = 0.575$ ✓

16(i) $\frac{dx}{dt} = 1-\cos 2t = 2\sin^2 t,\ \frac{dy}{dt} = 2\sin t\cos t$

$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{4\sin^4 t + 4\sin^2 t\cos^2 t} = 2\sin t$

$S = 2\pi\int_0^{\pi} y\cdot\frac{ds}{dt}\,dt = 2\pi\int_0^{\pi}\sin^2 t\times 2\sin t\,dt$

$S = 4\pi\int_0^{\pi}\sin^3 t\,dt \Rightarrow a = 4$ ✓

(ii) $S = 4\pi\int_0^{\pi}\sin^3 t\,dt = \pi\int_0^{\pi}(3\sin t - \sin 3t)dt$

$= \pi\left[-3\cos t + \frac{1}{3}\cos 3t\right]_0^{\pi}$

$= \pi\left[3 - \frac{1}{3} - \left(-3+\frac{1}{3}\right)\right] = \frac{16\pi}{3}$ ✓

17(i) $x^2+2xy = y^3-2$

diff w.r.t x

$2x + 2\left(x\frac{dy}{dx}+y\right) = 3y^2\frac{dy}{dx}$

$(3y^2-2x)\frac{dy}{dx} = 2x+2y$ — (1)

$x=-1, y=1 \Rightarrow \frac{dy}{dx} = 0$ ✓

(ii) $x = -1$

$\Rightarrow y^3+2y-3 = 0$

$(y-1)(y^2+y+3) = 0$

There only one real root $y=1$

$(-1,+1)$ is the only point on the curve for $x=-1$ ✓

(iii) diff (1) again

$(3y^2-2x)\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(6y\frac{dy}{dx}-6\right)$

$= 2 + 2\frac{dy}{dx}$

$x=-1, y=1, \frac{dy}{dx}=0$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{2}{5}$ or $A_2 = \frac{2}{5}$ ✓

(iv) $I_n = \int_{-1}^{0} x^n\cdot\frac{d^n y}{dx^n}dx$

$= \left[x^n\cdot\frac{d^{n-1}y}{dx^{n-1}}\right]_{-1}^{0} - n\int_{-1}^{0} x^{n-1}\cdot\frac{d^{n-1}y}{dx^{n-1}}dx$

$= (-1)^{n+1}A_{n-1} - nI_{n-1}$ ✓

(v) $I_3 = (-1)^4 A_2 - 3I_2$

$= A_2 - 3[A_1 - 2I_1]$

$= \frac{2}{5} + 6I_1$ ✓

Answers

18.(i) $I_n = [x(x^2-1)^n]_1^{\sqrt2} - 2n\int_1^{\sqrt2} x^2(x^2-1)^{n-1}dx$

$= \sqrt2 - 2n\int_1^{\sqrt2}(x^2-1+1)(x^2-1)^{n-1}dx$

$I_n = \sqrt2 - 2nI_n - 2nI_{n-1}$

$\Rightarrow (2n+1)I_n = \sqrt2 - 2nI_{n-1}$ ✓

(ii) Put $x=\sec\theta \Rightarrow \frac{dx}{d\theta} = \sec\theta\tan\theta$

$\sec\theta=\sqrt2 \Rightarrow \theta=\frac{\pi}{4}$; $\sec\theta=1\Rightarrow\theta=0$

$I_n = \int_0^{\frac{\pi}{4}}(\sec^2\theta-1)^n \cdot \tan\theta\sec\theta\, d\theta$

$= \int_0^{\frac{\pi}{4}}\tan^{2n+1}\theta\sec\theta\,d\theta$ ✓

(iii) $\int_0^{\frac{\pi}{4}}\frac{\sin^7\theta}{\cos^8\theta}d\theta = I_3$

$I_0 = (\sqrt2-1)$, $I_1 = \frac{2-\sqrt2}{3}$

$\Rightarrow 3I_1 = \sqrt2 - 2I_0 \Rightarrow I_1 = \frac{2-\sqrt2}{3}$

$I_2 = \frac{\sqrt2}{5}-\frac45 I_1 = \frac{7\sqrt2-8}{15}$

$I_3 = \frac{\sqrt2}{7}-\frac67 I_2 = \frac{16-9\sqrt2}{35}$ ✓

19.(i) $I_n = \int_0^{\frac{\pi}{2}}x^n\sin x\,dx$

$= [-x^n\cos x]_0^{\pi/2} + \int_0^{\frac{\pi}{2}} nx^{n-1}\cos x\,dx$

$= 0 + [nx^{n-1}\sin x]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} n(n-1)x^{n-2}\sin x\,dx$

$I_n = n(\frac{\pi}{2})^{n-1} - n(n-1)I_{n-2}$

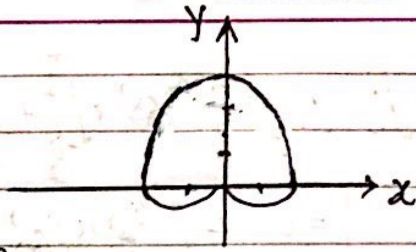
$\Rightarrow I_n + n(n-1)I_{n-2} = n(\frac12\pi)^{n-1}$ ✓

(ii) $I_1 = \int_0^{\frac{\pi}{2}}x\sin x\,dx = [-x\cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\cos x\,dx$

$= [\sin x]_0^{\frac{\pi}{2}} = 1$ ✓

$n=3 \Rightarrow I_3 = 3(\frac{\pi}{2})^2 - 3\times2\times1 = \frac34\pi^2 - 6$ ✓

20.(i)



(ii) $\int_{-\pi}^{\pi}\frac{a^2}{2}(1+2\sin\theta+\sin^2\theta)d\theta$

$A = \frac{a^2}{2}\int_{-\pi}^{\pi}(1+2\sin\theta+\frac12-\frac12\cos2\theta)d\theta$

$= \frac{a^2}{2}\left[\frac{3\theta}{2} - 2\cos\theta - \frac14\sin2\theta\right]_{-\pi}^{\pi}$

$= \frac{3\pi a^2}{2}$ ✓

(iii) show that $r=0,\ \theta=-\frac{\pi}{2}$,
when $r=2a$, $\theta=\frac{\pi}{2}$ and $\frac{dr}{d\theta}=a\cos\theta$

$S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{a^2(1+2\sin\theta+a^2\sin^2\theta)+a^2\cos^2\theta}\,d\theta$

$= \sqrt{2}a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1+\sin\theta}\,d\theta$ ✓

(iv) $u = 1+\sin\theta \Rightarrow \frac{du}{d\theta} = \cos\theta = \sqrt{1-(u-1)^2}$

$= \sqrt{2u-u^2}$

or $S = \sqrt2 a\int_0^2\frac{1}{\sqrt{2-u}}du$

$= \sqrt2 a\left[-2(2-u)^{\frac12}\right]_0^2 = 4a$ ✓

21(i) xx

(ii) $\frac{ds}{dx} = \sqrt{1+(\frac{dy}{dx})^2} = \sqrt{\frac14(e^{2x}+2+e^{-2x})}$

$= \frac12(e^x+e^{-x})$ ✓

$S = 2\pi\int_0^4\frac14(e^x+e^{-x})^2dx$

$= \frac{\pi}{2}\int_0^4(e^{2x}+2+e^{-2x})$

$= \frac{\pi}{2}\left[\frac{e^{2x}}{2}+2x-\frac{e^{-2x}}{2}\right]_0^4$

$= \frac{\pi}{2}\left[\frac{e^8}{2}+8-\frac{e^{-8}}{2}\right]$ ✓

Answers

22(i) $\frac{dx}{dt}^2 + \frac{dy}{dt}^2 = (t^{3/2} - t^{-1/2})^2 + 4t$

$= (t^{3/2}+t^{-1/2})^2$

$s = \int_1^4 (t^{3/2}+t^{-1/2})\,dt$

$= [\frac{2}{5}t^{5/2} + 2t^{1/2}]_1^4 =$

$= (\frac{64}{5}+4) - (\frac{2}{5}+2) = \frac{72}{5}$ ✓

(ii) $S = \int_1^4 y\cdot(\frac{ds}{dt})\,dt$

$= 2\pi\int_1^4 \frac{4}{3}t^{3/2}(t^{3/2}+t^{-1/2})\,dt$

$= \frac{8}{3}\pi\int_1^4 (t^3+t)\,dt$

$= \frac{8}{3}\pi[\frac14 t^4+\frac12 t^2]_1^4$

$= \frac{8\pi}{3}[(64+8)-(\frac14+\frac12)] = 190\pi$ ✓

23 (i) $I_n = \int_0^2 (4+x^2)^{-n}\,dx$

$\frac{d}{dx}(x(4+x^2)^{-n}) = (4+x^2)^{-n} - 2nx^2(4+x^2)^{-n-1}$

Integrating w.r.t x

$[x(4+x^2)^{-n}]_0^2 = I_n - 2n\int_0^2 (4+x^2-4)(4+x^2)^{-n-1}\,dx$

$\Rightarrow 2\cdot 8^{-n} = I_n - 2nI_n + 8nI_{n+1}$

$\Rightarrow 8n\cdot I_{n+1} = (2n-1)I_n + 2\cdot 8^{-n}$

(ii) $I_1 = \int_0^2 \frac{1}{4+x^2}\,dx = [\frac12\tan^{-1}\frac{x}{2}]_0^2 = \frac12\cdot\frac{\pi}{4} = \frac{\pi}{8}$

$8I_2 = \frac{\pi}{8} + \frac14 \Rightarrow I_2 = \frac{\pi}{64}+\frac{1}{32}$

$16\cdot I_3 = \frac{3\pi}{64}+\frac{3}{32}+\frac{1}{32}$

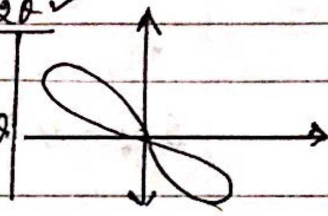
$\Rightarrow I_3 = \frac{3\pi}{1024}+\frac{1}{128}$ ✓

24. $r = 2a\cos(2\theta+\frac12\pi)$ for $0\le\theta\le 2\pi$

(i) $r = 2a[\cos2\theta\cos\frac{\pi}{2} - \sin2\theta\sin\frac{\pi}{2}]$

$= -2a\sin2\theta$ ✓

(ii)



$r = -4a\sin\theta\cos\theta$

$= -4a\frac{y}{r}\cdot\frac{x}{r}$

$\Rightarrow -4axy = r^3 = (\sqrt{x^2+y^2})^3$

$\Rightarrow (x^2+y^2)^{3/2} = -4axy$ ✓

(iii) Area of one loop

$= \frac12\int_{\pi/2}^{\pi} 4a^2\sin^2 2\theta\,d\theta$

$= a^2\int_{\frac{\pi}{2}}^{\pi} 2\sin^2 2\theta\,d\theta$

$= a^2\int_{\frac{\pi}{2}}^{\pi}(1-\cos4\theta)\,d\theta$

$= a^2[\theta - \frac14\sin4\theta]_{\frac{\pi}{2}}^{\pi} = \frac12\pi a^2$ ✓

(iv) $y = r\sin\theta = -2a\sin2\theta\cdot\sin\theta$

$\frac{dy}{d\theta} = -2a[2\cos2\theta\sin\theta + \sin2\theta\cos\theta] = 0$

$\Rightarrow 2\cos2\theta\sin\theta = -\sin2\theta\cos\theta$

$\Rightarrow 2\tan\theta = -\tan2\theta$ ✓

25(i) $I_n = \int_0^{\frac{\pi}{4}}\sec^n x\,d\theta$, $n>0$

Now $I_2 = \int_0^{\frac{\pi}{4}}\sec^2 x\,dx = [\tan x]_0^{\frac{\pi}{4}} = 1$ ✓

(ii) $I_n = \int_0^{\frac{\pi}{4}}\sec^{n-2}x\sec^2 x\,dx$

$= [\sec^{n-2}x\cdot\tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}(n-2)\sec^{n-3}x(\sec x\tan x)\tan x\,dx$

$= [\sec^{n-2}x\tan x]_0^{\frac{\pi}{2}} - (n-2)\int_0^{\frac{\pi}{4}}\sec^{n-2}x(\sec^2 x-1)\,dx$

$\Rightarrow (n-1)I_n = 2^{\frac12 n-1} + (n-2)I_{n-2}$ ✓

(Continued →)

continued→

Answers

25(iii)

Volume of revolution $= \pi\int_0^{\frac{1}{4}\pi} y^2 dx$

$= \int_0^{\frac{1}{4}\pi} \sec^6 x\, dx$

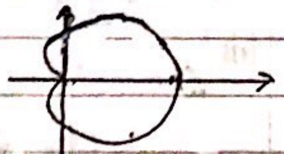
$3I_4 = 2 + 2\times 1 \Rightarrow I_4 = 4/3$

$5I_6 = 4 + 4\times\frac{4}{3} \Rightarrow I_6 = 28/15$

∴ Volume of revolution $= \frac{28\pi}{15}$ ✓

$r = a(1+\cos\theta),\ 0 \le \theta < 2\pi$

26(i)



(ii) $r = a(1+\cos\theta)$

$\Rightarrow \sqrt{x^2+y^2} = a\left[1+\frac{x}{\sqrt{x^2+y^2}}\right]$

$\Rightarrow x^2+y^2 = a(1+\sqrt{x^2+y^2})$ ✓

(iii) Surface area $= \frac{a^2}{2}\int_0^{\frac{\pi}{3}}(1+2\cos\theta+\cos^2\theta)\,d\theta$

$= \frac{a^2}{2}\int_0^{\pi/3}\left(\frac{3}{2}+2\cos\theta+\frac{\cos 2\theta}{2}\right)d\theta$

$= \frac{a^2}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi/3}$

$= \frac{a^2}{16}\left[4\pi + 9\sqrt{3}\right]$ ✓

(iv) Arc length $= \int_0^{\frac{1}{3}\pi}\sqrt{a^2(1+2\cos\theta+\cos^2\theta)+a^2(-\sin\theta)^2}\,d\theta$

$= a\int_0^{\pi/3}\sqrt{2+2\cos\theta}\,d\theta$

$= a\left[4\sin\theta/2\right]_0^{\frac{\pi}{3}}$

$= 2a$ ✓

$r^2 = 8\,\mathrm{cosec}\,2\theta,\ 0<\theta<\frac{\pi}{2}$

27(i) $r^2 = \frac{4}{\sin\theta\cos\theta}$

$\Rightarrow r\sin\theta \cdot r\cos\theta = 4 \Rightarrow xy = 4$ ✓

(ii) Sketch the curve.

(iii) $\frac{1}{2}\int_{\frac{\pi}{6}}^{\pi/3} 8\,\mathrm{cosec}\,2\theta\,d\theta = \left[2\ln|\tan\theta|\right]_{\frac{\pi}{6}}^{\pi/3}$

$= 2\left\{\ln|\sqrt{3}| - \ln\left|\frac{1}{\sqrt{3}}\right|\right\} = 2\ln 3$

$\text{or} = \ln 9$ ✓

28(i) $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,\sin^2 x\,dx$

Now $\frac{d}{dx}(\cos^{n-1}x\,\sin^3 x)$

$= -(n-1)\cos^{n-2}x\,\sin^4 x + 3\cos^n x\,\sin^2 x$

$\Rightarrow [\cos^{n-1}x\,\sin^3 x]_0^{\frac{\pi}{2}} = -\int_0^{\frac{\pi}{2}}(n-1)\cos^{n-2}x\cdot\sin^2 x(1-\cos^2 x)\,dx + 3I_n$

$\Rightarrow 0 = -(n-1)I_{n-2} + (n-1)I_n + 3I_n$

$\Rightarrow (n+2)I_n = (n-1)I_{n-2}$ ✓

(ii) $I_0 = \int_0^{\frac{\pi}{2}}\sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{1}{2}\pi}(1-\cos 2x)\,dx$

$= \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$

$I_2 = \frac{1}{4}\times\frac{\pi}{4}$; $I_4 = \frac{3}{6}\times\frac{1}{4}\times\frac{\pi}{4} = \frac{\pi}{32}$ ✓

29(i) $x = e^{2t}\cos 2t,\ y = e^{2t}\sin 2t$

$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$\frac{dx}{dt} = 2e^{2t}\cos 2t - 2e^{2t}\sin 2t$ —

and $\frac{dy}{dt} = 2e^{2t}\cos 2t + 2e^{2t}\sin 2t$

$\Rightarrow \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4e^{4t}[\cos^2 2t - 2\cos 2t\sin 2t + \sin^2 2t + \cos^2 2t + 2\cos 2t\sin 2t + \sin^2 2t]$

$= 8e^{4t}$

$s = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sqrt{2}e^{2t}\,dt = \sqrt{2}\left[e^{2t}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \sqrt{2}(e^{\pi} - e^{-\pi})$

$= 2\sqrt{2}\sinh\pi$ or 32.7 ✓

(continued →)

FP-2

(continued →)

Answers

29(ii) $S = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y \frac{ds}{dt} \cdot dt$

$= \int_{-\pi/2}^{\pi/2} e^{2t} \sin 2t \cdot 2\sqrt{2} e^{2t} dt$

$S = 4\sqrt{2}\pi \int_{-\pi/2}^{\frac{\pi}{2}} e^{4t} \cdot \sin 2t\, dt$ —— (1)

Consider $I \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{4t} \sin 2t\, dt$

$= \left[-e^{4t} \frac{\cos 2t}{2} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2e^{4t} \cos 2t\, dt$

$= \frac{e^{2\pi}}{2} - \frac{e^{-2\pi}}{2} + 2 \left\{ \left[e^{4t} \frac{\sin 2t}{2} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2e^{4t} \sin 2t\, dt \right\}$

$= \frac{e^{2\pi} - e^{-2\pi}}{2} + 0 - 4I \Rightarrow I = \frac{e^{2\pi} - e^{-2\pi}}{10}$

from (1) $\Rightarrow S = 4\sqrt{2}\pi \cdot \frac{(e^{2\pi} - e^{-2\pi})}{10} = \frac{2\sqrt{2}}{5}\pi (e^{2\pi} - e^{-2\pi})$

$= \frac{4\sqrt{2}\pi}{5} \sinh 2\pi$ or 952 ✓

$y = -\ln(1-x^2)$

30(i) $\frac{dy}{dx} = \frac{2x}{1-x^2}$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4x^2}{(1-x^2)^2} = \frac{1 - 2x^2 + x^4 + 4x^2}{(1-x^2)^2}$

$= \frac{(1+x^2)^2}{(1-x^2)^2}$ ✓

(ii) $\frac{1+x^2}{1-x^2} \equiv \frac{1}{(1+x)} + \frac{1}{(1-x)} - 1$ ✓

(iii) $S = 2 \int_0^{\frac{1}{2}} \left(\frac{1}{1+x} + \frac{1}{1-x} - 1 \right) dx$

$= 2 \left[\ln\left(\frac{1+x}{1-x}\right) - x \right]_0^{1/2}$

$= 2\ln 3 - 1$ ✓

$I_n = \int_0^2 x^n (4-x^2)^{\frac{1}{2}} dx, \quad n \geq 1$

31(i) $\frac{d}{dx} \left(x^n (4-x^2)^{3/2} \right)$

$= -3x^{n+1}(4-x^2)^{\frac{1}{2}} + nx^{n-1}(4-x^2)^{3/2}$

$= -3x^{n+1}(4-x^2)^{\frac{1}{2}} + nx^{n-1}(4-x^2)(4-x^2)^{1/2}$

$\Rightarrow \left[x^n (4-x^2)^{3/2} \right]_0^2 = 0 = -3I_{n+1} + 4nI_{n-1} - nI_{n+1}$

$\Rightarrow (n+3) I_{n+1} = 4n I_{n-1}$ ✓

(ii) $I_1 = \left[-\frac{1}{3}(4-x^2)^{3/2} \right]_0^2 = 8/3$

$5I_3 = 8I_1 \Rightarrow I_3 = \frac{64}{15}$ ✓

---

32. $r = \frac{1}{1-\cos\theta}, \quad 0 < \theta < 2\pi$

(i) $\sqrt{x^2+y^2} = \frac{1}{1 - \frac{x}{\sqrt{x^2+y^2}}}$

$\Rightarrow \sqrt{x^2+y^2} - x = 1$

$\Rightarrow y^2 = 1 + 2x$ ✓

(ii) Sketch: Parabola symmetrical about x-axis. Intercepts at $(-\frac{1}{2}, 0)$ and $(0, \pm 1)$

shading the correct area.

$\frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{(1-\cos\theta)^2} d\theta$ as the area of the region between parabola and the y-axis.

$= 2 \int_{-\frac{1}{2}}^{0} (1+2x)^{\frac{1}{2}} dx$

$= 2 \left[\frac{1}{3} (1+2x)^{3/2} \right]_{-\frac{1}{2}}^{0}$

$= 2/3$ ✓

Answers

33. (i) $x = e^u$ & $y = f(x)$

$x\frac{dy}{dx} = x\frac{dy}{du}\times\frac{du}{dx} = x\frac{dy}{du}\times\frac{1}{x} = \frac{dy}{du}$ ✓

(ii) $x^2\frac{d^2y}{dx^2} = x^2\frac{d}{dx}\left(\frac{dy}{dx}\right) = x^2\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{du}\right)$

$= x^2\left[-\frac{1}{x^2}\frac{dy}{du} + \frac{1}{x}\frac{d}{du}\left(\frac{dy}{du}\right)\cdot\frac{du}{dx}\right]$

$= -\frac{dy}{du} + x\frac{d^2y}{du^2}\cdot\frac{1}{x} = \frac{d^2y}{du^2} - \frac{dy}{du}$

Substituting in the diff. eqn.

$\frac{d^2y}{du^2} - \frac{dy}{du} + 3\frac{dy}{du} + 17y = 34u + 21$

$\Rightarrow \frac{d^2y}{du^2} + 2\frac{dy}{du} + 17y = 34u + 21$ ✓

(iii) $m^2 + 2m + 17 = 0 \Rightarrow m = -1 \pm 4i$

$\Rightarrow y_c = e^{-u}(A\cos 4u + B\sin 4u)$

$y_p = cu + d \Rightarrow y' = c$ and $y'' = 0$

$\Rightarrow 2c + 17cu + 17d = 34u + 21$

$\Rightarrow c = 2,\ d = 1 \Rightarrow y_p = 2u + 1$

$\Rightarrow y = \frac{1}{x}\left[A\cos(4\ln x) + B\sin(4\ln x)\right] + 2\ln x + 1$

$y = 0,\ x = 1 \Rightarrow A + 1 = 0 \Rightarrow A = -1$ or $u = 0, y = 0$

$\frac{dy}{dx} = -\frac{1}{x^2}\left[A\cos(4\ln x) + B\ln[\ln x]\right] + \frac{1}{x}\left[-A\sin(4\ln x)\cdot\frac{4}{x} + B\cos(4\ln x)\right] + \frac{2}{x}$

Now $\frac{dy}{dx} = -1, x = 1 \Rightarrow -1 = 1 + 4B + 2$

$\Rightarrow B = -1$

or all in terms of u

$y = 2\ln x + 1 - \frac{1}{x}\left[\cos(4\ln x) + \sin(4\ln x)\right]$

34. $x = 1 - 3t^2,\ y = t(1 - 3t^2),\ 0 \le t \le \frac{1}{\sqrt{3}}$

(a) $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 36t^2 + 1 - 18t^2 + 81t^4$

$= 1 + 18t^2 + 81t^4 = (1 + 9t^2)^2$ ✓

(b)(i) $s = \int_0^{\frac{1}{\sqrt{3}}}(1 + 9t^2)\,dt = \left[t + 3t^3\right]_0^{\frac{1}{\sqrt{3}}} = \frac{2}{\sqrt{3}}$ ✓

(ii) $S = 2\pi\int_0^{\frac{1}{\sqrt{3}}} t(1 - 3t^2)(1 + 9t^2)\,dt$

$= 2\pi\int_0^{\frac{1}{\sqrt{3}}}(t + 6t^3 - 27t^5)\,dt = 2\pi\left[\frac{1}{2}t^2 + \frac{3}{2}t^4 - \frac{9}{2}t^6\right]_0^{\frac{1}{\sqrt{3}}} = \frac{\pi}{3}$ ✓

$C_1: r = \frac{1}{\sqrt{2}},\quad 0 \le \theta \le 2\pi$

36(i) $C_2: r = \sqrt{\sin\theta/2};\ 0 \le \theta \le \pi$

for intersection of ① & ②

$\sin\theta/2 = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$, Intersect $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{3}\right)$ ✓

(ii) C_1: Circle, centre at pole and radius $\frac{1}{\sqrt{2}}$.

C_2: Curve from $(0,0)$ to $(1,\pi)$ ✓

(iii) Area $= \frac{1}{6}\pi\times\frac{1}{2} - \frac{1}{2}\int_0^{\frac{\pi}{3}}\sin\frac{\theta}{2}\,d\theta$

$= \frac{1}{12}\pi - \frac{1}{2}\left[-2\cos\theta/2\right]_0^{\pi/3}$

$= \left(\frac{1}{12}\pi + \frac{\sqrt{3}}{2} - 1\right)$ ✓

34(c) Substitute $t = \frac{y}{x} \Rightarrow x = 1 - 3\frac{y^2}{x^2}$

and $x = r\cos\theta,\ y = r\sin\theta$

$\Rightarrow r\cos\theta = 1 - 3\tan^2\theta$

$\Rightarrow r = \sec\theta\left[1 - 3\tan^2\theta\right]$ ✓

for $0 \le \theta \le \frac{\pi}{6}$

Now $A = \frac{1}{2}\int_0^{\frac{\pi}{6}}\sec^2\theta(1 - 6\tan^2\theta + 9\tan^4\theta)\,d\theta$

$= \frac{1}{2}\left[\tan\theta - 2\tan^3\theta + \frac{9}{5}\tan^5\theta\right]_0^{\frac{\pi}{6}}$

$= \frac{4}{45}\sqrt{3}$ ✓

(continued)

Answers

37.(i) $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x\,dx,\ n>0$

$= \left[-x^n \cos x\right]_0^{\pi/2} + \int_0^{\frac{\pi}{2}} n x^{n-1} \cos x\,dx$

$= 0 + \left[n x^{n-1} \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} n(n-1) x^{n-2} \sin x\,dx$

$\therefore I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1) I_{n-2}$ ✓

(ii) $I_0 = \int_0^{\frac{\pi}{2}} \sin x\,dx = \left[-\cos x\right]_0^{\frac{\pi}{2}} = 1$

$I_2 = \pi - 2,$

$I_4 = 4\left(\frac{\pi}{2}\right)^3 - 12(\pi-2) = \frac{1}{3}\pi^3 - 12\pi + 24$

$x = 4t + 2t^{3/2},\ y = 4t - 2t^{3/2},\ 0 \le t \le 4$

38. (i) $\frac{dx}{dt} = 4 + 3t^{1/2};\ \frac{dy}{dt} = 4 - 3t^{1/2}$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 32 + 18t$

$S = \int_0^4 (32+18t)^{1/2} dt = \left[\frac{1}{27}(32+18t)^{3/2}\right]_0^4$

$= \frac{1}{27}\left(104^{3/2} - 32^{3/2}\right)$

$= 32.6$ ✓

(ii) $\int_0^{32} y\,dx = \int y \frac{dx}{dt}\,dt = \int_0^4 t(4-2\sqrt{t})(4+3\sqrt{t})\,dt$

$= \int_0^4 \left(16t + 4t^{3/2} - 6t^2\right)dt$

$= \left[8t^2 + \frac{8}{5}t^{5/2} - 2t^3\right]_0^4 = 51.2$ ✓

$M.V = \frac{\int_b^a y\,dx}{b-a} = \frac{51.2}{32} = 1.6$ ✓

39. $r = e^{4\theta}$ $S = \int_0^{\alpha} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

$= \int_0^{\alpha} \sqrt{e^{8t} + 16e^{8\theta}}\,d\theta = \int_0^{\alpha} \sqrt{17}\,e^{4\theta}\,d\theta$

$= \left[\frac{\sqrt{17}}{4} e^{4\theta}\right]_0^{\alpha}$

$\Rightarrow \frac{\sqrt{17}}{4} e^{4\alpha} - \frac{\sqrt{17}}{4} = 2015$ (Given)

$\Rightarrow e^{4\alpha} = 1955.837 \Rightarrow \alpha = 1.89$ ✓

$I_n = \int_0^{\frac{\pi}{2}} \frac{\sin 2n\theta}{\cos\theta}\,d\theta,\ n<0$

40(i) $I_n + I_{n-1} = \int_0^{\frac{\pi}{2}} \frac{\sin 2n\theta + \sin(2n-2)\theta}{\cos\theta}\,d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{2\sin(2n-1)\theta \cdot \cos\theta}{\cos\theta}\,d\theta$

$= \int_0^{\frac{\pi}{2}} 2\sin(2n-1)\theta\,d\theta = \left[-\frac{2\cos(2n-1)\theta}{(2n-1)}\right]_0^{\frac{\pi}{2}}$

$= 0 - \left[\frac{-2}{(2n-1)}\right] = \frac{2}{(2n-1)}$ ✓

(ii) $I_1 = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\cos\theta}\,d\theta = \int_0^{\frac{\pi}{2}} 2\sin\theta\,d\theta$

$= \left[-2\cos\theta\right]_0^{\frac{\pi}{2}} = 0 - (-2) = 2$

$I_2 = 2/3 - 2 = -4/3$

$\Rightarrow I_3 = 2/5 + \frac{4}{3} = \frac{26}{15},$

$I_4 = \frac{2}{7} - \frac{26}{15} = \frac{-152}{105}$ ✓

41.(i) $I_n = \int_1^e (\ln x)^n\,dx$ for $n \ge 0$

$\int_1^e \ln x\,dx = (x\ln x - x)$

$I_n = \int_1^e (\ln x)^{n-1} \cdot \ln x\,dx$

$= \left[(\ln x)^{n-1} \cdot (x\ln x - x)\right]_1^e$

$- \int_1^e (n-1)(\ln x)^{n-2} \cdot \frac{1}{x}(x\ln x - x)\,dx$

$= 0 - \int_1^e (n-1)(\ln x)^{n-2}(\ln x - 1)\,dx$

$= (n-1)\left[I_{n-2} - I_{n-1}\right]$ ✓

(ii) $I_0 = [x]_1^e = (e-1)$

$I_1 = [x\ln x - x]_1^e = 1$

$I_2 = 1 \times (e-1-1) = (e-2)$

$I_3 = 2(I_1 - I_2) = 2(1-(e-2))$

$= (6-2e)$

$MV = \frac{I_3}{e-1} = \frac{(6-2e)}{(e-1)}$ ✓

Answers

42. $r = a(1-\cos\theta), \quad 0 \le \theta < 2\pi$

(ii) $S = 2 \times \frac{1}{2} a^2 \int_{\frac{\pi}{2}}^{\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$

$= a^2 \int_{\frac{\pi}{2}}^{\pi} \left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$

$= a^2 \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_{\pi/2}^{\pi}$

$= a^2\left(\frac{3}{4}\pi + 2\right)$ ✓

(iii) $\left(\frac{ds}{d\theta}\right)^2 = a^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)$

$= 2a^2(1-\cos\theta)$

$= 2a^2 \times 2\sin^2\theta/2$

$= 4a^2 \sin^2\theta/2$ ✓

Now arc length

$S = 2\int_{\frac{\pi}{2}}^{\pi} 2a \sin\theta/2 \, d\theta$

$= 4a\left[-2\cos\theta/2\right]_{\frac{\pi}{2}}^{\pi}$

$= \underline{4\sqrt{2}a}$ ✓

← X —— X →