IG-0606
Additonal Maths

Differentiation 1
Notes

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IG-Maths
0606

Differentiation-Notes

Every day I leave for school at around 7:30 am, to reach the school at 8:00 am. School is approx. 20 km away from my house.

One day, as I crossed a red-lightsignal just before the school, all of a sudden, a traffic inspector appeared and asked me to stop the car and issued me a speed challan. I was surprised, as the speed limit allowed is 60km/hr and I was driving at a speed of 40km/hr.

I tried to explain to the traffic inspector as follows-

Distance=20km

Time =30min=1/2hr

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{20}{1/2} = 40\text{km/hr}$$

Which is very much within the speed limit i.e. 60km/hr

I am still wondering why at all the speed challan was issued? Traffic cop was not listening to my argument that my speed 40km/hrwas within limit.

If you have an answer, please help me?

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Differentiation

Sir Isaac Newton and Gottfried Leibniz both independently developed the concept of calculus in the middle of 17th Century.

Differential calculus is about finding the rate of change of one quantity with respect to another quantity. Graphically speaking differential calculus is about finding the "gradient" (or slope) of a tangent to the graph of a function $y = f(x)$

Example 1. Given, the distance travelled by a car is given a function of time, then how to find the instantaneous rate of change of distance with time i.e. velocity of car.

Example 2. Given area of a square 'A' and length of a side 'x', both can vary, then what is the rate of change of area 'A' with change in length of side 'x'.

we know $A = x^2$; $A + \delta A = (x+\delta x)^2$

Let δx denotes a very small change in x and δA denotes the corresponding change in area, $\frac{\delta A}{\delta x} = ?$

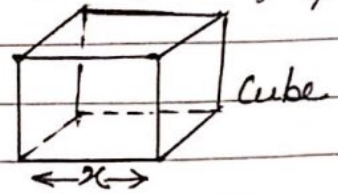
x	A	$x+\delta x$	$A+\delta A$	δx	δA	$\delta A / \delta x$
1	1	1·001	1·002001	0·001	0·002001	$\frac{0.002001}{0.001} = 2.001 = 2\times 1$ approx.
2	4	2·001	4·004001	0·001	0·004001	$\frac{0.004001}{0.001} = 4.001 = 2\times 2$ approx
3	9	3·001	9·006001	0·001	0·006001	$\frac{0.006001}{0.001} = 6.001 = 2\times 3$ approx.
⋮	⋮	⋮	⋮	⋮	⋮	⋮ ⋮ = ⋮
x	x^2	$(x+\delta x)$	$(x+\delta x)^2 = x^2+2x\cdot\delta x + \delta x^2$	δx	$2x\delta x+\delta x^2 = \delta x(2x+\delta x)$	$\frac{\delta x(2x+\delta x)}{\delta x} = (2x+\delta x) = 2\times x$ approx. as δx is very small.

$\therefore$ Rate of change of $A = x^2$ with respect to x is $2x$, denoted by $\frac{dA}{dx} = 2x$ ✓ or $\frac{d}{dx}x^2 = 2x$ ✓

Rate of change of area of square at $x=8$; $\left(\frac{dA}{dx}\right)_{x=8} = 2\times 8 = 16$

Differenentiation

Example 3. Given the volume of a cube 'V' and the length of its one side 'x', both can vary, then what is the rate of change of Volume 'V' with change in x.



$$V = x^3$$

$$x \to x + \delta x$$

$$V \to V + \delta V$$

what is the value of $\frac{\delta V}{\delta x}$ when δx is very small.

x	V	$x+\delta x$	$V+\Delta V$	Δx	ΔV	$\delta V / \delta x$
1	1	1.001	1.003003	0.001	0.003003	$\frac{0.003003}{0.001} = 3.003 = 3 \times 1^2$ approx
2	8	2.001	8.012006	0.001	0.012006	$\frac{0.012006}{0.001} = 12.006 = 3 \times 2^2$ approx
3	27	3.001	27.027009	0.001	0.027009	$\frac{0.027009}{0.001} = 27.009 = 3 \times 3^2$ approx
⋮	⋮	⋮	⋮	⋮	⋮	⋮
x	x^3	$x+\delta x$	$(x+\delta x)^3$	Δx	$3x^2 \cdot \delta x + 3x \cdot \delta x^2 + \delta x^3$ $= \delta x(3x^2 + 3x \cdot \delta x + \delta x^2)$	$= \frac{\cancel{\delta x}(3x^2 + 3x \cdot \delta x + \delta x^2)}{\cancel{\delta x}} = 3 \cdot x^2$ as δx is very small.

∴ Rate of change of V with respect to $x = 3x^2$

or differential of Volume with respect to x, denoted by,

$V(x) = x^3$
$\Rightarrow V'(x) = 3x^2$ or $\frac{dV}{dx} = 3x^2$ or $\frac{d}{dx} x^3 = 3x^2$

Rate of change of Volume at $x = 4$; $\left(\frac{dV}{dx}\right)_{x=4} = 3 \times 4^2 = 48$ ✓

In general.

$y = x^n$ $\frac{dy}{dx} = n \cdot x^{n-1}$	$\frac{d}{dx} x^n = n \cdot x^{n-1}$	$f(x) = x^n$ $f'(x) = n x^{n-1}$

$\frac{dy}{dx}$ denotes the derivative of y w.r.t. x.

Differentiation

§ Differentiation Formulae:

1(i) $\frac{d}{dx} x^n = n x^{n-1}$ or $\begin{cases} y = x^n \\ \frac{dy}{dx} = n x^{n-1} \end{cases}$ or $\begin{cases} f(x) = x^n \\ f'(x) = n \cdot x^{n-1} \end{cases}$

(ii) $y = a f(x)$

$\frac{dy}{dx} = a \cdot f'(x)$

(iii) $\frac{dc}{dx} = 0$ (c is a constant)

(iv) $\frac{d}{dx} ax = a \left\{ \because \frac{d}{dx} x = 1 \right.$

(v) $\frac{d}{dx} a x^n = a n x^{n-1}$

(vi) $\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$

(vii) $\frac{d}{dx} \frac{1}{x} = \frac{d}{dx} x^{-1} = -1 x^{-2} = \frac{-1}{x^2}$

(viii) $y = a f(x) + b g(x) - c h(x)$

$\frac{dy}{dx} = a f'(x) + b g'(x) - c h'(x)$

Example: $y = x^5 - 3x^4 + 7x^2 + 8$

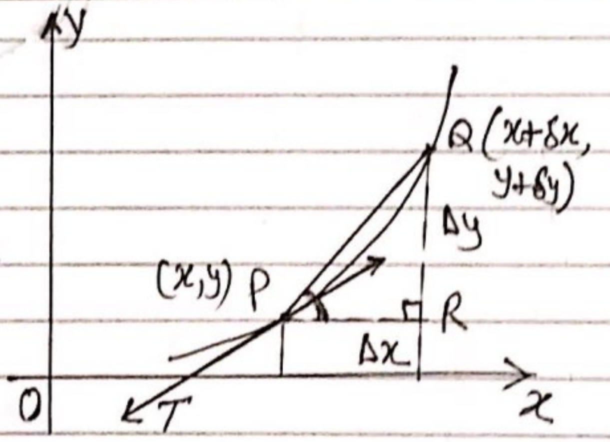
$\frac{dy}{dx} = 5x^4 - 3 \times 4x^3 + 7 \times 2x + 0$

$= 5x^4 - 12x^3 + 14x$ ✓

§ Geometric interpretation of $\frac{dy}{dx}$.

Given a function $y = f(x)$

Then $\frac{dy}{dx}$ or $f'(x)$ denotes the gradient (or slope) of the tangent to the curve



$y = f(x)$ at any point (x, y) on the curve.

$f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\Delta y}{\Delta x}$ = Gradient of the tangent to the curve at any point (x, y).

"PT is the tangent at P."

Differentiation

Example 4 Differentiate with respect to x (or w.r.t x)

(i) $\frac{1}{x^3}$ (ii) $x^2(1+x)$ (iii) $\frac{1+x}{x^2}$ (iv) $\frac{x^2+5x}{3\sqrt{x}}$

(i) $\frac{d}{dx}\frac{1}{x^3} = \frac{d}{dx}x^{-3} = -3x^{-3-1} = -3x^{-4} = \frac{-3}{x^4}$ ✓

Since $\frac{d}{dx}x^n = n \cdot x^{n-1}$

(ii) $\frac{d}{dx}x^2(1+x) = \frac{d}{dx}(x^2+x^3) = 2x+3x^2$ ✓

(iii) $\frac{d}{dx}\frac{1+x}{x^2} = \frac{d}{dx}\left(\frac{1}{x^2}+\frac{x}{x^2}\right) = \frac{d}{dx}\left(x^{-2}+x^{-1}\right) = -2x^{-3}-1x^{-2}$

$= \frac{-2}{x^3} - \frac{1}{x^2}$ ✓

(iv) $\frac{d}{dx}\left(\frac{x^2+5x}{3\sqrt{x}}\right) = \frac{1}{3}\frac{d}{dx}\left(\frac{x^2}{x^{1/2}}+\frac{5x}{x^{1/2}}\right) = \frac{1}{3}\frac{d}{dx}\left(x^{3/2}+5x^{1/2}\right)$

$= \frac{1}{3}\left[\frac{3}{2}x^{1/2} + 5\times\frac{1}{2}x^{-\frac{1}{2}}\right]$

$= \frac{1}{3}\left[\frac{3}{2}\sqrt{x} + \frac{5}{2\sqrt{x}}\right]$ ✓

Example 5. Find the value of $\frac{dy}{dx}$ at the given points.

(a) $y = x^4$ at $x=2$

$\frac{dy}{dx} = 4x^3$

$\left(\frac{dy}{dx}\right)_{x=2} = 4\times 2^3 = 32$ ✓

(b) $y = 5x^3 - 3x^2 + 7x + 6$ at $x=5$

$\frac{dy}{dx} = 5\times 3x^2 - 3\times 2x + 7\times 1 + 0$

$= 15x^2 - 6x + 7$

$\left(\frac{dy}{dx}\right)_{x=5} = 15\times 5^2 - 6\times 5 + 7 = 375 - 30 + 7 = 352$ ✓

Example 6. Given $f(x) = 4\sqrt{x}$ find $f'(9)$

$f'(x) = 4\times\frac{1}{2\sqrt{x}} = \frac{2}{\sqrt{x}}$

$\therefore f'(9) = \frac{2}{\sqrt{9}} = \frac{2}{3}$ ✓

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Differentiation

§ To find the derivative of Product of two functions:

$$\boxed{\frac{d}{dx}(u\cdot v) = u\cdot\frac{dv}{dx} + v\cdot\frac{du}{dx}} \quad \left\{ \begin{array}{l} u = f(x) \\ v = g(x) \end{array} \right.$$

Example 7.

Given $y = x^3\cdot(2x^4+7x+1)$, find $\frac{dy}{dx}$.

Using product rule,

Solution:

$$\frac{dy}{dx} = x^3\cdot\frac{d}{dx}(2x^4+7x+1) + (2x^4+7x+1)\cdot\frac{d}{dx}x^3$$

$$= x^3\cdot(8x^3+7) + (2x^4+7x+1)\times 3x^2$$

$$= 8x^6 + 7x^3 + 6x^6 + 21x^3 + 3x^2$$

$$= 14x^6 + 28x^3 + 3x^2 \checkmark$$

Alternate method (direct method)

$$y = x^3(2x^4+7x+1)$$

$$y = 2x^7 + 7x^4 + x^3 \checkmark$$

$$\therefore \frac{dy}{dx} = 2\times7x^6 + 7\times4x^3 + 3x^2$$

$$= 14x^6 + 28x^3 + 3x^2 \checkmark$$

Note: It is not always possible to take the product and differentiate by direct method.
for example: $(x^2\cdot\sin x)$; to be done later.

Example 8 $y = \sqrt{x}\,(x^3+5)$ find $\frac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = \sqrt{x}\cdot\frac{d}{dx}(x^3+5) + (x^3+5)\frac{d}{dx}\sqrt{x}$$

$$= \sqrt{x}(3x^2) + (x^3+5)\times\frac{1}{2\sqrt{x}}$$

$$= 3x^{5/2} + \frac{1}{2}\left(\frac{x^3+5}{\sqrt{x}}\right)$$

$$= 3x^{5/2} + \frac{1}{2}x^{5/2} + \frac{5}{2\sqrt{x}} = \frac{7}{2}x^{5/2} + \frac{5}{2\sqrt{x}} \checkmark$$

$\because$
$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Differentiation

§ To find the derivative of quotient of two functions:

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\cdot\frac{du}{dx} - u\cdot\frac{dv}{dx}}{v^2}} \quad \left\{\begin{array}{l} u = f(x) \\ v = g(x) \end{array}\right.$$

Example 9. $y = \frac{(x^3-5x)}{(2x-3)}$ find $\frac{dy}{dx}$

Solution: $\frac{dy}{dx} = \frac{(2x-3)\times\frac{d}{dx}(x^3-5x) - (x^3-5x)\frac{d}{dx}(2x-3)}{(2x-3)^2}$

$$= \frac{(2x-3)(3x^2-5) - (x^3-5x)\times 2}{(2x-3)^2}$$

$$= \frac{(6x^3-10x-9x^2+15) - (2x^3-10x)}{(2x-3)^2}$$

$$= \frac{4x^3-9x^2+15}{(2x-3)^2} \checkmark$$

Example 10. Find the slope (gradient) of the curve $y = \frac{x}{x-1}$ at the point (2,2).

Solution: $y = \frac{x}{x-1}$

$$\frac{dy}{dx} = \frac{(x-1)\cdot\frac{d}{dx}x - x\cdot\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$= \frac{(x-1)\times 1 - x\times 1}{(x-1)^2}$$

or $\frac{dy}{dx} = \frac{-1}{(x-1)^2}$

$\therefore$ Gradient of the curve at the point (2,2) is

$$\left(\frac{dy}{dx}\right)_{x=2} = \frac{-1}{(2-1)^2} = -1 \checkmark$$

Differentiation

§ The Chain Rule :

$y = f(u)$ and $u = h(x)$

$\frac{dy}{du} = f'(u)$ $\frac{du}{dx} = h'(x)$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$

§ $y = (ax+b)^n$

$$\boxed{\frac{dy}{dx} = na(ax+b)^{n-1}}$$

$y = (ax+b)^n$

$y = u^n$; $u = ax+b$

$\frac{dy}{du} = nu^{n-1}$; $\frac{du}{dx} = a$

$\frac{dy}{dx} = nu^{n-1} \times a$

$= na(ax+b)^{n-1}$ ✓

Example 11. $y = (3x+4)^7$

$\frac{dy}{dx} = 7(3x+4)^{7-1} \times 3$

$= 21(3x+4)^6$

Example 12.

$y = (5x^3 - 4x + 1)^6$ find $\frac{dy}{dx}$.

Solution: $y = u^6$; $u = 5x^3 - 4x + 1$

$\frac{dy}{du} = 6u^5$; $\frac{du}{dx} = 5 \times 3x^2 - 4x^1$

$= 15x^2 - 4$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 6u^5 \times (15x^2 - 4)$

$= 6(15x^2-4)(5x^3-4x+1)^5$

In short:

$\frac{dy}{dx} = 6(5x^3-4x+1)^5 \times \frac{d}{dx}(5x^3-4x+1)$

$= 6(5x^3-4x+1)^5 \times (15x^2-4) = 6(15x^2-4)(5x^3-4+1)^5$ ✓

Example 13: Find the equation of the normal to the curve $y = \frac{2x-1}{\sqrt{x^2+5}}$, at the point where $x = 2$, give your answer in the form $ax+by=c$ where a, b and c are integers. $\boxed{SP-20/01/Q6}$ ---- [8]

Solution: $y = \frac{2x-1}{\sqrt{x^2+5}}$ ---- ①

$$\frac{dy}{dx} = \frac{\sqrt{x^2+5}\cdot\frac{d}{dx}(2x-1) - (2x-1)\cdot\frac{d}{dx}\sqrt{x^2+5}}{(\sqrt{x^2+5})^2}$$ (Using quotient rule)

$$= \frac{\sqrt{x^2+5}\times 2 - (2x-1)\times\frac{1}{2\sqrt{x^2+5}}\times\frac{d}{dx}(x^2+5)}{(x^2+5)}$$

let $g(x) = \sqrt{x^2+5}$ (chain rule)

$g = \sqrt{u}$

$\frac{dg}{dx} = \frac{1}{2\sqrt{u}}\cdot\frac{du}{dx}$

$$= \frac{2\sqrt{x^2+5} - \frac{(2x-1)}{\cancel{2}\sqrt{x^2+5}}\times\cancel{2}x}{(x^2+5)}$$

$$= \frac{2(x^2+5) - x(2x-1)}{(x^2+5)^{3/2}}$$

$$\frac{dy}{dx} = \frac{10+x}{(x^2+5)^{3/2}}$$

$$\left(\frac{dy}{dx}\right)_{x=2} = \frac{10+2}{(4+5)^{3/2}} = \frac{12}{27} = \frac{4}{9}$$ $(m_1 \times m_2 = -1)$

$\therefore$ Gradient of tangent $m_1 = \frac{4}{9}$

so the gradient of Normal $m_2 = -\frac{9}{4}$

On the curve ①, $x = 2 \Rightarrow y = 1$

$\therefore$ The equation of normal to the curve at $(2,1)$

$$y - y_1 = m_2(x - x_1)$$

$$y - 1 = -\frac{9}{4}(x-2)$$

$$4y - 4 = -9x + 18$$

or $9x + 4y = 22$ ✓

Example 14. The point P lies on the curve $y = 3x^2 - 7x + 11$. The normal to the curve at 'P' has equation $5y + x = k$. Find the coordinate of P and the value of k. $\boxed{S-17/22/Q4}$ ---[6]

Solution: Equation of normal to the curve at P, $5y + x = k$ —— (1)

or $y = -\frac{1}{5}x + \frac{k}{5}$

$\therefore$ Gradient of normal $m_1 = -\frac{1}{5}$

let the gradient of tangent at P, $= m$.

$m_1 \times m = -1$

$-\frac{1}{5} \times m = -1 \Rightarrow m = 5$

$\therefore$ Gradient of tangent $m = 5$ —— (2)

Now Given eqn of curve $y = 3x^2 - 7x + 11$ —— (3)

diff. $\frac{dy}{dx} = 6x - 7$ —— (4)

frm (2) & (4) Gradient of tangent $6x - 7 = 5$

$\Rightarrow x = 2$ and $y = 9$ from (3)

P(2,9) lies on the normal (1) | $\Rightarrow$ P(2,9) ✓

$\therefore 5 \times 9 + 2 = k$ [$5y + x = k$ —— (1)

$\therefore k = 47$ ✓

Example 15. The normal to the curve $y = \sqrt{4x+9}$, at the point where $x = 4$, meets the x-axis and y-axis at the points A and B. Find the coordinates of the mid point of AB. ----[7] $\boxed{S-17/13/Q5}$

Solution:

$y = \sqrt{4x+9}$ —— (1)

$\frac{dy}{dx} = \frac{1}{2\sqrt{4x+9}} \times 4 = \frac{2}{\sqrt{4x+9}}$

$\therefore \left(\frac{dy}{dx}\right)_{x=4} = \frac{2}{\sqrt{4 \times 4 + 9}} = \frac{2}{5}$ is the gradient of tangent.

$\therefore$ gradient of Normal $m = -\frac{5}{2}$; from (1) $x = 4 \Rightarrow y = 5$

$\therefore$ Equation of Normal at (4,5), $y - 5 = -\frac{5}{2} + 15 \Rightarrow y = -\frac{5}{2}x + 15$ —— (2)

$\therefore$ A(6,0), B(0,15), $\therefore$ Mid point of AB is $(3, \frac{15}{2})$ ✓

Differentiation

Example 16. The curve with equation $y = x^3+2x^2-7x+2$, passes through the point $A(-2, 16)$, Find.

(i) the equation of the tangent to the curve at the point A, ---[3]

(ii) the coordinates of the point where the tangent meets the curve again. [W-16/21/Q5] ---[5]

Solution: $y = x^3+2x^2-7x+2$ —— ①

$$\frac{dy}{dx} = 3x^2+4x-7$$

at $A(-2,16)$, $\left(\frac{dy}{dx}\right)_{x=-2} = 3(-2)^2+4(-2)-7 = -3$ is the gradient of the tangent at $A(-2,16)$

∴ Equation of tangent at $A(-2, 16)$.

$$y-16 = -3(x+2)$$

or $\underline{y = -3x+10}$ ———— ② ✓

(ii) Solving ① & ②

$$x^3+2x^2-7x+2 = -3x+10$$

or $x^3+2x^2-48-8=0$

$$(x+2)(x+2)(x-2)=0$$

∴ $x = 2$, $x=-2$ already there.

∴ tangent intersects the curve again at $x=2$

fr. ① $y=4$

∴ $\underline{\text{Required point } (2,4)}$ ✓

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Differentiation

§ Small increments and approximations:

$y = f(x)$

$x \to x + \delta x$

$y \to y + \delta y$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

Now we say that for very small values of δx & δy

approximately. $\frac{\delta y}{\delta x} = \frac{dy}{dx}$

or $\boxed{\delta y = \frac{dy}{dx} \times \delta x}$

Example 17. It is given that $y = (x-4)(3x-1)^{5/3}$

(i) show that $\frac{dy}{dx} = (3x-1)^{2/3}(Ax+B)$ where A and B are integers to be found. ---[5]

(ii) Hence find, in terms of h, where h is small, the approximate change in y when x increases from 3 to $3+h$. [W-17/13/Q8] ---[3]

Solution: $y = (x-4)(3x-1)^{5/3}$

(i) $\frac{dy}{dx} = (x-4) \times \frac{5}{\cancel{3}}(3x-1)^{2/3} \times \cancel{3} + (3x-1)^{5/3} \times 1$ [Product Rule]

$= (3x-1)^{2/3}\left[5(x-4) + (3x-1)^1\right]$

$= (3x-1)^{2/3}(8x-21)$ ✓

(ii) $\left(\frac{dy}{dx}\right)_{x=3} = (3\times3-1)^{2/3}(8\times3-21) = 4\times3 = 12$ ✓

Now $\frac{\delta y}{\delta x} = \left(\frac{dy}{dx}\right)_{x=3}$

when x increases $3 \to 3+h$

$\delta y = \left(\frac{dy}{dx}\right)_{x=3} \times \delta x$

$\delta x = h$

$\delta y = 12 \times h$ ✓

Differentiation

Example 18. Variable x and y are related by equation $y = x\sqrt{x}$.

(a) Find $\frac{dy}{dx}$ --- [2]

(b) Hence find the approximate change in x when y increases from 8 by the small amount 0.015. [SP-20/01/Q2] --- [3]

Solution (a) $y = x\sqrt{x} = x^{3/2}$ —— ①

$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$ ✓

Now when $y = 8$ from ① $x = 4$ ✓

(b) $\left(\frac{dy}{dx}\right)_{x=4} = \frac{3}{2}\sqrt{4} = \frac{3}{2} \times 2 = 3$ —— ①

Now given change in y is $\delta y = 0.015$

we know $\frac{\delta y}{\delta x} = \frac{dy}{dx}$ for small value of δx & δy

Given $\delta y = 0.015$ at $x = 8$, $\frac{\delta y}{\delta x} = \left(\frac{dy}{dx}\right)_{x=4}$

or $\frac{0.015}{\delta x} = 3$ from ①

$\Rightarrow \delta x = \frac{0.015}{3} = 0.005$

$\therefore \delta x = 0.005$ ✓

Example 19. A curve has equation $y = 6x - x\sqrt{x}$

(iii) Find the approximate change in y when x increases from 4 to $4+h$, where h is small. [S-17/13/Q11] --- [3]

Solution: $y = 6x - x\sqrt{x}$

or $y = 6x - x^{3/2}$

diff. $\frac{dy}{dx} = 6 - \frac{3}{2}x^{1/2}$

$\left(\frac{dy}{dx}\right)_{x=4} = 6 - \frac{3}{2}(4)^{1/2} = 6 - 3$

$= 3$ ✓ —— ①

we know for small values of δx & δy

$\frac{\delta y}{\delta x} = \frac{dy}{dx}$

Given change in x is h at $x = 4$

$\therefore \frac{\delta y}{h} = \left(\frac{dy}{dx}\right)_{x=4}$

$\Rightarrow \delta y = 3 \times h$ from ①

$= 3h$ ✓

$\therefore$ the approximate change in $y = 3h$ ✓

Differentiation

§ Rate of Change:

Given $y = f(x)$

The rate of change of y w.r.t x is denoted by $\frac{dy}{dx}$.

Now given the rate of change of x with 't' (third variable) i.e. $\frac{dx}{dt}$ is given, then:

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}}$$

$$\boxed{\text{Note: } \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}}$$

Example 20. The volume, V, and surface area, S, of a sphere of radius r, are given by $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$, respectively. The volume of a sphere increases at rate $200\ cm^3$ per second. At the instant when the radius of the sphere is 10 cm, find,

(i) the rate of increase of radius of the sphere, ---[4]

(ii) the rate of increase of the surface area of the sphere. ---[3]

$\boxed{M\text{-}18/22/Q12}$

Solution: $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\left(\frac{dV}{dr}\right)_{r=10} = 4\pi \times 10^2 = 400\pi$$

$$\therefore \left(\frac{dr}{dV}\right)_{r=10} = \frac{1}{400\pi} \quad \text{—(1)}$$

(i) To find $\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$$= \left(\frac{dr}{dV}\right)_{r=10} \times \left(\frac{dV}{dt}\right)_{r=10}$$

$$= \frac{1}{400\pi} \times 200 \text{ (Given)}$$

$$\therefore \frac{dr}{dt} = \frac{1}{2\pi} = 0.159 \quad \text{—(2)}$$

(ii) $S = 4\pi r^2$

$$\frac{dS}{dr} = 8\pi r$$

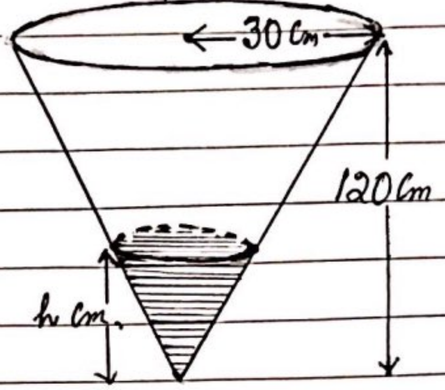
$$\left(\frac{dS}{dr}\right)_{r=10} = 8\pi \times 10 = 80\pi \quad \text{—(3)}$$

To find $\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$

$$= 80\pi \times \frac{1}{2\pi} \quad \text{from (2) \& (3)}$$

$$= 40\ cm^2/sec.$$

Example 21. The diagram shows a container in the shape of a cone of height 120 cm and radius 30 cm. Water is poured into the container at a rate of $20\pi\ cm^3 s^{-1}$.



(i) At the instant when the depth of water in the cone is h cm, the volume of water in the cone is $V cm^3$.
Show that $V = \frac{\pi h^3}{48}$ ----[3]

(ii) Find the rate at which h is increasing when $h = 50$ cm.

(iii) Find the rate at which the circular area of water's surface is increasing when $h = 50$ cm. $\boxed{W-13/23/Q10}$ ---[4]

Solution: Let the radius of water surface $= r$ and height $= h$

(i)

$$\frac{h}{120} = \frac{r}{30} \Rightarrow r = \frac{30}{120}h$$

$$\text{or } r = \frac{h}{4} \checkmark$$

Volume of water $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h$$

$$V = \frac{1}{48}\pi h^3 \checkmark$$

(ii)

$$\frac{dV}{dh} = \frac{1}{48}\pi \times 3h^2$$

$$= \frac{\pi h^2}{16}$$

$$\therefore \left(\frac{dV}{dh}\right)_{h=50} = \frac{\pi \times 50^2}{16}$$

$$= \frac{2500\pi}{16} \longrightarrow (1)$$

or $\frac{dh}{dV} = \frac{16}{2500\pi}$ and $\frac{dV}{dt} = 20\pi$

Now $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$$= \frac{16}{2500\pi} \times 20\pi$$

$$= 0.128 \text{ cm/sec} \longrightarrow (2)$$

(iii) Circular Area of water's surface $A = \pi r^2 = \pi\left(\frac{h}{4}\right)^2$

$$\text{or } A = \frac{\pi h^2}{16}$$

diff.

$$\frac{dA}{dh} = \frac{\pi h}{8}$$

$$\left(\frac{dA}{dh}\right)_{h=50} = \frac{\pi \times 50}{8} \longleftarrow (3)$$

To find: $\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt}$

$$= \frac{\pi}{8} \times 50 \times 0.128$$

(from (2) & (3))

$$= 0.8\pi \text{ or } 2.51 \text{ cm}^2/\text{sec.}$$

Differentiation

Example 22. Find the rate of change of the area of a circle with respect to its radius, when the radius is 3cm.

Solution: Let 'A' is the area of circle of radius 'r'.

$$A = \pi r^2$$

diff.

$$\frac{dA}{dr} = 2\pi r$$

$$\left(\frac{dA}{dr}\right)_{r=3} = 2\pi \times 3 = 6\pi$$

∴ The area of circle is increasing at a rate of $6\pi\ cm^2/sec$.

Example 23. A stone is dropped into a quiet lake and waves move in a circle at a speed of 3.5 cm/sec. At the instant when the radius of the circular wave is 7.5 cm, how fast is the enclosed area increasing?

Solution: Let 'A' is the area of circular wave of radius 'r' at any instant 't'.

$$A = \pi r^2 \quad \text{and} \quad \frac{dr}{dt} = 3.5\ cm/sec \text{ —— (1)}$$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$\left(\frac{dA}{dr}\right)_{r=7.5} = 2\pi \times 7.5 = 15\pi \text{ —— (2)}$$

Now

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$= 15\pi \times 3.5$$

$$= 52.5\pi\ cm^2/sec.$$

Differentiation

§ Second Derivative:

Given $y = f(x)$

Then first derivative is $\frac{dy}{dx}$ or $f'(x)$ or $\frac{d}{dx}f(x)$

and Second Derivative is $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ or $f''(x)$

Example: 24. Find the second derivative of the following functions:

(a) $y = 4x^3 - 7x^2 + 5x + 13$

$\frac{dy}{dx} = 12x^2 - 14x + 5$

$\frac{d^2y}{dx^2} = 24x - 14$ ✓

(b) $f(x) = x^5 + \frac{1}{x}$

or $f(x) = x^5 + x^{-1}$

diff. $f'(x) = 5x^4 - x^{-2}$

Diff again $f''(x) = 20x^3 + 2x^{-3}$

$= 20x^3 + \frac{2}{x^3}$ ✓

(c) $y = \frac{3x-1}{(x+1)}$

$\frac{dy}{dx} = \frac{(x+1)\frac{d}{dx}(3x-1) - (3x-1)\cdot\frac{d}{dx}(x+1)}{(x+1)^2}$

$= \frac{(x+1)\times 3 - (3x-1)}{(x+1)^2}$

$= \frac{4}{(x+1)^2}$ Continued →

→ $\frac{dy}{dx} = \frac{4}{(x+1)^2} = 4(x+1)^{-2}$

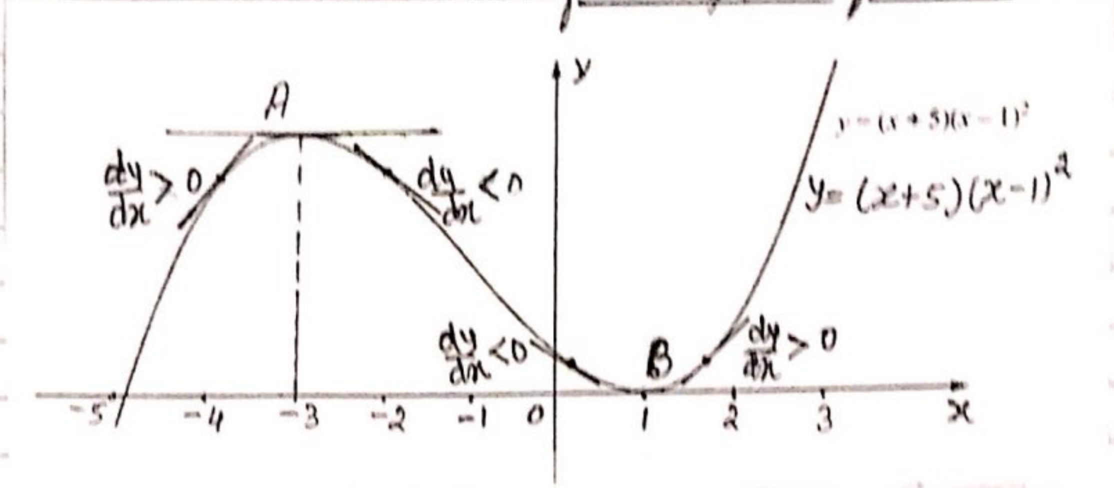
diff again

$\frac{d^2y}{dx^2} = 4\times(-2)(x+1)^{-3}$

$= \frac{-8}{(x+1)^3}$ ✓

Differentiation

§ Stationary points and Maximum point/Minimum point: and point of Inflexion.



Given a function $y = f(x)$

Let the given diagram shows the graph of $y = f(x)$.

The tangents to the curve at point A and B are parallel to x-axis.

or Gradient of the tangents at A and B is zero.

or $\frac{dy}{dx}$ at A and B $= 0$

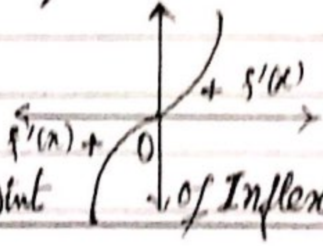
§ The points A and B are called stationary points of the function. To determine the nature of stationary points

Here A is maximum point and B is the minimum point.

§ At Maximum point 'A' the gradient $\frac{dy}{dx}$ (or $f'(x)$) changes sign from + to −ve.

§ At Minimum point B, the gradient changes sign from −ve to + ($f'(x) < 0$ to $f'(x) > 0$).

Note: For $y = x^3$, $\frac{dy}{dx} = 0$ at origin and $\frac{dy}{dx} = 3x^2$, gradient changes sign from + to + is called the point of Inflexion

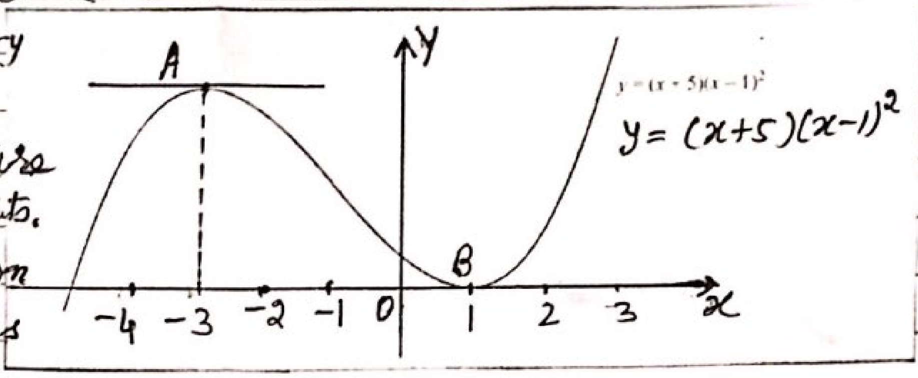


Example 25. Given $y=(x+5)(x-1)^2$

(a) Find the stationary points.

(b) Determine the nature of these stationary points.

(c) Find the maximum and minimum values of the function.



Solution: $y=(x+5)(x-1)^2$ —— ①

(a) $\frac{dy}{dx}=(x+5)\cdot 2(x-1)+(x-1)^2\times 1$ (Product Rule)

$= (x-1)[2(x+5)+(x-1)]$

or $\frac{dy}{dx}=3(x-1)(x+3)$ —— ②

for stationary point $\frac{dy}{dx}=0$

$\Rightarrow 3(x-1)(x+3)=0 \Rightarrow x=1,-3$

∴ The stationary points are $x=-3$ and $x=1$

(b) To check at which stationary point we have max/min. Using first derivative test.

at $x=-3$

on the left $\left(\frac{dy}{dx}\right)_{x=-3.5}=3(-3.5-1)(-3.5+3)=3(-)(-)=+ve$

on the right $\left(\frac{dy}{dx}\right)_{x=-2.5}=3(-2.5-1)(-2.5+3)=3(-)(+)=-ve$

at $x=-3$ gradient $\left(\frac{dy}{dx}\right)$ changes sign from + to −ve, hence there is max. at $x=-3$

Again at $x=1$

On the left of $x=1$, $\left(\frac{dy}{dx}\right)_{x=0.5}=3(0.5-1)(0.5+3)=3(-)(+)=-ve$

On the right of $x=1$, $\left(\frac{dy}{dx}\right)_{x=1.5}=3(1.5-1)(1.5+3)=3(+)(+)=+$

∴ at $x=1$ $\frac{dy}{dx}$ changes sign from −ve to +, hence Min at $x=1$.

(c) Max value at $x=-3$, from ① $(-3+5)(-3-1)^2=32$ is Max value at $x=-3$

Min. Value at $x=1$, from ① $(1+5)(1-1)^2=0$ is Min Value at $x=1$

Differentiation

Example 26. Show that the curve $y=(3x^2+8)^{5/3}$ has only one stationary point. Find the coordinates of this stationary point and determine its nature. $\boxed{S\text{-}17 | 11 | Q7}$ ---[8]

Solution: $y = (3x^2+8)^{5/3}$ —— ①

$\frac{dy}{dx} = \frac{5}{3}(3x^2+8)^{5/3-1} \times \frac{d}{dx}(3x^2+8)$ [using chain rule]

$= \frac{5}{3}(3x^2+8)^{2/3}(6x)$

$= 10x(3x^2+8)^{2/3}$ —— ②

for stationary point $\frac{dy}{dx}=0 \Rightarrow 10x(3x^2+8)^{2/3}=0$

$\Rightarrow x=0$ as $3x^2+8=0$ has no solution.

$\therefore$ Only solution is $x=0$

$\therefore$ stationary point is $(0,32)$ ✓

[$x=0$ fr① $y=(0+8)^{5/3} = 32$]

To determine the nature of stationary point at $x=0$; from ②

On the left of $x=0$, $\left(\frac{dy}{dx}\right)_{-0.5} = 10(-)(+) = -ve$

on the right of $x=0$, $\left(\frac{dy}{dx}\right)_{0.5} = 10(+)(+) = +ve$

$\therefore \frac{dy}{dx}$ changes sign $-ve$ to $+$ at $x=0$, $\therefore$ Minimum.

$\therefore$ Minimum at $x=0$

Example 27. A curve has equation $y = 6x - x\sqrt{x}$

(i) Find the coordinates of the stationary point of the curve. ---[4]

(ii) determine the nature of this stationary point. ---[2]

Solution: $y = 6x - x^{3/2}$ —— ①

(i) $\frac{dy}{dx} = 6 - \frac{3}{2}x^{1/2}$ —— ②

for stationary point $\frac{dy}{dx}=0 \Rightarrow 6-\frac{3}{2}x^{1/2}=0$ from ②

$\Rightarrow x^{1/2}=4$

$\Rightarrow x=16$ ✓ and $y=32$ fr①

$\therefore$ The stationary point is $(16,32)$ ✓

(ii) Now on the left of 16, $\left(\frac{dy}{dx}\right)_{x=15} = 6-\frac{3}{2}\sqrt{15} = +ve$

on the right of 16, $\left(\frac{dy}{dx}\right)_{x=17} = 6-\frac{3}{2}\sqrt{17} = -ve$

$\frac{dy}{dx}$ changes sign from $+$ to $-ve$, hence there is a Max. at $(16,32)$ ✓

Differentiation.

§ To determine the nature of a stationary point using second derivative $\frac{d^2y}{dx^2}$ or $f''(x)$:

Given $y = f(x)$ ——— (1)

find $\frac{dy}{dx} = f'(x)$ ——— (2)

and $\frac{d^2y}{dx^2} = f''(x)$ ——— (3)

To find the stationary points, $\frac{dy}{dx}$ (or $f'(x)$) $= 0$

Solve, let $x = x_1, x_2 \ldots$ are stationary value.

Now to determine the nature at $x = x_1$

find $f''(x_1) = -ve$ then there is Max. at $x = x_1$

again find $f''(x_2) = +ve$ then there is Min. at $x = x_2$

$\boxed{\text{Note:}}$ If at a stationary point let $x = x_1$

$f''(x_1) = 0$

Then the second derivative method fails

Apply the first derivative method to find to confirm.

Max/Min/or point of Inflexion.

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Differentiation

Example 28. A curve has equation $y = \frac{x}{x^2+1}$.

(i) Find the coordinates of stationary points of the curve. ---[5]

(ii) Show that $\frac{d^2y}{dx^2} = \frac{px^3+qx}{(x^2+1)^3}$, where p and q are integers to be found. M-16/22/Q11

and determine the nature of the stationary points of the curve. --[5]

Solution: $y = \frac{x}{x^2+1}$ ---①

(i) Diff. $\frac{dy}{dx} = \frac{(x^2+1)\cdot\frac{d}{dx}x - x\cdot\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$ (Quotient Rule)

$$= \frac{(x^2+1)\times 1 - x\times 2x}{(x^2+1)^2} = \frac{(1-x^2)}{(x^2+1)^2} \text{ --- ②}$$

for stationary point $\frac{dy}{dx} = 0 \Rightarrow \frac{1-x^2}{(x^2+1)^2} = 0 \Rightarrow 1-x^2 = 0$

$\Rightarrow x = 1, -1$

$y = 0.5; -0.5$

∴ Stationary points are $(1, 0.5)$ and $(-1, -0.5)$ ✓

(ii) diff. ②

$$\frac{d^2y}{dx^2} = \frac{(x^2+1)^2\cdot\frac{d}{dx}(1-x^2) - (1-x^2)\cdot\frac{d}{dx}(x^2+1)^2}{\{(x^2+1)^2\}^2}$$

$$= \frac{(x^2+1)^2\cdot(-2x) - (1-x^2)\times 2(x^2+1)\cdot 2x}{(x^2+1)^4}$$

$$= \frac{(x^2+1)[-2x(x^2+1) - 4x\cdot(1-x^2)]}{(x^2+1)^4}$$

Using Chain Rule.

$\frac{d}{dx}(x^2+1)^2$

$= 2(x^2+1)\frac{d}{dx}(x^2+1)$

$= 2(x^2+1)\times 2x$

$$\therefore \frac{d^2y}{dx^2} = \frac{2x^3-6x}{(x^2+1)^3} \text{ --- ③}$$

Now To Check the nature of stationary point.

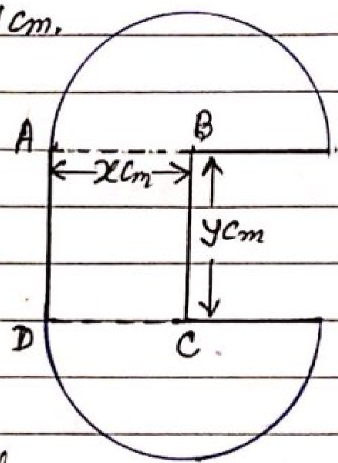
at $(1, 0.5)$, $\left(\frac{d^2y}{dx^2}\right)_{x=1} = \frac{2\times 1^3 - 6\times 1}{(1^2+1)^3} = \frac{-4}{8} < 0 \Rightarrow$ There is a Max. at $(1, 0.5)$ ✓

and at $(-1, -0.5)$, $\left(\frac{d^2y}{dx^2}\right)_{x=-1} = \frac{2(-1)^3 - 6(-1)}{[(-1)^2+1]^3} = \frac{4}{8} > 0 \Rightarrow$ There is a Min at $(-1, -0.5)$ ✓

Example 29. The diagram shows a badge, made of thin sheet metal, consisting of two semi-circular pieces, centres B and C, each of radius x cm. They are attached to each other by a rectangular piece of thin sheet metal, ABCD, such that AB and CD are radii of the semicircular pieces and AD = BC = y cm.

(a) Given that the area of the badge is 20cm^2, show that the perimeter, P cm, of the badge is given by, $P = 2x + \frac{40}{x}$ --- [4]

(b) Given that x can vary, find the minimum value of P, justify that this value is minimum. ---[5]



[SP-20/01/Q7]

Solution: Area of the badge = 2× semicircle + rectangle

(a)
$$= 2 \times \frac{1}{2}\pi x^2 + xy$$
$$= \pi x^2 + xy = 20 \quad \text{given}$$
$$\Rightarrow y = \frac{20 - \pi x^2}{x} \quad \text{——— ①}$$

Perimeter of the badge,
$$P = 2\pi x + 2x + 2y$$
$$= 2\pi x + 2x + 2\left(\frac{20 - \pi x^2}{x}\right) \quad \text{from ①}$$
$$= \cancel{2\pi x} + 2x + \frac{40}{x} - \cancel{2\pi x}$$
$$\therefore P = 2x + \frac{40}{x} \quad \text{——— ②}$$

(b)
$$\frac{dP}{dx} = 2 - \frac{40}{x^2} \quad \text{——— ③} \qquad \left(\frac{d}{dx}\frac{1}{x} = \frac{d}{dx}x^{-1} = -1x^{-2} = -\frac{1}{x^2}\right)$$

for stationary Point $\frac{dP}{dx} = 0 \Rightarrow 2 - \frac{40}{x^2} = 0 \Rightarrow x^2 = 20 \Rightarrow x = 2\sqrt{5}$ ✓

Now diff ③ $\frac{d^2P}{dx^2} = -40(-2x^{-3}) = \frac{80}{x^3}$ ——— ④

Now to Check the nature of stationary Point at $x = 2\sqrt{5}$
$$\left(\frac{d^2P}{dx^2}\right)_{x=2\sqrt{5}} = \frac{80}{(2\sqrt{5})^3} > 0 \quad \text{is +ve} \qquad \text{fm ④}$$

∴ The Perimeter is minimum for $x = 2\sqrt{5}$ ✓

fm ② Minimum value of $P = 2 \times 2\sqrt{5} + \frac{40}{2\sqrt{5}} = 4\sqrt{5} + 4\sqrt{5} = 8\sqrt{5}$
$= 17.9$ ✓

Example 30: The volume of a closed cylinder of base radius x cm. and height h cm is 500 cm^3.

(i) Express h in terms of x. ---[1]

(ii) Show that the total surface area of the cylinder is given by,

$$A = 2\pi x^2 + \frac{1000}{x} \quad cm^2$$ ---[2]

(iii) Given that, x can vary, find the stationary value of A, and show that this value is a minimum. $\boxed{W\text{-}17/22/Q6}$ --[5]

Solution: Volume of Cylinder, $\pi x^2 h = 500$ {$\because V = \pi r^2 h$ and $r = x$ given

(i) or $h = \dfrac{500}{\pi x^2}$ —① ✓

(ii) Total Surface area $= 2\pi x h + 2\pi x^2$

$= 2\pi x \times \dfrac{500}{\pi x^2} + 2\pi x^2$ [from ① $h = \dfrac{500}{\pi x^2}$

$\therefore \quad A = 2\pi x^2 + \dfrac{1000}{x}$ —② ✓

(iii) diff ② $\dfrac{dA}{dx} = 4\pi x - \dfrac{1000}{x^2}$ —③

for stationary value of Area 'A' $\dfrac{dA}{dx} = 0$

$\Rightarrow 4\pi x - \dfrac{1000}{x^2} = 0$

$\Rightarrow x = \sqrt[3]{\dfrac{1000}{4\pi}} = 4.3$ ✓

from ② the stationary value of A (at $x = 4.3$) $= 2\pi(4.3)^2 + \dfrac{1000}{4.3}$

$= 349 cm^2$ ✓

To check the nature of the station value of A,

diff ③ $\dfrac{d^2A}{dx^2} = 4\pi + \dfrac{2000}{x^3}$

at $x = 4.3$; $\left(\dfrac{d^2A}{dx^2}\right)_{x=4.3} = 4\pi + \dfrac{2000}{(4.3)^2} > 0$ or +ve

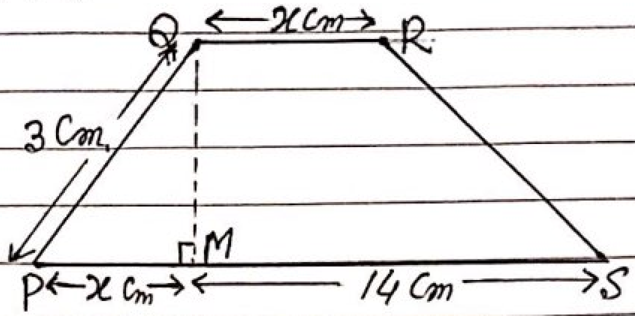
$\therefore$ The stationary value of the surface area 'A' is minimum.

Differentiation

Example 31, (i) Show that the area, $A\,cm^2$, of the trapezium PQRS is given by $A = (7+x)\sqrt{9-x^2}$ ---[2]

(ii) Given that x can vary, find the stationary value of A. ---[7]

W-16/21/Q7



Solution: (i) Area of Trapezium $= \frac{1}{2} \times$ sum of parallel sides $\times$ distance between the parallel side

$= \frac{1}{2} \times (PS + QR) \times QM$ (here $QM \perp PS$)

$= \frac{1}{2}(14 + x + x) \times \sqrt{3^2 - x^2}$

$A = (7+x)\sqrt{9-x^2}$ ——— ①

(ii) diff ①

$$\frac{dA}{dx} = \sqrt{9-x^2} + (7+x) \times \frac{1}{2}(9-x^2)^{-1/2} \times (-2x)$$

$$= \sqrt{9-x^2} - \frac{(7x+x^2)}{\sqrt{9-x^2}} = \frac{9-x^2-7x-x^2}{\sqrt{7-x^2}}$$

or $\dfrac{dA}{dx} = \dfrac{9-7x-2x^2}{\sqrt{7-x^2}}$ ——— ②

for stationary value of A, $\dfrac{dA}{dx} = 0$

or $\dfrac{9-7x-2x^2}{\sqrt{7-x^2}} = 0$ (from ②)

or $2x^2 + 7x - 9 = 0$

$(x-1)(2x+9) = 0$

$x = 1$ or $x = -\frac{9}{2}$ ✗

∴ stationary value at a point $x = 1$

fr① stationary value $= (7+1)\sqrt{9-1^2} = 8\sqrt{8} = 16\sqrt{2}$ ✓

$= 16\sqrt{2}\ cm^2$.

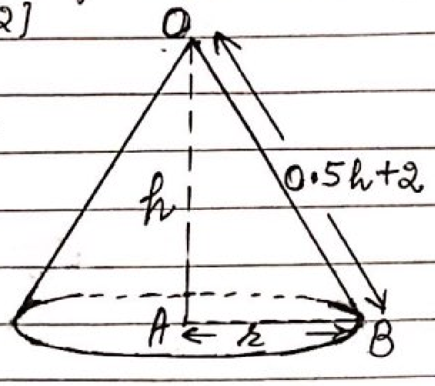
Example 32. In this question all lengths are in metres.
A conical tent is to be made with height h, base radius r, and slant height $0.5h+2$, as shown the diagram. [W-16/23/Q7]

(i) Show that $r^2 = 2h+4-0.75h^2$ ---[2]

The volume of tent V, is given by $\frac{1}{3}\pi r^2 h$.

(ii) Given that h can vary, find correct to 2 decimal places, the value of h, which gives a stationary value of V. --- [5]

(iii) Determine the nature of stationary value ---[2]



Solution: In rt triangle OAB, using Pythagoras Theorem,

$$h^2+r^2 = (0.5h+2)^2$$
$$\Rightarrow h^2+r^2 = 0.25h^2+4+2h$$
$$\Rightarrow r^2 = 2h+4-0.75h^2 \quad \text{—— (1)}$$

(ii) Volume $V = \frac{1}{3}\pi r^2 h$

$$= \frac{\pi}{3}(2h+4-0.75h^2)\cdot h$$
$$V = \frac{\pi}{3}(2h^2+4h-0.75h^3)$$

diff:
$$\frac{dV}{dh} = \frac{\pi}{3}(4h+4-2.25h^2) \quad \text{—— (2)}$$

for stationary value of V, $\frac{dV}{dh} = 0$

from (2)
$$\Rightarrow \frac{\pi}{3}(4h+4-2.25h^2) = 0$$
$$\Rightarrow 2.25h^2-4h-4 = 0 \quad \times 4$$
$$\Rightarrow 9h^2-16h-16 = 0$$
$$h = \frac{16 \pm \sqrt{(-16)^2-4\times 9(-16)}}{2\times 9}$$

∴ Stationary Value of $h = 2.49$ ✓ only positive value.

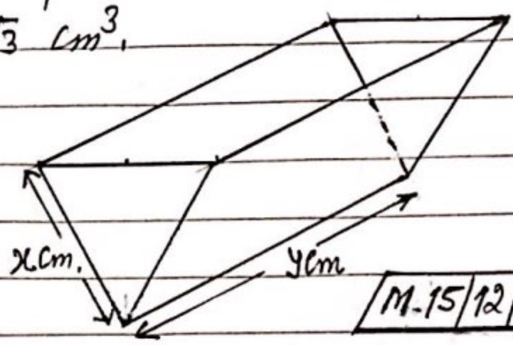
(iii) diff (2)
$$\frac{d^2V}{dh^2} = \frac{\pi}{3}(4-4.5h)$$
$$\left(\frac{d^2V}{dh^2}\right)_{h=2.49} = \frac{\pi}{3}(4-4.5\times 2.49) < 0 \quad \therefore \text{Max.}$$

∴ V is Max. at $h = 2.49$ ✓

Example 33. The diagram shows an empty container in the form of an open triangular prism. The triangular faces are equilateral with a side of x cm, and the length of each rectangular face is y cm. The container is made from thin sheet metal. When full, the container holds $200\sqrt{3}$ cm^3.

(i) Show that A cm^2, the total area of the thin sheet metal used, is given by,

$$A = \frac{\sqrt{3}}{2}x^2 + \frac{1600}{x} \quad \text{--- [5]}$$



M-15/12/Q9

(ii) Given that x and y can vary, find the stationary value of A and determine its nature. --- [6]

Solution: Area of equilateral triangular face $= \frac{\sqrt{3}}{4}x^2$

(i) Volume of prism $= \frac{\sqrt{3}}{4}x^2 \times y = 200\sqrt{3}$ (given)

$$\therefore x^2 y = 800 \quad \text{——} (1)$$

Total area of thin sheet, $A = 2 \times \frac{\sqrt{3}}{4}x^2 + 2xy$

or $A = \frac{\sqrt{3}}{2}x^2 + \frac{1600}{x}$ —(2) [$\because$ from (1) $xy = \frac{800}{x}$

(ii) Diff. A (from (2)); $\frac{dA}{dx} = \sqrt{3}x - \frac{1600}{x^2}$ —— (3)

for stationary value of A, $\frac{dA}{dx} = 0$

$$\Rightarrow \sqrt{3}x - \frac{1600}{x^2} = 0$$

$$\Rightarrow x^3 = \frac{1600}{\sqrt{3}} = 923.76$$

$$\Rightarrow x = \sqrt[3]{923.76}$$

$$\text{or } x = 9.74$$

from (2) $\therefore$ Stationary value of $A = \frac{\sqrt{3}}{2}(9.74)^2 + \frac{1600}{9.74}$

$$= 246 \text{ cm}^2$$

To determine the nature of the stationary value of A.

diff (3)

$$\frac{d^2A}{dx^2} = \sqrt{3} + \frac{3200}{x^3}$$

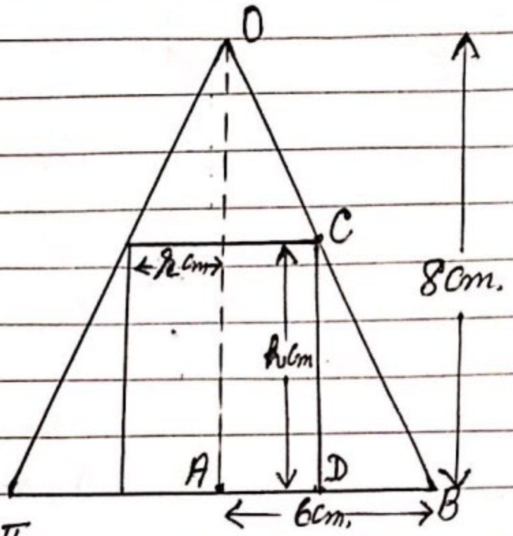
$$\left(\frac{d^2A}{dx^2}\right)_{x=9.74} = \sqrt{3} + \frac{3200}{(9.74)^3} > 0$$

$$= +ve$$

$\therefore$ Area is Minimum at $x = 9.74$

Differentiation

Example 34: A cone of height 8cm, and base radius 6cm, is placed over a cylinder of radius 'r' cm and height 'h' cm and is in contact with the cylinder along the cylinder upper rim. The arrangement is symmetrical and the diagram shows a vertical cross-section through the vertex of the cone.



(i) Use similar triangles to express h in terms of r. ---[2]

(ii) Hence show that the volume, $V\,cm^3$, of cylinder is given by,

$$V = 8\pi r^2 - \frac{4}{3}\pi r^3 \quad ---[1]$$

(iii) Given that 'r' can vary, find the value of 'r', which gives a stationary value of V. Find this stationary value of V in terms of π, and determine its nature. --- [6]

W-15/21/Q7

Solution: In similar triangles CDB and ΔOAB

(i) $\frac{CD}{OA} = \frac{DB}{AB} \Rightarrow \frac{h}{8} = \frac{6-r}{6}$

$\Rightarrow h = \frac{4}{3}(6-r)$ ——— ①

Volume of cylinder $V = \pi r^2 h$

$= \pi r^2 \cdot \frac{4}{3}(6-r)$

or $V = 8\pi r^2 - \frac{4}{3}\pi r^3$ ✓ ——— ②

(ii) diff. V w.r.t. r

$\frac{dV}{dr} = 16\pi r - 4\pi r^2$ ——— ③

$= 4\pi r(4-r)$

for stationary value of volume 'V'

$\frac{dV}{dr} = 0 \Rightarrow 4\pi r(4-r) = 0$

$\Rightarrow r = 4$ ✓ or $r = 0$ ✗

fr ② Stationary value of V at $r = 4$

$V = 8\pi \times 4^2 - \frac{4}{3} \times \pi \cdot 4^3 = \underline{\underline{\frac{128\pi}{3}}}$ ✓

Now to find the nature of stationary value of V, at $r=4$

diff ③

$\frac{d^2V}{dr^2} = 16\pi - 8\pi r$

$\left(\frac{d^2V}{dr^2}\right)_{r=4} = 16\pi - 8\pi \times 4 = -16\pi < 0$

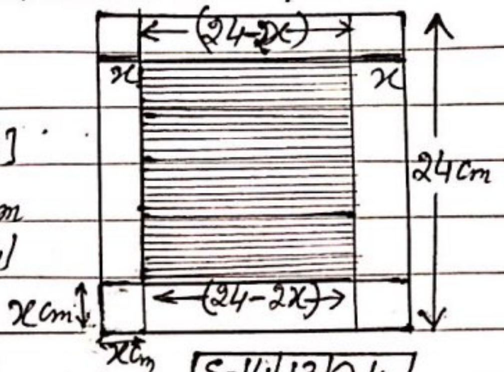
$\therefore$ Stationary value of V is Maximum.

Differentiation

Example 35. The diagram shows a thin square sheet of metal measuring 24cm by 24cm. A square of side x cm. is cut off from each corner. The remainder is then folded to form an open box, x cm deep, whose square base is shown shaded in the diagram.

(i) Show that the volume, V cm^3, of the box is given by $V = 4x^3 - 96x^2 + 576x$ ---[2]

(ii) Given that x can vary, find the maximum volume of the box. ---[4]



S-14/13 Q4

Solution (i) Volume of box $V = l\,b\cdot h$

$$V = (24-2x)(24-2x)\cdot x$$
$$= x(24-2x)^2 = x(4x^2 - 96x + 576)$$
$$\therefore\ V = 4x^3 - 96x^2 + 576x \quad \text{———} (1)$$

(ii) diff. (1)

$$\frac{dV}{dx} = 12x^2 - 192x + 576 \quad \text{———} (2)$$

for stationary value of V, $\frac{dV}{dx} = 0$

$$\Rightarrow 12x^2 - 192x + 576 = 0$$
$$\Rightarrow \cancel{12}(x^2 - 16x + 48) = 0$$
$$\Rightarrow (x-4)(x-12) = 0$$
$$\Rightarrow x = 4 \text{ or } x = 12^{\times}$$

To check the nature of stationary value at $x=4$.

diff. (2) $\frac{d^2V}{dx^2} = 24x - 192$
$$= 24(x-8)$$

$$\left(\frac{d^2V}{dx^2}\right)_{x=4} = 24(4-8) = -96 < 0, \quad \therefore \text{Maximum}$$

$\therefore$ Volume of box is Max. at $x=4$.

from (1) Max $V = 4(24-2\times4)^2$ $\quad [V = x(24-2x)^2]$
$$= 4\times16^2 = 1024$$
$$\therefore\ \text{Max } V = 1024 \text{ cm}^3. \checkmark$$