IG-0606 Additional Maths

Circular Measure Notes

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IG- Maths 0606

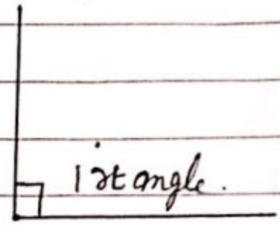
Circular measure

P-1

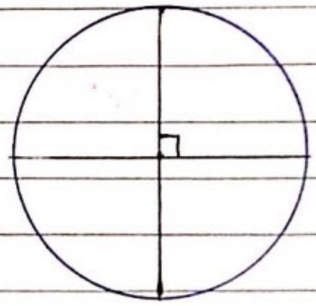
§ Measurement of angles:

English System (or degree system). Unit = degree (°)

1 right angle = 90°



Complete angle at the centre of a circle = 4 rt angles = 360°



§ $\dfrac{\text{Circumference of a circle}}{\text{diameter of the circle}} = \pi$ (constant) $= 3.14159\cdots$ (an irrational number)

Circumference of a circle $= \pi d$ (d = diameter of circle)

or $C = 2\pi r$ (r = radius of circle).

§ Circular measure:

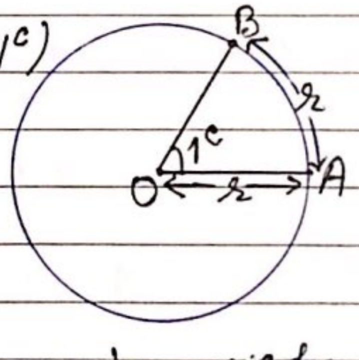
Unit = radian (or 1^c)

A radian is the angle subtended by an arc $\widehat{AB}$ of length 'r' at the centre of a circle of radius r.

length of arc $\widehat{AB} = r$

radius $OA = r$

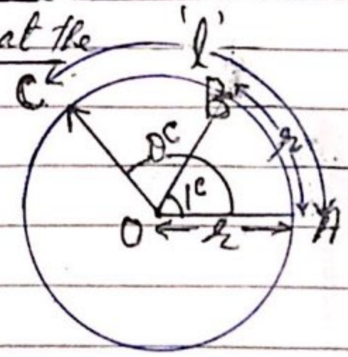
angle $AOB = 1$ radian (or 1^c) (^{c} for circular)



§ Angle subtended by an arc of length 'l' at the centre of a circle of radius 'r' is θ^c:

→ We know that the angles at the centre of a circle are proportional to the lengths of the corresponding arcs: $\dfrac{\angle AOC}{\angle AOB} = \dfrac{\text{arc } AC}{\text{arc } AB}$

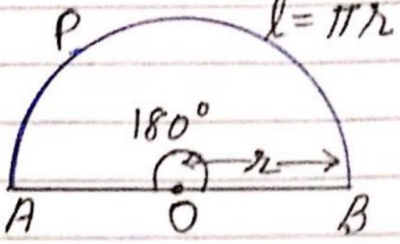
or $\boxed{\dfrac{\theta^c}{1^c} = \dfrac{l}{r}}$ or $\boxed{l = r\theta^c}$



Circular measure

§ Relation between angle measure in "Degree and Radian"

Consider a semicircle $\widehat{APB}$ of radius 'r'.



Length of Semicircular arc $\widehat{APB}$

$$l = \pi r$$

angle in circular measure $\theta^c = \frac{l}{r} = \frac{\pi \cancel{r}}{\cancel{r}}$

$\therefore \theta^c = \pi^c = 180°$ [straight angle]

$$\boxed{\pi^c = 180°}$$

or $\boxed{180° = \pi^c}$

$\pi^c = 180°$

$\therefore 1^c = \frac{180°}{\pi} = \frac{180°}{3.14159} = 57.2958°$

$\boxed{1° = \frac{\pi}{180}^c}$ $\boxed{1^c = 57.2958°}$

For some Particular Angles:

$30° = \frac{\pi}{3}^c$; $45° = \frac{\pi}{4}^c$; $60° = \frac{\pi}{3}^c$, $90° = \frac{\pi}{2}^c$

Example 1. Express angle $40°$ in radians.

$$40° = 40 \times \frac{\pi}{180}^c = 0.6981^c ✓$$

Example 2.

Express 1.4^c in degrees.

$$1.4^c = 1.4 \times \frac{180}{\pi}° = 80.214° ✓$$

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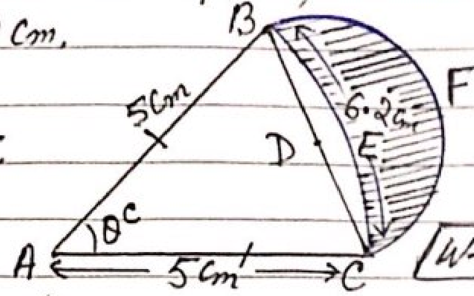
Circular measure

Example 3: The diagram shows an isosceles triangle ABC, where AB = AC = 5cm. The arc BEC is part of the circle with centre A and has length 6·2 cm. The point D is the mid point of the line BC. The arc BFC is a semicircle, centre D.



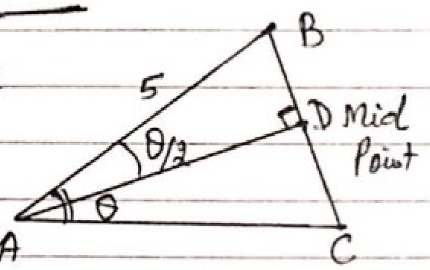
W-17/12/Q10

(i) Show that angle BAC is 1·24 radians, [1]

(ii) Find the perimeter of the shaded region. (BECFB) [3]

Solution: (i) angle $BAC = \theta = \frac{l}{r} = \frac{6\cdot2}{5} = 1\cdot24^c$ [$\because l = r\theta^c$]

(ii) $AD \perp BC$ (In isosceles triangle, line joining the vertex to the midpoint of base is perpendicular to base)



In rt Δ ADB, $\frac{BD}{AB} = \sin\theta/2 = \sin\left(\frac{1\cdot24}{2}\right) = \sin(0\cdot62)^c$

$= 0\cdot581$

$\therefore BD = AB \times 0\cdot581 = 5 \times 0\cdot581 = 2\cdot91$cm

$\therefore$ length of arc $BFC = \pi r = \pi \times 2\cdot91 = 9\cdot13$ cm.

$\therefore$ Perimeter of shaded region $= \widehat{BEC} + \widehat{BFC}$

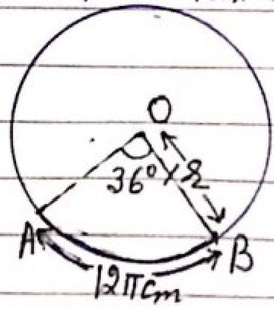
$= 6\cdot2 + 9\cdot13$

$= 15\cdot3$ cm

Example 4 Arc $\widehat{AB}$ of a circle, radius 'r' and centre O, subtends $36°$ at the centre. Find the radius 'r', given length of arc $AB = 12\pi$cm.

Solution: Length of arc $l = 12\pi$

angle $AOB = 36° = 36 \times \frac{\pi}{180}^c = \frac{\pi}{5}^c$



we know. $l = r\theta^c$

$\Rightarrow r = \frac{l}{\theta^c} = \frac{12\pi}{\frac{\pi}{5}} = 60$cm.

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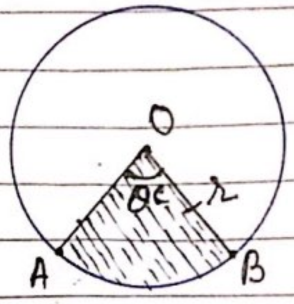
Circular measure

P-4

§ **Area of a Sector of a circle of radius 'r', centre O, angle at the centre θ^c.**

Area of a sector is proportional to the angle subtended at the centre:

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\text{Sector angle}}{\text{Total angle at the centre O}}$$

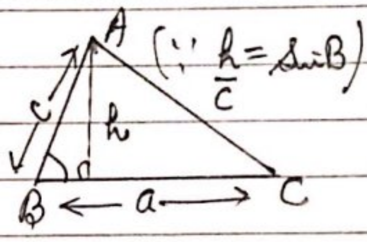


or $\dfrac{\text{Area of Sector}}{\pi r^2} = \dfrac{\theta}{2\pi}$

$\Rightarrow$ Area of Sector $= \dfrac{\theta}{2\pi} \times \pi r^2 = \dfrac{1}{2} r^2 \theta$

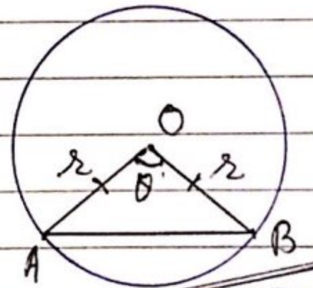
$$\boxed{\text{Area of Sector} = \tfrac{1}{2} r^2 \theta}$$

§ **Area of a triangle (SAS)**



Area of triangle $= \frac{1}{2} ac \sin B$

§ **Area of a triangle formed by the chord AB and radii OA and OB.**
Given angle $AOB = \theta$. (SAS)

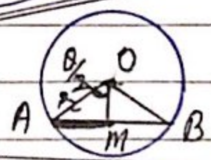


Area of $\Delta AOB = \frac{1}{2} r \times r \times \sin\theta$

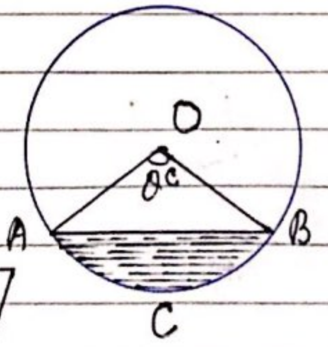
$$\boxed{\text{ar of } \Delta AOB = \tfrac{1}{2} r^2 \sin\theta}$$

§ $\boxed{\text{Length of Chord } AB = 2r \sin\theta/2}$ $AB = 2 \cdot AM = 2r \sin\theta/2$

$AM = r \sin\theta/2$



§ **Area of Segment of a circle:**



Area of Segment ACB = Area of Sector OACB − area of triangle OAB.

$= \frac{1}{2} r^2 \theta^c - \frac{1}{2} r^2 \sin\theta$

$\therefore$ $\boxed{\text{Area of segment of a circle} = \tfrac{1}{2} r^2 (\theta^c - \sin\theta)}$

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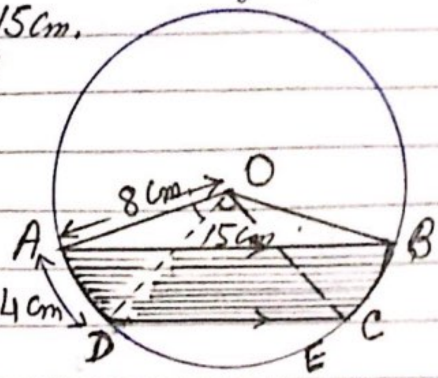
Example 5: The diagram shows a circle, centre O, radius 8 cm. The points A, B, C and D lie on the circumference of the circle such that AB is parallel to DC. The length of arc AD is 4 cm and the length of chord AB is 15 cm.

(i) Find, in radians, angle AOD. ---[1]

(ii) Hence show that DOC = 1.43 rad. correct to 2 decimal places. ---[3]

(iii) Find the perimeter of the shaded region. ---[3]

(iv) Find the area of the shaded region --[4]



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Solution: Let angle AOB $= \theta^c$

length of chord AB $= 2r\sin\theta/_2 = 2\times 8\sin\theta/_2 = 15$ (given)

$\Rightarrow \sin\theta/_2 = \frac{15}{16}$

$\therefore$ angle AOB $= 2\sin^{-1}\frac{15}{16} = 2.43 \longrightarrow (a)$

(i) angle AOD $= \frac{l}{r} = \frac{\text{arc } AD}{r} = \frac{4}{8} = 0.5^c$ ✓ $\left[l = r\theta^c \Rightarrow \theta^c = \frac{l}{r} \right.$

(ii) angle DOC = angle AOB − 2(angle AOD)

$= 2.43 \times 2 \times 0.5$ [frm (a)

$= 1.43^c$ ✓

(iii) Lenth of Chord DC $= 2r\sin\left(\frac{DOC}{2}\right) = 2\times 8\times \sin\left(\frac{1.43}{2}\right)$

$= 16\times \sin 0.715 = 10.49$

$\therefore$ Perimeter of the shaded region $= \overset{\frown}{AD} + DC + \overset{\frown}{BC} + AB$

$= 4 + 10.49 + 4 + 15$

$= 33.5$ cm ✓

(iv) Area of the shaded region = Area of segment AEB − Area segment DEC

$= \frac{1}{2}\times 8^2(2.43 - \sin 2.43) - \frac{1}{2}\times 8^2(1.43 - \sin 1.43)$

$= 42.8\ cm^2$ ✓

[$\because$ Area of segment of a circle $= \frac{1}{2}r^2(\theta^c - \sin\theta)$]

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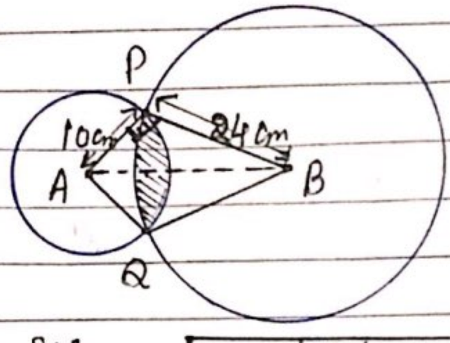
Example 6: The diagram shows a circle, centre A, radius 10 cm, intersecting a circle, centre B, radius 24 cm. The two circles intersect at the points P and Q. The radii AP and AQ are tangents to the circle with centre B. The radii BP and BQ are tangents to the circle with centre A.



(i) Show that angle PAQ is 2.35 radians correct to 3 s.f. ----[2]

(ii) Find angle PBQ in radians ---[1]

(iii) Find the perimeter of the shaded region. ---[3]

(iv) Find the area of the shaded region. ---[4]

[W-17/11/Q9]

Solution: AP is tangent to the circle with centre B.

$\therefore$ AP $\perp$ PB (tangent $\perp$ radius segment)

$\therefore \angle APB$ is a rt angle.

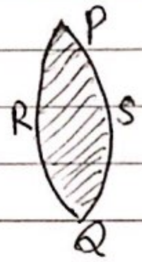
(i) $\therefore$ In rt triangle APB, $\tan PAB = \frac{24}{10} = 2.4$

angle PAB = 1.776

$\therefore$ Angle PAQ = 2 × angle PAB = 2 × 1.776 = 2.35 rad. ✓

(ii) Following same steps as in part (i), angle PBQ $= 0.790^c$ ✓

(iii)



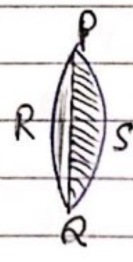
Perimeter of the shaded region = length of arc PSQ + arc PRQ

$= 10 \times 2.352 + 24 \times 0.790$ $[\because l = r\theta^c]$

$= 42.5$ cm ✓

(iv) Area of the shaded region =

area of segment PRQ + area of segment PSQ



$= \frac{1}{2} \times 24^2 \times (0.79 - \sin 0.79)$

$+ \frac{1}{2} \times 10^2 \times (2.352 - \sin 2.352)$

$= 22.94 + 82.1$

$= 105 \text{ cm}^2$ ✓

[$\because$ Area of segment of a circle $= \frac{1}{2} r^2 \times (\theta^c - \sin\theta)$]

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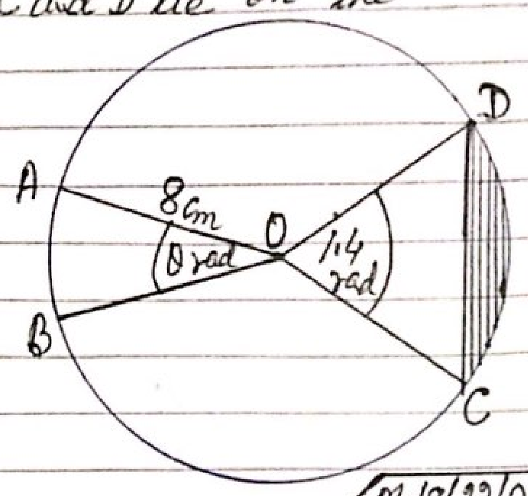
Example 7: The diagram shows a circle with centre O and radius 8 cm. The points A, B, C and D lie on the circumference of the circle.
Angle AOB = θ radians and angle COD = 1.4 rad. The area of sector AOB is 20 cm^2.

(i) Find angle θ ---[2]

(ii) Find the length of arc AB ---[2]

(iii) Find the area of the shaded segment. ---[3]



M-18/22/Q7

Solution: (i) area of sector AOB $= \frac{1}{2} r^2 \theta = 20$

or $\frac{1}{2} \times 8^2 \times \theta = 20 \Rightarrow \theta = \frac{20}{32} = \frac{5}{8} = 0.625$ rad. ✓

(ii) length of arc AB $= r\theta = 8 \times \frac{5}{8}$ $[l = r\theta]$

$= 5$ cm ✓

(iii) Area of shaded segment $= \frac{1}{2} r^2 (\theta - \sin\theta)$

$= \frac{1}{2} \times 8^2 \times (1.4 - \sin 1.4)$

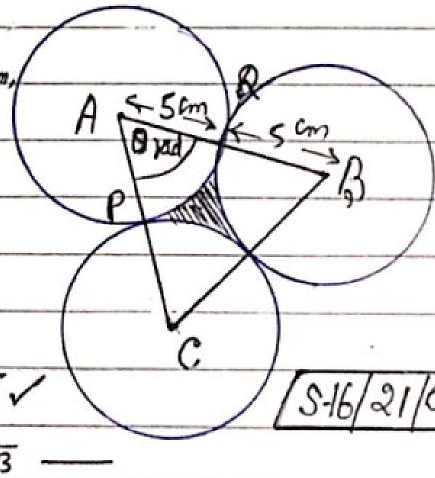
$= 32 \times (1.4 - 0.985)$

$= 13.26$ cm^2 ✓

Example 8. The diagram shows three circles with centres A, B and C, each of radius 5 cm. Each circle touches the other two circles. Angle BAC is θ radians.

i) Write down the value of θ ---[1]

ii) Find the area of the shaded region --[4]



S-16/21/Q4

Solution (i) AB = BC = CA = 5 × 2 = 10 cm.

∴ Triangle ABC is equilateral ⇒ angle $\theta = \frac{\pi}{3}$ ✓

(ii) Area of Triangle ABC $= \frac{1}{2} \times 10^2 \times \sin\frac{\pi}{3} = 25\sqrt{3}$

Area of sector APQ $= \frac{1}{2} r^2 \theta = \frac{1}{2} \times 5^2 \times \frac{\pi}{3} = \frac{25\pi}{6}$

∴ Area of shaded region = Area triangle − area of 3 sectors $= 25\sqrt{3} - 3 \times \frac{25\pi}{6}$

$= 4.03$ cm^2

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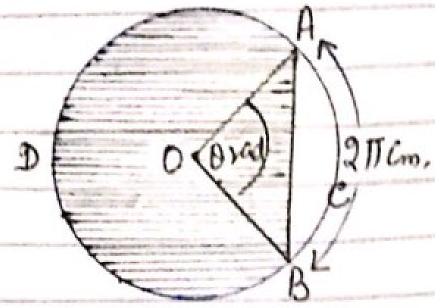
Example 9. The diagram shows a circle, centre O of radius r cm. and chord AB, angle AOB = θ radians. The length of major arc AB is 5 times of the length of minor arc AB.
The minor arc AB has length 2π cm.

(i) Find the value of θ and of r. ---[2]

(ii) Calculate the exact perimeter of the shaded segment. ---[2]

(iii) Calculate the exact area of the shaded segment.



[S-17/23/Q8]

Solution: (i) Circumference $= 2\pi + 5\times 2\pi = 12\pi$

$\therefore\ 2\pi r = 12\pi$

$\Rightarrow r = 6$ cm. ✓

Now length of arc ACB. $l = r\theta$

or $2\pi = 6\times\theta \Rightarrow \theta = \frac{\pi}{3}$ ✓

(ii) Perimeter of the shaded segment

$= $ length of major arc ADC + AB

$= 5\times 2\pi + 2\times 6\times \sin\left(\frac{\pi}{3}\times\frac{1}{2}\right)$

$= 10\pi + 12\sin\frac{\pi}{6}$ [∵ length chord $= 2r\sin\frac{\theta}{2}$]

$= (10\pi + 6)$ cm.

(iii) Area of the shaded segment = area of major sector + area of triangle OAB

$= \frac{1}{2}r^2\times(2\pi-\theta) + \frac{1}{2}r^2\sin\theta$

$= \frac{1}{2}\times 6^2\times\left(2\pi-\frac{\pi}{3}\right) + \frac{1}{2}\times 6^2\times\sin\frac{\pi}{3}$

$= 18\times\frac{5\pi}{3} + 18\times\frac{\sqrt{3}}{2}$

$= \underline{(30\pi + 9\sqrt{3})\ \text{cm}^2}$ ✓

← X — X — X →