NOTES

IG_Maths
0580

Algebra-3

Derived Functions Notes

Syllabus - 2020-22

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IG-Maths 0580

Derived Functions — Algebra-3

§1. Derived Functions:

Given a function $y = f(x)$,
Then derived function denoted by $\frac{dy}{dx}$ (or $f'(x)$) is a function, which gives the instant rate of change of $f(x)$ with respect to x.

Example: Consider $y = x^2$

$x \to x + \Delta x$
$y \to y + \Delta y$
$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$
or $\lim_{x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

for $y = x^2$:
Δx = small change in x
Δy = corresponding change in y
rate of change $= \frac{\Delta y}{\Delta x}$

x	y	$x+\Delta x$	$y+\Delta y$	Δx	Δy	$\frac{\Delta y}{\Delta x}$
1	1	1.001	1.002001	0.001	0.002001	$\frac{0.002001}{0.001} = 2.001 = 2\times 1$ approx
2	4	2.001	4.004001	0.001	0.004001	$\frac{0.004001}{0.001} = 4.001 = 2\times 2$
3	9	3.001	9.006001	0.001	0.006001	$\frac{0.006001}{0.001} = 6.001 = 2\times 3$ approx.

In general:
for $y = x^2$, $\frac{dy}{dx} = 2x$
or $\left(\frac{d}{dx} x^2 = 2x\right)$

$\left(\frac{dy}{dx}\right)_{x=1} = 2\times 1 = 2$
$\left(\frac{dy}{dx}\right)_{x=2} = 2\times 2 = 4$
$\left(\frac{dy}{dx}\right)_{x=3} = 2\times 3 = 6$

Again for $y = x^3$

x	y	$x+\Delta x$	$y+\Delta y$	Δx	Δy	$\frac{\Delta y}{\Delta x}$
1	1	1.001	1.003003	0.001	0.003003	$\frac{0.003003}{0.001} = 3.003 = 3\times 1^2$ approx.
2	8	2.001	8.012006	0.001	0.012006	$\frac{0.012006}{0.001} = 12.006 = 3\times 2^2$
3	27	3.001	27.027009	0.001	0.027009	$\frac{0.027009}{0.001} = 27.009 = 3\times 3^2$ approx.

$y = x^3$
$\therefore$ Here $\frac{dy}{dx} = 3x^2$ or $\left(\frac{d}{dx} x^3 = 3x^2\right)$ $\left[\left(\frac{dy}{dx}\right)_{x=2} = 3\times 2^2 = 12 \text{ etc}\right]$

§2. Differentiation Formula:

1(i) $\frac{d}{dx}x^n = n \cdot x^{n-1}$ or $\begin{cases} y = x^n \\ \frac{dy}{dx} = n \cdot x^{n-1} \end{cases}$ or $\begin{cases} f(x) = x^n \\ f'(x) = n \cdot x^{n-1} \end{cases}$

(ii) $y = a \cdot f(x)$

$\frac{dy}{dx} = a \cdot f'(x)$

(iii) $\frac{d}{dx}c = 0$ (c is a constant)

(iv) $\frac{d}{dx}ax = a$

(Note: n is a positive integer.)

(v) $\frac{d}{dx}ax^n = a \cdot nx^{n-1}$

Example 1: Find the derivative of the following:

(i) $y = x^5$

$\therefore \frac{dy}{dx} = 5x^4$

(ii) $y = ax + b$

$\therefore \frac{dy}{dx} = a$

(iii) $y = 7x^3$

$\therefore \frac{dy}{dx} = 7 \times 3x^2 = 21x^2$

(iv) $y = 3x^7 + 5x^4 - 4x^3 + 9$

$\frac{dy}{dx} = 3 \times 7x^6 + 5 \times 4x^3 - 4 \times 3x^2 + 0$

$= 21x^6 + 20x^3 - 12x^2$

(v) $f(x) = \frac{3}{2}x^6$ find $f'(x)$ at $x = 2$

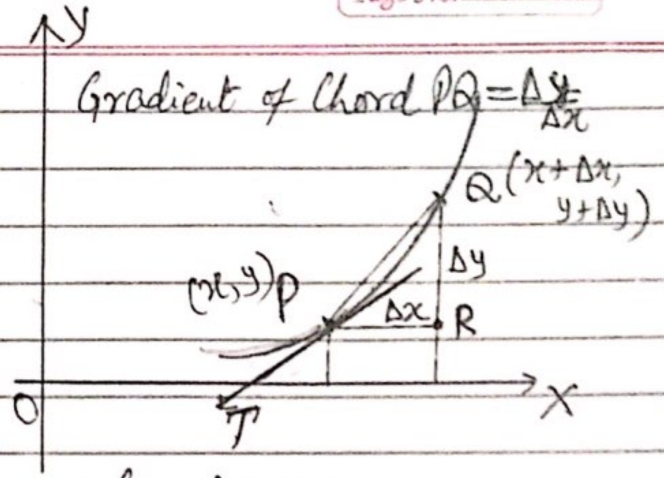
$f'(x) = \frac{3}{2} \times 6x^5 = 9x^5$

at $x = 2$, $f'(2) = 9 \times 2^5 = 288$ ✓

§3. Geometric Meaning of $\frac{dy}{dx}$:

Given a function $y = f(x)$.
Then $\frac{dy}{dx}$ or $f'(x)$ denotes the

Gradient of the tangent to the curve $y=f(x)$ at any point (x,y) on the curve.



when $\Delta x \to 0$
$\Delta y \to 0$
Chord PQ → tangent 'PT' at P

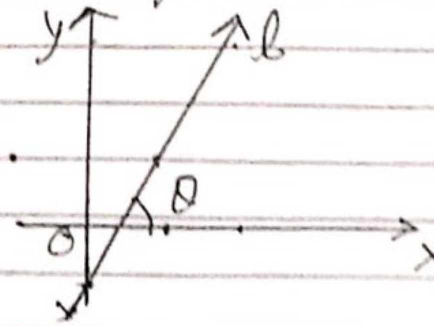
$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

= Gradient of the tangent to curve at any point (x,y).

§4. Gradient of line:

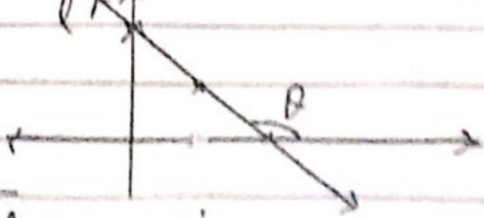
Given the equation of a line $y = mx + c$

differentiating, $\frac{dy}{dx} = m$ = gradient of line.

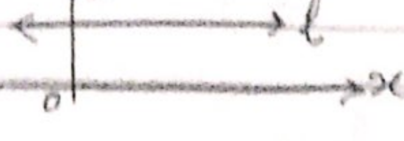


Note: Case I. if $m > 0$
Example $y = 2x - 1$
$\frac{dy}{dx} = 2 > 0$
line is inclined at, $0 < \theta < 90$
an acute angle with x-axis.

Case II. If $m < 0$, line is inclined at an obtuse angle.
Example: $y = -x + 2$
$\frac{dy}{dx} = -1$
$90 < \theta < 180$.



Case III $m = 0$, line is parallel to x-axis.
Eqn of line $y = 1$, $\frac{dy}{dx} = 0$
$l \parallel$ x-axis.



Example2: Find the gradients of the following curves, at given point on the curve.

(i) $y = 3x^2 - 5x + 7$ at $(1,5)$

Solution:

diff. $\frac{dy}{dx} = 6x - 5$

at $(1,5)$. $\left(\frac{dy}{dx}\right)_{x=1} = 6\times1 - 5 = 1$

$\therefore$ Gradient of the tangent to the curve at $(1,5)$ is $\underline{1}$.

(ii) $y = x^3 - 3x^2 + 5x + 2$ at $(2,8)$

$\frac{dy}{dx} = 3x^2 - 6x + 5$

$\left(\frac{dy}{dx}\right)_{(2,8)} = 3\times2^2 - 6\times2 + 5 = 5$

(iii) $y = 5 + 3x - 3x^3$ at $x = 1$

$\frac{dy}{dx} = 0 + 3 - 3\times3x^2$

$= 3 - 9x^2$

$\left(\frac{dy}{dx}\right)_{x=1} = 3 - 9\times1^2 = \underline{-6}$

Example3: Find the equation of tangent to the curve,

$y = x^2 + 1$ at $(1,2)$

$\frac{dy}{dx} = 2x$

Gradient of tangent $m = \left(\frac{dy}{dx}\right)_{(1,2)} = 2\times1 = 2$

$\therefore$ Equation of tangent at $(1,2)$

$y - 2 = 2(x - 1)$

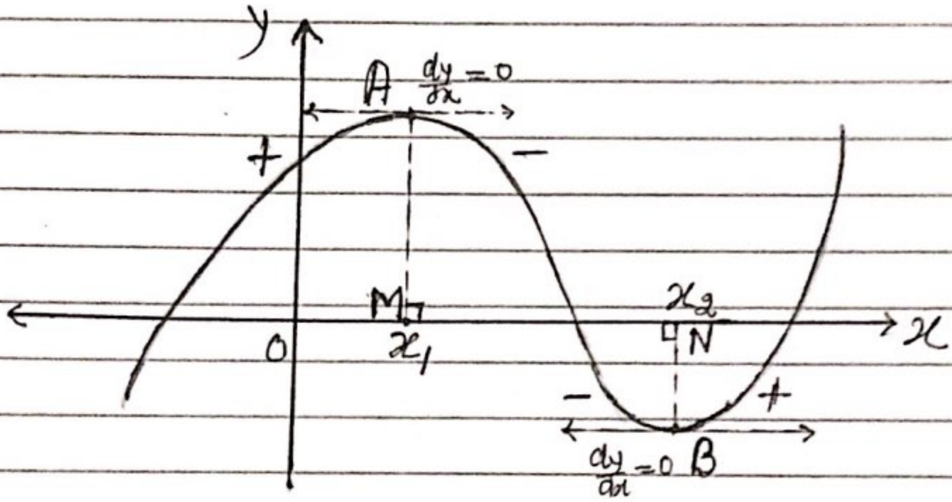
or $\underline{y = 2x}$

Equation of tangent at (x_1, y_1)

$y - y_1 = m(x - x_1)$

§ Turning points (or Stationary points) and Maxima and minima:

Given a function: $y = f(x)$



Let us observe the graph of the function $y=f(x)$.

The points A and B on the curve are called the turning points (or stationary points) or the points of maxima/minima,

(i) Tangents to curve at A (and B) are parallel to x-axis. or gradient $m = f'(x) = 0$

(ii) Point A is maxima, gradient of the curve changes from + to −ve as we cross point A, from left to right,

(iii) Point B is minimum, gradient of the curve changes from −ve to +ve as we cross the point B, from left to right.

(iii)' [If $\frac{dy}{dx}$ does not change the sign, then no max/Min (we have a point of Inflexion)]

(iv) To find the turning points on a curve $y=f(x)$, find $f'(x)$ and solve $f'(x)=0$,
Let the solution (turning point) are $x_1, x_2, --$ then test at x_1/x_2 one by one the change in gradient.

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Derived Functions.

Turning points, Maxima/Minima

Example 4: A curve has equation $y = x^3 - 6x^2 + 16$

(a) Find the coordinates of the two turning point. ---[6]

(b) Determine whether each of the turning points is a maximum or a minimum, give reasons for your answers. [3]

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Solution: Given the equation of curve,

$$y = x^3 - 6x^2 + 16 \quad \text{---(i)}$$

differentiating $\frac{dy}{dx} = 3x^2 - 12x$ ——

$$= 3x(x-4) \quad \text{---(ii)}$$

for any turning point, $\frac{dy}{dx} = 0$

from (ii) $3x(x-4=0$

$\Rightarrow$ $x = 0$ or $x = 4$ ——(iii)

from (i) Turning points $(0,16)$ and $(4,-16)$ ✓

Testing for Max/Min:

(i) for Point $(0,16)$

take a value $x<0$, let -0.1 put in (ii)

$$\left(\frac{dy}{dx}\right)_{x=-.1} = 3(-0.1)(-0.1-4) = 3\times0.1\times4.1 > 0$$

Again $x>0$, let $x = 0.1$ $\left(\frac{dy}{dx}\right)_{x=0.1} = 3(0.1)(0.1-4) = 3\times0.1\times(-3.9) < 0$

$\therefore$ At $x=0$ the gradient changes from + to − as we cross from left to right, $\therefore$ Maxima at $(0,16)$ ✓

(ii) Now at $(4,-16)$

Take $x<4$, let $x = 3.9$ put in (ii)

$$\left(\frac{dy}{dx}\right)_{x=3.9} = 3\times3.9(3.9-4) = 3\times3.9(-0.1) < 0$$

Again $x>4$ let $x = 4.1$ put in (ii)

$$\left(\frac{dy}{dx}\right)_{x=4.1} = 3\times4.1(4.1-4) = 3\times4.1\times0.1 > 0$$

$\therefore$ A $x = 4$ the gradient changes from −ve to +ve as we cross from left to right, $\therefore$ Minima at $(4,-16)$ ✓

Turning points, Maxima/Minima/Point of Inflexion

Example 5: Find all the stationary points and hence determine whether at these point are max/min, $f(x)=x^3-6x^2+12x$

Solution: Given:

$$f(x)=x^3-6x^2+12x \quad \text{---(i)}$$

differentiating, $f'(x)=3x^2-12x+12$

or $= 3(x^2-4x+4)$

or $f'(x) = 3(x-2)^2$ ——— (ii)

Now for turning point $f'(x)=0$

or $3(x-2)^2=0 \Rightarrow x=2$ ✓

Now To Check Max/Min at $x=2$. from (ii)

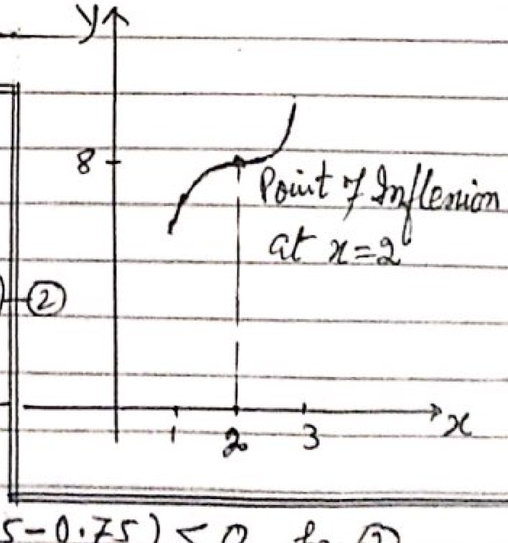
take a point on the left of $x=2$, $x=1.9 \Rightarrow f'(1.9)=3(1.9-2)^2>0$

Again point on the right of $x=2$, $x=2.1 \Rightarrow f'(2.1)=3(2.1-2)^2>0$

here we find that $f'(x)$ **doesnot** change sign as we cross $x=2$ from left to right.

Hence $x=2$ is neither a maxima nor a minima.

It is a point of Inflexion.



Example 6: Given $y=2x^2-3x+5$ find the maxima/minima.

Solution $y=2x^2-3x+5$ —(1)

diff. $\frac{dy}{dx}=2\times 2x-3\times 1=4x-3=4(x-\frac{3}{4})$ (2)

for turning point $\frac{dy}{dx}=0 \Rightarrow 4x-3=0$

$\Rightarrow x=\frac{3}{4}$ ✓ $=0.75$

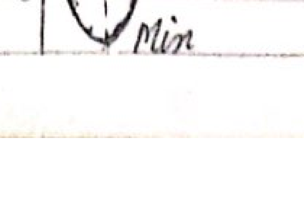
Check Max/Min at $x=3/4=0.75$

choose $x<0.75$ let $x=0.5$ ∴ $\left(\frac{dy}{dx}\right)_{x=0.5}=4(0.5-0.75)<0$ from (2)

Again $x>0.75$ let $x=1$, $\left(\frac{dy}{dx}\right)_{x=1}=4(1-0.75)>0$ from (2)

∴ $\frac{dy}{dx}$ changes sign from −ve to + at $x=\frac{3}{4}$

∴ There is Minima at $x=\frac{3}{4}$ ✓



Derived Functions

Turning point (Stationary Point), Maxima/Minima.

Example 7: A curve has equation $y = x^3 - 6x^2 + 9x$

(a) Find the coordinates of two turning points. --- [6]

(b) Determine whether each of the turning points is a maximum or a minimum, Give reasons for your answers. --- [3]

(C) Find the maximum and the minimum values. -- [2]

Solution: Given $y = x^3 - 6x^2 + 9x$ ——— (i)

(a) differentiating, $\frac{dy}{dx} = 3x^2 - 12x + 9$

$= 3(x^2 - 4x + 3)$

or $\frac{dy}{dx} = 3(x-1)(x-3)$ ——— (ii)

For turning points $\frac{dy}{dx} = 0 \Rightarrow 3(x-1)(x-3) = 0$

$\Rightarrow x = 1, 3$ are the turning pots.

(b) Check at $x=1$,

Consider value of x, $x<1$, let $x = 0.9$

frm (ii) $\left(\frac{dy}{dx}\right)_{x=0.9} = 3(0.9-1)(0.9-3) = + > 0$

and value of x, $x>1$, let $x = 1.1$

frm (ii) $\left(\frac{dy}{dx}\right)_{x=1} = 3(1.1-1)(1.1-3) = - < 0$

$\therefore$ at $x=1$, $\frac{dy}{dx}$ changes sign from + to -ve as crosses from left to right of $x=1$, $\therefore$ Max at $x=1$ ✓

Again at $x=3$:

Consider $x<3$, let $x = 2.9$, frm (ii) $\left(\frac{dy}{dx}\right)_{2.9} = 3(2.9-1)(2.9-3) = -ve < 0$

and $x>3$, let $x = 3.1$, frm (ii) $\left(\frac{dy}{dx}\right) x=3.1 = 3(3.1-1)(3.1-3) = + > 0$

$\therefore$ As we cross $x=3$ from left to right $\frac{dy}{dx}$ changes sign from -ve to +, $\therefore$ Minimum at $x=3$ ✓

(c) frm (i) $x = 1$; $y = 1 - 6 + 9 = 4$, $\therefore$ Max. Value at $x=1$ is $= 4$ ✓

frm (i) and for $x=3$, $y = 27 - 54 + 27 = 0$, $\therefore$ at $x=3$, Min Value $= 0$ ✓