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FP-2

Further Pure Maths-2

Matrices Exercise

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FP-2

Matrices

1. (a) (i) Find the set of values of a for which the system of equations,

$$x - 2y - 2z + 7 = 0$$
$$2x + (a-9)y - 10z + 11 = 0$$
$$3x - 6y + 2az + 29 = 0$$

has a unique solution. ---[4]

(ii) Given that $a = -3$, show that the system of equations in part (i) has no solution. Interpret this situation geometrically. ---[3]

(b) The matrix A is given by, $\boxed{SP-20/02/Q8}$

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

(i) Find the eigenvalues of A. ---[4]

(ii) Use the characteristic equation of A to find A^{-1}. ---[4]

2. (a) Find the value of a for which the system of equations:

$$3x + y + z = 0$$
$$ax + 6y - z = 0$$
$$ay - 2z = 0$$

does not have a unique solution. ---[3]

The matrix A given by,

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 6 & -1 \\ 0 & 0 & -2 \end{pmatrix}$$

(b) Use the characteristic equation of A to find the inverse of A^2. ---[4]

(c) Find the matrix P and a diagonal matrix D, such that $A^5 = PDP^{-1}$ ---[7]

$\boxed{S-20/21/Q8}$

3. The matrix A is given by.

$$A = \begin{pmatrix} 5 & -1 & 7 \\ 0 & 6 & 0 \\ 7 & 7 & 5 \end{pmatrix}$$

(a) Find the eigenvalues of A. ---[4]

(b) Use the characteristic equation of A to find A^{-1}. $\boxed{S-20/23/Q3}$ ---[4]

4. It is given that e is an eigenvector of the matrix A, with corresponding eigenvalue λ.

(i) Show that e is an eigenvector of A^2, with corresponding eigenvalue λ^2. ---[2]

The matrices A and B are given by

$$A = \begin{pmatrix} n & 1 & 3 \\ 0 & 2n & 0 \\ 0 & 0 & 3n \end{pmatrix} \text{ and } B = (A + nI)^2,$$

where I is the 3×3 identity matrix and n is a non-zero integer.

(ii) Find, in terms of n, a non-singular matrix P and a diagonal matrix D such that $B = PDP^{-1}$ --[8]

/S-19/11/Q9/

5. A 3×3 matrix A has distinct eigenvalues 2, 1, 3, with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

respectively, where b is a positive constant.

(i) Find A in terms of b. ---[9]

(ii) Find $A^{-1}\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$ --[2]

(iii) It is given that, $A^n\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$ and $A^n\begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ b^{-1} \end{pmatrix}$

Find the values of n and b. --[3]

/S-19/13/Q 11(i)/

6. The matrix M is defined by.

$$M = \begin{pmatrix} 2 & m & 1 \\ 0 & m & 7 \\ 0 & 0 & 1 \end{pmatrix}$$

where $m \neq 0, 1, 2$

(i) Find a matrix P and a diagonal matrix D such that $M = PDP^{-1}$ --[7]

(ii) Find M^7P --[3]

/W-19/11/Q8/

7. It is given that e is an eigenvector of the matrix A, with corresponding eigenvalue λ.
(i) Write down another eigenvector of A corresponding to λ -- [1]
(ii) Write down an eigenvector and corresponding eigenvalue of A^n, where n is a positive integer. --[2]

$$\text{let } A = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 7 & 0 \\ 4 & 8 & 1 \end{pmatrix}$$

(iii) Find a matrix P and a diagonal matrix D such that, $A^n = PDP^{-1}$ -- [7]
(iv) Determine the set of values of the real constant k such that $\sum_{n=1}^{\infty} k^n(A^n - kA^{n+1}) = kA$ ---[4]
$\boxed{S-18/11/Q11(ii)}$

8. It is given that e is an eigenvector of a matrix A with corresponding eigenvalue λ.
(i) Show that e is an eigenvector of A^3 and state the corresponding eigenvalue. ---[3]
It is given that $A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$
(ii) Find a matrix P and a diagonal matrix D such that
$$A^3 + I = PDP^{-1}$$
where I is the 2×2 identity matrix. ---[5]
$\boxed{S-18/13/Q5}$

9. It is given that λ is an eigenvalue of the matrix A with e as a corresponding eigenvector, and μ is an eigenvalue of the matix B for which e is also a corresponding eigenvector.
(i) Show that $\lambda+\mu$ is an eigenvalue of the matrix A+B with e as a corresponding eigenvector. ---[2]

(continued →)

(continued →)

9→ The matrix A is given by $A=\begin{pmatrix}2 & 0 & 1\\ -1 & 2 & 3\\ 1 & 0 & 2\end{pmatrix}$

has $\begin{pmatrix}1\\2\\1\end{pmatrix}$, $\begin{pmatrix}1\\4\\-1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$ as eigenvectors.

(ii) Find the corresponding eigenvalues. ---[3]

The matrix B has eigenvalues 4, 5 and 1 with corresponding eigenvectors $\begin{pmatrix}1\\2\\1\end{pmatrix}$, $\begin{pmatrix}1\\4\\-1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$ respectively.

(iii) Find a matrix P and a diagonal matrix D such that $(A+B)^3 = PDP^{-1}$ $\boxed{W\text{-}18/11/Q5}$ ---[3]

10. It is given that.
$$A=\begin{pmatrix}2 & 3 & 1\\ 0 & -2 & 1\\ 0 & 0 & 1\end{pmatrix}$$

(i) Find the eigenvalue of A corresponding to the eigenvector $\begin{pmatrix}1\\0\\0\end{pmatrix}$ --[1]

(ii) Writedown the negative eigenvalue of A and find a corresponding eigenvector. ---[3]

(iii) Find an eigenvalue and a corresponding eigenvector of the matrix $A+A^6$ $\boxed{W\text{-}18/12/Q2}$ ---[2]

11(i) Find the values of k for which the set of linear equations:
$$x+3y+kz=4$$
$$4x-2y-10z=-5$$
$$x+y+2z=1$$
has no unique solution. ---[3]

(ii) For this value of k, find the set of possible solutions, giving your answer in the form. $\begin{pmatrix}x\\y\\z\end{pmatrix} = \mathbf{a}+t\mathbf{b}$

where **a** and **b** are vectors and t is a scalar. ---[3]

$\boxed{S\text{-}17/11/Q4}$

12. The matrix A, given by,
$$A = \begin{pmatrix} 1 & 2 & -2 \\ 6 & 4 & -6 \\ 6 & 5 & -7 \end{pmatrix}$$
has eigenvalues 1, −1 and −2

(i) Find a set of corresponding eigenvectors. ---[4]

(ii) The matrix B is given by $B = A - 2I$, where I is the 3×3 identity matrix. Write down the eigenvalues of B, and state a set of corresponding eigenvectors. $\boxed{S\text{-}17 / 11 / Q5}$ ---[2]

13. The matrix A is given by,
$$A = \begin{pmatrix} 6 & -8 & 7 \\ 7 & -9 & 7 \\ 6 & -6 & 5 \end{pmatrix}$$

(i) Given that $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of A, find the corresponding eigenvalue. ---[2]

(ii) Given that −1 is an eigenvalue of A, find the corresponding eigenvector. ---[2]

(iii) It is given that determinant of A is equal to the product of the eigenvalues of A. Use this result to find the third eigenvalue of A, and find also a corresponding eigenvector. ---[3]

(iv) Write down matrices P and D such that $P^{-1}AP = D$, where D is a diagonal matrix, and hence find the matrix A^n in terms of n, where n is a positive integer. ---[6]

$\boxed{S\text{-}17 / 13 / Q10}$

14(i) The vector **e** is an eigenvector of matrix A, with corresponding eigenvalue λ, and also an eigenvector of the matrix B, with corresponding eigenvalue μ. Show that **e** is an eigenvector of the matrix AB with corresponding eigenvalue $\lambda\mu$. ---[3]

(ii) Find the eigenvalues and corresponding eigenvectors of the matrix A, where
$$A = \begin{pmatrix} 0 & 1 & 3 \\ 3 & 2 & -3 \\ 1 & 1 & 2 \end{pmatrix}.$$ ---[6]

(Continued→)

(Continued→)

14(iii) The matrix B, where
$$B=\begin{pmatrix}3 & 6 & 1\\ 1 & -2 & -1\\ 6 & 6 & -2\end{pmatrix}$$
has eigenvectors $\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}$. Find the eigenvalue of the matrix AB, and state corresponding eigenvectors. ---[4]

$\boxed{W-17/11/Q11}$

15(i) Write down the eigenvalues of the matrix A, where
$$A=\begin{pmatrix}-2 & 1 & 1\\ 0 & -1 & 2\\ 0 & 0 & 1\end{pmatrix}$$
and find corresponding eigenvectors. ---[4]

(ii) Find a Matrix P and a diagonal matrix D such that $P^{-1}AP=D$, and hence find the matrix A^n, where n is a positive integer. $\boxed{S-16/11/Q10}$ --[8]

16(i) Find the two values of the constant k, the equations, have no unique solution.
$$kx+y+z=2$$
$$x+ky+z=-1$$
$$x+y+kz=-1$$
---[4]

(ii) Show that, for one of these values of k, the equations have no solution, and solve the equations for the other value of k. $\boxed{S-16/13/Q3}$ --[3]

17(i) It is given that 1 and 4 are eigenvalues of matrix A, where
$$A=\begin{pmatrix}1 & -3 & -3\\ -8 & 6 & -3\\ 8 & -2 & 7\end{pmatrix}$$
Find eigenvectors corresponding to each of these eigenvalues. --[3]

(Continued→)

(Continued →)

17 (ii) Given further that $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of A, find the corresponding eigenvalues. --[2]

(iii) Write down matrices P and D such that $P^{-1}AP = D$, where D is a diagonal matrix, and find P^{-1} --[5]

(iv) Write down a matrix C such that $C^2 = D$, and deduce a matrix B such that $B^2 = A$. $\boxed{S\text{-}16/13/Q\,11}$ --[4]

18. Find a matrix A whose eigenvalues are $-1, 1, 2$ and for which corresponding eigenvectors are,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$ --[7]

respectively.

$\boxed{W\text{-}16/11/Q3}$

19. Find the value of constant k for which the system of equations:

$$2x - 3y + 4z = 1$$
$$3x - y \quad\quad = 2$$
$$x + 2y + kz = 1$$

(i) does not have a unique solution. --[2]

(ii) For this value of k, solve the system of equations. --[4]

$\boxed{S\text{-}15/11/Q2}$

20. The matrix A is given by. $A \begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix}$

(i) The matrix A has an eigenvector $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Find the corresponding eigenvalue --[2]

(ii) The matrix A also has eigenvalues 4 and 6. Find the corresponding eigenvectors. --[3]

(iii) Hence find a matrix P such that $A = PDP^{-1}$, where D is a diagonal matrix which is to be determined. --[2]

(iv) The matrix B is such that $B = QAQ^{-1}$ where $Q = \begin{pmatrix} 4 & 11 & 5 \\ 1 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

By using the expression PDP^{-1} for A, find the set of eigenvalues and a corresponding set of eigenvectors for B. $\boxed{S\text{-}15/11/Q\,10}$ --[5]

21. One of the eigenvalues of the matrix M, where

$$M = \begin{pmatrix} 3 & -4 & 2 \\ -4 & \alpha & 6 \\ 2 & 6 & -2 \end{pmatrix},$$

is -9, find the value of α. --[3]

Find

(i) the other two eigenvalues, λ_1 and λ_2 of M, where $\lambda_1 > \lambda_2$ ---[5]

(ii) corresponding eigenvectors for all three eigenvalues of M. --[3]

It is given that $x = a e_1 + b e_2$, where e_1 and e_2 are eigenvectors of M corresponding to the eigenvalues λ_1 and λ_2 respectively, and a and b are scalar constants. Show that $Mx = p e_1 + q e_2$, expressing p and q in terms of a and b. --[3]

$\boxed{S\text{-}15/13/Q11}$

22. The matrix A, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 10 & -7 & 10 \\ 7 & -5 & 8 \end{pmatrix}$$

(i) has eigenvalues 1 and 3, find the corresponding eigenvectors. ---[3]

(ii) It is given that $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of A. Find the corresponding eigenvalue. --[2]

(iii) Find a diagonal matrix D and matrices P and P^{-1}, such that $P^{-1}AP = D$. $\boxed{W\text{-}15/11/Q6}$ --[5]

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(FP-2)

Matrices

Answers

1 (a)(i) $\begin{vmatrix} 1 & -2 & -2 \\ 2 & a-9 & -10 \\ 3 & -6 & 2a \end{vmatrix} = 0$

$\Rightarrow 2a(a-9) - 60 + 2(4a+30) - 2(-12 - 3a + 27) = 0$

$\Rightarrow 2a^2 - 4a - 30 = 0 \Rightarrow a = 5$, or -3.

for Unique Solu. $a \neq 5,\ a \neq -3$

(ii) Ist and IIIrd equations:

$x - 2y - 2z + 7 = 0$

and $x - 2y - 2z + \frac{29}{3} = 0$

are inconsistent $\Rightarrow$ No solution

These two equations represent parallel planes, other equations represent non-parallel planes which intersect each other in two lines.

(b) (i) $\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(\lambda-2)(\lambda-3) = 0$

$\Rightarrow$ eigenvalues are, $\lambda = 1, 2, 3$.

(ii) A satisfies the characteristic equation, $A^3 - 6A^2 + 11A - 6I = 0$

multiply throughout by A^{-1}

$\Rightarrow 6A^{-1} = A^2 - 6A + 11I$

$\therefore A^{-1} = \frac{1}{6}\begin{pmatrix} 4 & -1 & 2 \\ -2 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix}$ ✓

$\left[\text{as } A^2 = \begin{pmatrix} -1 & 5 & 10 \\ -2 & 6 & 10 \\ -4 & 4 & 9 \end{pmatrix} \right]$

2 (a) $\begin{vmatrix} 3 & 1 & 1 \\ a & 6 & -1 \\ 0 & a & -2 \end{vmatrix} = 0$

$\Rightarrow a^2 + 5a - 36 = 0 \Rightarrow a = 4, -9$

(b) $(\lambda-3)(\lambda-6)(\lambda+2)$

$= \lambda^3 - 7\lambda^2 + 36 = 0$

$36I = 7A^2 - A^3$

$\Rightarrow 36(A^{-1})^2 = 7I - A$

$\Rightarrow (A^{-1})^2 = \frac{1}{36}\begin{pmatrix} 4 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 9 \end{pmatrix}$

(c) Eigenvalues of A are 3, 6 and -2

$\lambda = 3$: $\begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 0 & 3 & -1 \end{vmatrix} = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = 6$: $\begin{vmatrix} i & j & k \\ -3 & 1 & 1 \\ 0 & 0 & -1 \end{vmatrix} = \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix}$

$\lambda = -2$: $\begin{vmatrix} i & j & k \\ 5 & 1 & 1 \\ 0 & 8 & -1 \end{vmatrix} = \begin{pmatrix} -9 \\ 5 \\ 40 \end{pmatrix}$

Thus $P = \begin{pmatrix} 1 & 1 & -9 \\ 0 & 3 & 5 \\ 0 & 0 & 40 \end{pmatrix}$

and $D = \begin{pmatrix} 243 & 0 & 0 \\ 0 & 7776 & 0 \\ 0 & 0 & -32 \end{pmatrix} = \begin{pmatrix} 3^5 & 0 & 0 \\ 0 & 6^5 & 0 \\ 0 & 0 & -2^5 \end{pmatrix}$

Answers

3(a) $A = \begin{pmatrix} 5 & -1 & 7 \\ 0 & 6 & 0 \\ 7 & 7 & 5 \end{pmatrix}$

$\begin{vmatrix} 5-\lambda & -1 & 7 \\ 0 & 6-\lambda & 0 \\ 7 & 7 & 5-7 \end{vmatrix} = 0$

$-\lambda^3 + 16\lambda^2 - 36\lambda - 144 = 0$

$\Rightarrow (\lambda+2)(\lambda-6)(\lambda-12) = 0$

$\Rightarrow \lambda = -2, 6, 12$ are eigenvalue

(b) $-A^3 + 16A^2 - 36A - 144I = 0$

$144A^{-1} = -A^2 + 16A - 36I$

$A^2 = \begin{pmatrix} 74 & 38 & 70 \\ 0 & 36 & 0 \\ 70 & 70 & 74 \end{pmatrix}$

$\Rightarrow A^{-1} = \frac{1}{24}\begin{pmatrix} -5 & -9 & 7 \\ 0 & 4 & 0 \\ 7 & 7 & -5 \end{pmatrix}$

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4 (i) $A^2 e = A(Ae) = \lambda Ae = \lambda^2 e$

(ii) Eigen values of A are n, $2n$ and $3n$

$\lambda = n$: $e_1 = \begin{vmatrix} i & j & k \\ 0 & 1 & 3 \\ 0 & n & 0 \end{vmatrix} = \begin{pmatrix} -3n \\ 0 \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = 2n$; $e_2 = \begin{vmatrix} i & j & k \\ -n & 1 & 3 \\ 0 & 0 & n \end{vmatrix} = \begin{pmatrix} n \\ n^2 \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ n \\ 0 \end{pmatrix}$

$\lambda = 3n$; $e_3 = \begin{vmatrix} i & j & k \\ -2n & 1 & 3 \\ 0 & -n & 0 \end{vmatrix} = \begin{pmatrix} 3n \\ 0 \\ 2n^2 \end{pmatrix} = t\begin{pmatrix} 3 \\ 0 \\ 2n \end{pmatrix}$

Eigen values of $A + nI$ are $2n$, $3n$ and $4n$.

Thus $P = \begin{pmatrix} 1 & 1 & 3 \\ 0 & n & 0 \\ 0 & 0 & 2n \end{pmatrix}$

and

$D = \begin{pmatrix} (2n)^2 & 0 & 0 \\ 0 & (3n)^2 & 0 \\ 0 & 0 & (4n)^2 \end{pmatrix}$

(continued →)

5(i) $P = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & b & -1 \end{pmatrix}$; $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$\det P = -b-1$

$P^{-1} = \frac{1}{b+1}\begin{pmatrix} b & -1 & -b \\ 1 & 1 & b \\ 1 & 1 & -1 \end{pmatrix}^T = \frac{1}{b+1}\begin{pmatrix} b & 1 & 1 \\ -1 & 1 & 1 \\ -b & b & -1 \end{pmatrix}$

$A = PDP^{-1}$

$A = \frac{1}{b+1}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & b & -1 \end{pmatrix}\begin{pmatrix} 2b & 2 & 2 \\ -1 & 1 & 1 \\ -3b & 3b & -3 \end{pmatrix}$

$= \frac{1}{b+1}\begin{pmatrix} 1+2b & 1 & 1 \\ -b & 3b+2 & -1 \\ 2b & -2b & 3+b \end{pmatrix}$ ✓

(ii) $A^{-1}\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = 2A^{-1}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

(iii) $A^n\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 2^2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow n = 2$ ✓

$A^n\begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ b^{-1} \end{pmatrix} \Rightarrow b = b^{-1} \Rightarrow b = 1$ ✓

Answers

6(i) Eigenvalues of (upperdiagonal matrix) A are 2, m, 1

(or from characteristic equation: $(\lambda-2)(\lambda-m)(\lambda-1)=0$)

$$\lambda=2: e_1=\begin{vmatrix} i & j & k \\ 0 & m-2 & 1 \\ 0 & 0 & -1 \end{vmatrix}=\begin{pmatrix} 2-m \\ 0 \\ 0 \end{pmatrix}=t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda=m: e_2=\begin{vmatrix} i & j & k \\ 2-m & m & 1 \\ 0 & 0 & 7 \end{vmatrix}=\begin{pmatrix} 7m \\ 7(m-2) \\ 0 \end{pmatrix}=t\begin{pmatrix} m \\ m-2 \\ 0 \end{pmatrix}.$$

$$\lambda=1: e_3=\begin{vmatrix} i & j & k \\ 0 & m-2 & -7 \\ 0 & 0 & m-1 \end{vmatrix}=\begin{pmatrix} 6m+1 \\ -7 \\ m-1 \end{pmatrix}=t\begin{pmatrix} 6m+1 \\ -7 \\ m-1 \end{pmatrix}$$

Thus $P=\begin{pmatrix} 1 & m & 6m+1 \\ 0 & m-2 & -7 \\ 0 & 0 & m-1 \end{pmatrix}$

and $D=\begin{pmatrix} 2 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(ii) $M^7P=PD^7P^{-1}P=PD^7$

$$=\begin{pmatrix} 1 & m & 6m+1 \\ 0 & m-2 & -7 \\ 0 & 0 & m-1 \end{pmatrix}\begin{pmatrix} 2^7 & 0 & 0 \\ 0 & m^7 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 2^7 & m^8 & 6m+1 \\ 0 & m^8-2m^7 & -7 \\ 0 & 0 & m-1 \end{pmatrix}$$

7(i) $2e$ [any scalar multiple of e, μe, $\mu\neq 0,1$]

(ii) Eigenvector e, Eigenvalue λ^n.

(iii) Eigenvalues (diagonal Matrix) $A: \lambda=3,7,1$

$$\lambda=3: e_1=\begin{vmatrix} i & j & k \\ 2 & 4 & 0 \\ 4 & 8 & -2 \end{vmatrix}=\begin{pmatrix} -8 \\ 4 \\ 0 \end{pmatrix}=t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda=7: e_2=\begin{vmatrix} i & j & k \\ -4 & 0 & 0 \\ 4 & 8 & -6 \end{vmatrix}=\begin{pmatrix} 0 \\ -24 \\ -32 \end{pmatrix}=t\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$$

$$\lambda=1: e_3=\begin{vmatrix} i & j & k \\ 2 & 6 & 0 \\ 4 & 8 & 0 \end{vmatrix}=\begin{pmatrix} 0 \\ 0 \\ -8 \end{pmatrix}=t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus $P=\begin{pmatrix} -2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 1 \end{pmatrix}$ and $D=\begin{pmatrix} 3^n & 0 & 0 \\ 0 & 7^n & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(iv) $$\sum_{n=1}^{N}\left(k^nA^n-k^{n+1}A^{n+1}\right)=kA-k^{N+1}A^{N+1}$$

$k^{N+1}\cdot A^{N+1}\to 0$ as $N\to\infty$

for $-\frac{1}{7}<k<\frac{1}{7}$ ✓

Answers

8 (i) $Ae = \lambda e$

$A^3 e = A^2(Ae) = \lambda A(Ae) = \lambda^2(Ae) = \lambda^3 e$

So eigenvalue is λ^3.

Special case: states eigenvalue is λ^3 B1

(ii) Eigenvalues of A are 2 and 3

Eigenvectors of A are $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$

So $P = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$

and $D = \begin{pmatrix}2^3+1 & 0\\ 0 & 3^3+1\end{pmatrix} = \begin{pmatrix}9 & 0\\ 0 & 28\end{pmatrix}$

Columns of P and D can be permuted but must match.

9 (i) $Ae = \lambda e$ and $Be = \mu e$

$\Rightarrow Ae + Be = \lambda e + \mu e$

$\Rightarrow (A+B)e = (\lambda+\mu)e$ ✓

(ii) $\begin{pmatrix}2 & 0 & 1\\ -1 & 2 & 3\\ 1 & 0 & 2\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}3\\6\\3\end{pmatrix}$

$\Rightarrow \lambda = 3k \Rightarrow \lambda = 1, 2$ ✓

(iii) $P = \begin{pmatrix}1 & 1 & 0\\ 2 & 4 & 1\\ 1 & -1 & 0\end{pmatrix}$

$D = \begin{pmatrix}7^3 & 0 & 0\\ 0 & 6^3 & 0\\ 0 & 0 & 3^3\end{pmatrix} = \begin{pmatrix}343 & 0 & 0\\ 0 & 216 & 0\\ 0 & 0 & 27\end{pmatrix}$ ✓

10 (i) 2

(ii) negative eigenvalue $= -2$

$A + 2I = \begin{pmatrix}4 & 3 & 1\\ 0 & 0 & 1\\ 0 & 0 & 3\end{pmatrix}$ $\begin{vmatrix} i & j & k\\ 4 & 3 & 1\\ 0 & 0 & 1\end{vmatrix}$

$\begin{pmatrix}3\\-4\\0\end{pmatrix}$ ✓

(iii) An eigenvalue of $A + A^6$ is $2 + 2^6 = 66$, 62 or 2

corresponding eigenvector is

$\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}3\\-4\\0\end{pmatrix}$ or $\begin{pmatrix}-6\\1\\3\end{pmatrix}$

11 (i) $\begin{vmatrix}1 & 3 & k\\ 2 & -1 & -5\\ 1 & 1 & 2\end{vmatrix} = 0 \Rightarrow \underline{k = 8}$ ✓

(ii) $x + 3y + 8z = 4$ —— ①

$4x - 2y - 10z = -5$ —— ②

from ① and ② $y = -3x$

$\therefore x + y + 2z = 1$ —— ③

Put $x = t$ and $y = -3t$ in ③

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\\frac{1}{2}\end{pmatrix} + t\begin{pmatrix}1\\-3\\1\end{pmatrix}$

or $\begin{pmatrix}\frac{1}{-2}\\\frac{3}{2}\\0\end{pmatrix} + t\begin{pmatrix}1\\-3\\1\end{pmatrix}$ ✓

Answers

12. (i) Eigenvectors are $i+2j+2k$, $i+j$ and $j+k$ for $\lambda = 1, -1, -2$ respectively.

(ii) Eigenvalues for B are $-1, -3$ and -4. Eigenvectors are the same as for A respectively.

13. (i)
$$\begin{pmatrix} -6 & -8 & 7 \\ 7 & -9 & 7 \\ 6 & -6 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} \Rightarrow \lambda = -2$$

(ii) $\lambda_2 = -1 \Rightarrow e_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

(iii) $\text{Det } A = 10 \Rightarrow \lambda_3 = 5$
$$\Rightarrow e_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(iv) $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ $D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

$\text{Det } P = -1$, $\text{Adj } P = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix}$

$$\Rightarrow P^{-1} = \begin{pmatrix} -1 & 2 & -1 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$A^n = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & - & 1 \end{pmatrix}\begin{pmatrix} (-2)^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 5^n \end{pmatrix}\begin{pmatrix} -1 & 2 & -1 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \ldots = \begin{pmatrix} 5^n + (-1)^n - (-2)^n & 2\cdot(-2)^n + (-1)^{n+1} - 5^n & 5^n - (-2)^n \\ 5^n - (-2)^n & 2\cdot(-2)^n - 5^n & 5^n - (-2)^n \\ 5^n - (-1)^n & (-1)^n - 5^n & 5^n \end{pmatrix}$$

14. (i) $Ae = \lambda e$ and $Be = \mu e$

$ABe = A\mu e = \mu Ae = \mu\lambda e = \lambda\mu e$

(ii) $(\lambda+1)(\lambda^2 - 5\lambda + 6) = 0$

$(\lambda+1)(\lambda-2)(\lambda-3) = 0 \Rightarrow \lambda = -1, 2, 3$

Eigenvectors are:
$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ resp.}$$

(iii)
$$\begin{pmatrix} 3 & 6 & 1 \\ 1 & -2 & -1 \\ 6 & 6 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix} \Rightarrow \mu_1 = -3$$

Eigenvalues of AB are 3, -4 and 12. Corresponding eigenvector are;
$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Answers

15 (i) Eigenvalues are; $-2, -1, 1$

eigenvectors are; $\begin{pmatrix}1\\0\\0\end{pmatrix}; \begin{pmatrix}1\\1\\0\end{pmatrix}; \begin{pmatrix}0\\1\\1\end{pmatrix}$

(ii) $A = \begin{pmatrix}-2 & 1 & 1\\0 & -1 & 2\\0 & 0 & 1\end{pmatrix}$

$P = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}; \quad D = \begin{pmatrix}-2 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}$

$P^{-1} = \begin{pmatrix}1 & -1 & 1\\0 & 1 & -1\\0 & 0 & 1\end{pmatrix}$

$(P^{-1}AP)^n = P^{-1}A^nP = D^n$

$$\Rightarrow A^n = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}(-2)^n & 0 & 0\\0 & (-1)^n & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & -1 & 1\\0 & 1 & -1\\0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}(-2)^n & (-2)^n & (-2)^n\\0 & (-1)^n & (-1)^{n+1}\\0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}(-2)^n & -(-2)^n+(-1)^n & (-2)^n+(-1)^{n+1}\\0 & (-1)^n & (-1)^{n+1}+1\\0 & 0 & 1\end{pmatrix}$$

16(i) $\begin{vmatrix}K & 1 & 1\\1 & K & 1\\1 & 1 & K\end{vmatrix} = 0 \Rightarrow K^3 - 3K + 2 = 0$

(ii) $\Rightarrow (K-1)^2(K+2) = 0 \Rightarrow K = 1, -2$

$K=1$: $x+y+z=2$ and $x+y+z=-1$ } Inconsistent

$K=-2$; $\Rightarrow x = z-1$

Hence $(x,y,z) = [(-1+t), t, t]$

17(i) $\lambda = 1$: $\begin{vmatrix}i & j & k\\0 & -3 & -3\\-8 & 5 & -3\end{vmatrix} = \begin{pmatrix}24\\24\\-24\end{pmatrix} \sim \begin{pmatrix}1\\1\\-1\end{pmatrix}$

$\lambda = 4$: $\begin{vmatrix}i & j & k\\-3 & -3 & -3\\-8 & 2 & -3\end{vmatrix} = \begin{pmatrix}15\\15\\-30\end{pmatrix} \sim \begin{pmatrix}1\\1\\-2\end{pmatrix}$

(ii) $\begin{pmatrix}1 & -3 & -3\\-8 & 6 & -3\\8 & -2 & 7\end{pmatrix}\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\9\\-9\end{pmatrix}$

$\Rightarrow \lambda = 9$ ✓

(iii) $P = \begin{pmatrix}1 & 1 & 0\\1 & 1 & 1\\-1 & -2 & -1\end{pmatrix}, D = \begin{pmatrix}1 & 0 & 0\\0 & 4 & 0\\0 & 0 & 9\end{pmatrix}$

$C = \begin{pmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{pmatrix}$

$P^{-1}AP = C^2 \Rightarrow A = PC^2P^{-1} = PCP^{-1} \cdot PCP^{-1}$

$(B = PCP^{-1})$

$PC = \begin{pmatrix}1 & 2 & 0\\1 & 2 & 3\\-1 & -4 & -3\end{pmatrix} \Rightarrow B = \begin{pmatrix}1 & -1 & 1\\-2 & 2 & -1\\2 & 0 & 3\end{pmatrix}$

$\text{Det}\, P = 1$; $\text{Adj}\, P = \begin{pmatrix}1 & 1 & 1\\0 & -1 & -1\\-1 & 1 & 0\end{pmatrix}$

$\Rightarrow P^{-1} = \begin{pmatrix}1 & 1 & 1\\0 & -1 & -1\\-1 & 1 & 0\end{pmatrix}$

(continued.)

Answers

18. $P=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$, $D=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&2\end{pmatrix}$

$P^{-1}AP=D \Rightarrow A=PDP^{-1}$

$P^{-1}=\begin{pmatrix}1&-1&1\\0&1&-1\\0&0&1\end{pmatrix}$

$PD=\begin{pmatrix}-1&1&0\\0&1&2\\0&0&2\end{pmatrix}$ or $DP^{-1}=\begin{pmatrix}-1&1&-1\\0&1&-1\\0&0&2\end{pmatrix}$

$A=\begin{pmatrix}-1&2&-2\\0&1&1\\0&0&2\end{pmatrix}$ ✓

19(i) $\begin{vmatrix}2&-3&4\\3&-1&0\\1&2&k\end{vmatrix}=0 \Rightarrow 7k=-28$
$\Rightarrow k=-4.$

(ii) Add Ist and 3rd $\Rightarrow 3x-y=2$

set $x=t$

$\Rightarrow y=3t-2,$

$z=\frac{7}{4}t-\frac{5}{4}$ ✓

20(i) $\begin{pmatrix}2&2&-3\\2&2&3\\-3&3&3\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}-3\\3\\-3\end{pmatrix} \Rightarrow$ ✓ $\lambda=-3$

(ii) $\lambda=4$: $\begin{vmatrix}i&j&k\\-2&2&-3\\-3&3&-1\end{vmatrix}=\begin{pmatrix}7\\7\\0\end{pmatrix}\sim\begin{pmatrix}1\\1\\0\end{pmatrix}$

$\lambda=6$: $\begin{vmatrix}i&j&k\\-4&2&-3\\-3&3&-3\end{vmatrix}=\begin{pmatrix}3\\-3\\-6\end{pmatrix}\sim\begin{pmatrix}1\\-1\\-2\end{pmatrix}$

(Continued →

$\Rightarrow P=\begin{pmatrix}1&1&1\\-1&1&-1\\1&0&2\end{pmatrix}$; $D=\begin{pmatrix}-3&0&0\\0&4&0\\0&0&6\end{pmatrix}$

(iii) $B=QAQ^{-1}=QPAP^{-1}Q^{-1};=QPA(QP)^{-1}$

$QP=\begin{pmatrix}-2&15&-17\\-1&5&-7\\0&3&-3\end{pmatrix}$

Eigenvalues are -3, 4 and 6

Eigenvectors are $\begin{pmatrix}2\\1\\0\end{pmatrix};\begin{pmatrix}15\\5\\3\end{pmatrix};\begin{pmatrix}17\\7\\3\end{pmatrix}$

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21. $\begin{vmatrix}12&-4&2\\-4&\alpha+9&6\\2&6&7\end{vmatrix}=0 \Rightarrow 80\alpha+80=0$
$\Rightarrow \alpha=-1$ ✓

(i) $\begin{vmatrix}3-\lambda&-4&2\\-4&-1-\lambda&6\\2&6&-2-\lambda\end{vmatrix}=0$

$\Rightarrow \lambda^3-63\lambda+162=0$

$\Rightarrow (\lambda+9)(\lambda-3)(\lambda-6)=0$

$\Rightarrow \lambda_1=6$ and $\lambda_2=3$ ✓

(ii) $\lambda=-9 \Rightarrow e\begin{pmatrix}1\\2\\-2\end{pmatrix}$

$\lambda_1=6 \Rightarrow e_1=\begin{pmatrix}-2\\2\\1\end{pmatrix}$ and $\lambda_2=3 \Rightarrow e_2=\begin{pmatrix}2\\1\\2\end{pmatrix}$

$Mx=M(ae_1+be_2)=aMe_1+bMe_2$
$=a\lambda_1e_1+b\lambda_2e_2$
$=6ae_1+3be_2$ ✓

Answers

22. (i) $\lambda=1$: $\begin{vmatrix} i & j & k \\ 10 & -8 & 10 \\ 7 & -5 & 7 \end{vmatrix} = \begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda = 3$: $\begin{vmatrix} i & j & k \\ -2 & 0 & 0 \\ 10 & -10 & 10 \end{vmatrix} = \begin{pmatrix} 0 \\ 20 \\ 20 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 0 & 0 \\ 10 & -7 & 10 \\ 7 & -5 & 8 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix} = -2\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \lambda = -2$

(iii) $D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $P = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$

$\text{Det } P = -1$

$\text{Adj } P = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ -2 & 1 & 2 \end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 2 & -1 & -2 \end{pmatrix}$ ✓

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