IG_0606
Additional Maths

Differentiation_2
Notes

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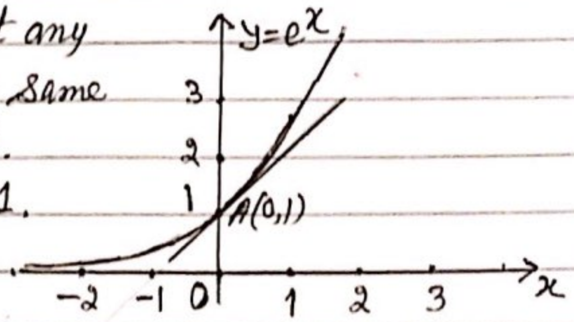
Differentiation - 2

§ Leonhard Eular (1701 - 1783). proved: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ (Constant) Irrational number.

or $\lim_{h\to 0}(1+h)^{\frac{1}{h}} = e = 2.71828\ldots$ (approx) —— (1)

{ use your Calculator to Check, $(1+.0000001)^{10000000} = 2.7182816\ldots$ }

What is so important about this number e. ?
'e' is called natural base to the exponentional function: e^x.
Consider the exponentional function $f(x) = e^x$: $x \in R$
It is observed that the gradient at any point on the graph of $y = e^x$ is same as the value of e^x at that point.
Example: Gradient at A(0, 1) is 1.



$\therefore\ y = e^x$

(i) $\frac{dy}{dx} = e^x$

(ii) $\frac{d}{dx}e^{ax+b} = a\cdot e^{ax+b}$

(iii) $\frac{d}{dx}e^{g(x)} = e^{g(x)} \times g'(x)$ using chain rule

$\lim\ y = e^x$; $\left\{\begin{array}{l} x \to x+\Delta x \\ y+\Delta y = e^{x+\Delta x} \\ \Delta y = e^{x+\Delta x} - e^x \\ \quad = e^x(e^{\Delta x}-1) \end{array}\right.$

$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$

$= \lim_{\Delta x\to 0}\frac{e^x(e^{\Delta x}-1)}{\Delta x}$

$= e^x \cdot \lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x}$

$= e^x \cdot \lim_{h\to 0}\frac{e^h-1}{h}$

$= e^x \times 1$ fr (2)

$\left\{\begin{array}{l} \text{frm (1) } h \text{ is small} \\ \lim_{h\to 0}(1+h)^{\frac{1}{h}} = e \\ \text{or } 1+h = e^h \\ \Rightarrow 1 = \frac{e^h-1}{h} \\ \text{or } \lim_{h\to 0}\frac{e^h-1}{h} = 1 \text{ —— (2)} \end{array}\right.$

$\therefore \frac{d}{dx}e^x = e^x$

Example(a). $\frac{d}{dx}e^{(5x+3)} = 5e^{(5x+3)}$

(b) $\frac{d}{dx}e^{x^3} = e^{x^3}\cdot\frac{d}{dx}x^3 = e^{x^3} \times 3x^2 = 3x^2\cdot e^{x^3}$ ✓

Differentiation-2

Example 1. Given that $y = \dfrac{e^{3x}}{4x^2+1}$, find $\dfrac{dy}{dx}$ ---[3]

$\boxed{S\text{-}17 / 11 / Q9(a)}$

Solution

$$\frac{dy}{dx} = \frac{(4x^2+1)\frac{d}{dx}e^{3x} - e^{3x}\cdot\frac{d}{dx}(4x^2+1)}{(4x^2+1)^2} \quad \text{(Quotient rule)}$$

$$= \frac{(4x^2+1)\cdot e^{3x} \times 3 - e^{3x} \times 8x}{(4x^2+1)^2}$$

$$= \frac{3e^{3x}}{(4x^2+1)} - \frac{8x}{(4x^2+1)^2}$$

Example 2. Show that $\dfrac{d}{dx}\left(e^{3x}\cdot\sqrt{4x+1}\right)$ can be written in the form,

$\dfrac{e^{3x}(px+q)}{\sqrt{4x+1}}$, where p and q are integers to be found. ---[5]

$\boxed{S\text{-}16 / 12 / Q6}$

Solution:

$$\frac{d}{dx}\left(e^{3x}\cdot\sqrt{4x+1}\right) = e^{3x}\cdot\frac{d}{dx}\sqrt{4x+1} + \sqrt{4x+1}\cdot\frac{d}{dx}e^{3x} \quad \text{(Product rule)}$$

$$= e^{3x}\cdot\frac{1}{\cancel{2}\sqrt{4x+1}} \times \cancel{4}^{\,2} + \sqrt{4x+1}\cdot e^{3x} \times 3$$

$$= \frac{2e^{3x} + 3(4x+1)e^{3x}}{(4x+1)^{1/2}}$$

$$= \frac{e^{3x}(2 + 12x + 3)}{(4x+1)^{1/2}}$$

$$= \frac{e^{3x}(12x+5)}{(4x+1)^{1/2}} \checkmark$$

Example 3. Differentiate w.r.t x, $x^4\cdot e^{3x}$ $\boxed{S\text{-}14 / 22 / Q7(i)}$ ---[2]

Solution: Let $y = x^4\cdot e^{3x}$

Diff. w.r.t x, $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^4\cdot e^{3x})$ (Product rule)

$$= x^4\cdot\frac{d}{dx}e^{3x} + e^{3x}\cdot\frac{d}{dx}x^4$$

$$= x^4\cdot e^{3x} \times 3 + e^{3x} \times 4x^3$$

$$= x^3 e^{3x}(3x+4) \checkmark$$

Differentiation 2.

§ **Natural Logarithms:** $\log_e x$ is specially denoted by $\ln x$

$\ln x$ is log. to the natural base 'e'.

§ **Derivative of $\ln x$.**

(i) $\frac{d}{dx}\ln x = \frac{1}{x}$ or $\begin{cases} y = \ln x \\ \frac{dy}{dx} = \frac{1}{x} \end{cases}$ or $f(x) = \ln x$, $f'(x) = \frac{1}{x}$

(ii) $\frac{d}{dx}\ln(ax+b) = \frac{a}{(ax+b)}$

Example:

differentiate w.r.t x

(a) $\ln(3x+2)$

$$\frac{d}{dx}\ln(3x+2) = \frac{1}{(3x+2)} \times 3 = \frac{3}{(3x+2)}$$

(b) $\ln(x^3-10)$

let $y = \ln(x^3-10)$

$$\frac{dy}{dx} = \frac{1}{(x^3-10)} \times \frac{d}{dx}(x^3-10) \quad \text{(using chain rule)}$$

$$= \frac{3x^2}{(x^3-10)}$$

(c) $x^5 \cdot \ln 5x$

let $f(x) = x^5 \cdot \ln 5x$

diff. w.r.t x $f'(x) = x^5 \cdot \frac{d}{dx}\ln 5x + \ln 5x \frac{d}{dx} x^5$

$$= x^5 \times \frac{1 \times 5}{5x} + \ln 5x \times 5x^4 \quad \text{(Using Product rule)}$$

$$= \underline{x^4 + 5x^4 \cdot \ln 5x}$$

(d) $\frac{\ln 3x}{x^3}$ let $f(x) = \frac{\ln 3x}{x^3}$

$$\therefore f'(x) = \frac{x^3 \cdot \frac{d}{dx}\ln 3x - \ln 3x \times \frac{d}{dx}x^3}{(x^3)^2} \quad \text{(Using quotient rule)}$$

$$= \frac{x^3 \times \frac{1}{3x} \times 3 - \ln 3x \times 3x^2}{x^6} = \frac{x^2 - 3x^2 \cdot \ln 3x}{x^6}$$

$$= \frac{(1 - 3\ln 3x)}{x^4}$$

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Example 4. It is given that $y = \frac{\ln(4x^2-1)}{(x+2)}$

(i) Find the value of x for which y is not defined. ---[2]

(ii) Find $\frac{dy}{dx}$ ---[3]

(iii) Hence find the approximate increase in y, when x increases from 2 to $(2+h)$, where h is small.

Solution:

(i) Denominator $x+2 \neq 0 \Rightarrow x \neq -2$

And $\ln(4x^2-1)$ is defined when $4x^2-1 > 0$

not defined for $4x^2-1 \leq 0$

or $(2x+1)(2x-1) \leq 0$

or $-\frac{1}{2} \leq x \leq \frac{1}{2}$

$\therefore$ y is not defined for -2; or $-\frac{1}{2} \leq x \leq \frac{1}{2}$ ✓

(ii) $y = \frac{\ln(4x^2-1)}{(x+2)}$

$$\frac{dy}{dx} = \frac{(x+2)\cdot\frac{d}{dx}\ln(4x^2-1) - \ln(4x^2-1)\times\frac{d}{dx}(x+2)}{(x+2)^2}$$

$$= \frac{(x+2)\times\frac{1}{(4x^2-1)}\times 8x - \ln(4x^2-1)\times 1}{(x+2)^2}$$

$$= \frac{8x^2+16x - (4x^2-1)\cdot\ln(4x^2-1)}{(4x^2-1)(x+2)^2}$$

(iii) $$\left(\frac{dy}{dx}\right)_{x=2} = \frac{8\times 2^2+16\times 2 - (4\times 2^2-1)\ln(4\times 2^2-1)}{(4\times 2^2-1)(2+2)^2}$$

$$= \frac{4}{15} - \frac{\ln 15}{16} = 0.0974 \checkmark \text{ --- (1)}$$

for small δx,

$$\frac{\delta y}{\delta x} = \frac{dy}{dx}$$

for x changes from 2 to $2+h$ $\Rightarrow$ $\frac{\delta y}{h} = \left(\frac{dy}{dx}\right)_2 = 0.0974$ fr (1)

$\therefore$ $\delta y = 0.0974h$ ✓

Differentiation-2

Example 5: (i) write $\ln\left(\frac{2x+1}{2x-1}\right)$ as the difference of two logarithms. ---[1]

(ii) A curve has equation, $y=\ln\frac{(2x+1)}{(2x-1)}+4x$ for $x>\frac{1}{2}$. Find $\frac{dy}{dx}$. --[4]

(iii) Hence find the x-coordinate of the stationary point on the curve. ---[2]

(iv) Determine the nature of this stationary point. [W-17/11/Q7] ---[2]

Solution: (i) $\ln\left(\frac{2x+1}{2x-1}\right)=\ln(2x+1)-\ln(2x-1)$ ✓ $\left[\because \log\frac{m}{n}=\log m-\log n\right]$

(ii) $y=\ln\left(\frac{2x-1}{2x+1}\right)+4x$

or $y=\ln(2x-1)-\ln(2x+1)+4x$ diff. w.r.t x

$$\frac{dy}{dx}=\frac{1}{(2x-1)}\times 2-\frac{2}{(2x+1)}+4$$

or $$\frac{dy}{dx}=\frac{16x^2-8}{4x^2-1}$$ ✓ —— ①

(iii) for stationary point $\frac{dy}{dx}=0$

$$\Rightarrow \frac{16x^2-8}{4x^2-1}=0$$

or $16x^2-8=0$

or $8(2x^2-1)=0 \Rightarrow x=\frac{1}{\sqrt{2}}$ or $\cancel{-\frac{1}{\sqrt{2}}}$

[as for $\ln(2x-1)$, $2x-1>0$, $x>\frac{1}{2}$]

(iv) To check the nature of stationary point.

diff ①

$$\frac{d^2y}{dx^2}=\frac{(4x^2-1)\times 32x-(16x^2-8)\times 8x}{(4x^2-1)^2}$$

$$=\frac{32x}{(4x^2-1)^2}$$ —②

Now $\left(\frac{d^2y}{dx^2}\right)_{x=\frac{1}{\sqrt{2}}}=\frac{32\times\frac{1}{\sqrt{2}}}{\left(4\times\left(\frac{1}{\sqrt{2}}\right)^2-1\right)^2}>0$ $\therefore$ Minimum ✓

$\therefore$ Given function has a Minimum at $x=\frac{1}{\sqrt{2}}$

Example 6. Find the equation of the normal to the curve $y=\ln(2x^2-7)$ at the point where the curve crosses the positive x-axis. Give your answer in the form $ax+by+c=0$ where a, b, and c are integers. [S-16/11/Q3] ---[5]

Solution: Given $y=\ln(2x^2-7)$ —— ①

Curve crosses x-axis where $y=0$

$\Rightarrow \ln(2x^2-7)=0$

$\Rightarrow 2x^2-7=1$ [$\because \ln 1=0$]

$\Rightarrow x=+2\checkmark, -2\times$

Now diff ① $\frac{dy}{dx}=\frac{1}{(2x^2-7)}\times 4x$ (Chain rule)

at $x=2$, $\left(\frac{dy}{dx}\right)=\frac{4\times 2}{2\times 2^2-7}=8$

$\therefore$ Gradient of normal at $x=2$, $m=-\frac{1}{8}$

and from ① $x=2\Rightarrow y=\ln 1=0$

$\therefore$ Equation of normal to the curve at $(2,0)$, $m=-\frac{1}{8}$

$y-y_1=m(x-x_1)$

or $y-0=-\frac{1}{8}(x-2)$

or $\underline{x+8y-2=0}\checkmark$

Example 7. (i) Find the equation of normal to the curve, $y = \frac{1}{2}\ln(3x+2)$ at the point 'P' where $x = -\frac{1}{3}$. --[4]

The normal to the curve at the point 'P' intersect the y-axis at the point 'Q'. The curve $y = \frac{1}{2}\ln(3x+2)$ intersects the y-axis at the point R.

(ii) Find the area of the triangle PQR. [W-16/11/Q5] --[3]

Solution: Given eqn of curve $y = \frac{1}{2}\ln(3x+2)$ —— (1)

when $x = -\frac{1}{3}$, $y = 0$; $\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{(3x+2)} \times 3 = \frac{3}{2(3x+2)}$

at $P(-\frac{1}{3}, 0)$; $\left(\frac{dy}{dx}\right)_{x=-\frac{1}{3}} = \frac{3}{2\left(3(-\frac{1}{3})+2\right)} = \frac{3}{2}$

$\therefore$ Gradient of Normal at P, $m = -\frac{2}{3}$

$\therefore$ Equation of Normal at $P(-\frac{1}{3}, 0)$

$$y - 0 = -\frac{2}{3}\left(x - (-\tfrac{1}{3})\right)$$

or $y = -\frac{2}{3}\left(x + \frac{1}{3}\right)$ —— (2)

Normal (2) intersects y-axis at Q $(x = 0)$

fr (2) $y = -\frac{2}{3}\left(0 + \frac{1}{3}\right) = -\frac{2}{9}$

$\therefore Q\left(0, -\frac{2}{9}\right)$

and Curve intersect y-axis at R.

$x = 0$ in (1) $y = \frac{1}{2}\ln(0+2) = \frac{1}{2}\ln 2$

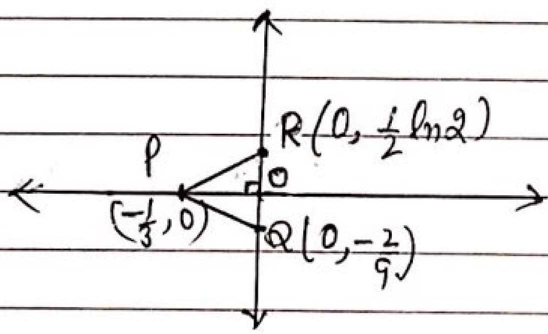
$R\left(0, \frac{1}{2}\ln 2\right)$

Area of Triangle PQR

$= \frac{1}{2} QR \times OP$ $[OP \perp QR]$

$= \frac{1}{2}\left(\frac{2}{9} + \frac{1}{2}\ln 2\right) \times \frac{1}{3}$

$= 0.0948$



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Differentiation - 2

§ Differentiation of Trigonometric function:

1\. (i) $\frac{d}{dx} \sin x = \cos x$

(ii) $\frac{d}{dx} \sin(ax+b) = a\cos(ax+b)$

2\. (i) $\frac{d}{dx} \cos x = -\sin x$

(ii) $\frac{d}{dx} \cos(ax+b) = -a\sin(ax+b)$

3\. (i) $\frac{d}{dx} \tan x = \sec^2 x$

(ii) $\frac{d}{dx} \tan(ax+b) = a\sec^2(ax+b)$

Example. Find the derivative.

(a) $\frac{d}{dx} \sin^3 x = \frac{d}{dx}(\sin x)^3$

$= 3\sin^2 x \cdot \frac{d}{dx} \sin x$

$= \underline{3\sin^2 x \cdot \cos x}$ ✓

(b) $\frac{d}{dx} \cos x^3$

$= \frac{d}{dx} \cos(x^3)$

$= -\sin x^3 \times \frac{d}{dx} x^3$

$= \underline{-3x^2 \cdot \sin x^3}$ ✓

(c) $\frac{d}{dx} \tan^2 5x$

$= \frac{d}{dx}(\tan 5x)^2$

$= 2\tan 5x \cdot \frac{d}{dx} \tan 5x$

$= 2\tan 5x \cdot \sec^2 5x \cdot \frac{d}{dx} 5x$

$= 2\tan 5x \sec^2 5x \times 5$

$= 10\tan 5x \sec^2 5x$, ✓

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Differentiation-2

Example 8. A curve has equation $y = 4 + 5\sin 3x$

(i) Find $\frac{dy}{dx}$ ---[2]

(ii) Hence find the equation of the tangent to the curve, $y = 4 + 5\sin x$ at the point where $x = \frac{\pi}{3}$. $\boxed{M\text{-}18/12/Q2}$ ---[3]

Solution: $y = 4 + 5\sin 3x$ —— (1)

(i) $\frac{dy}{dx} = 5\cos 3x \times 3$
$= 15\cos 3x$

(ii) at $x = \frac{\pi}{3}$, $\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{3}} = 15\cdot\cos \cancel{3}\cdot\frac{\pi}{\cancel{3}} = 15\cos\pi$ $[\cos\pi = -1]$
$= 15\times(-1)$
$= -15$ ✓

∴ Slope of the tangent $m = -15$ ✓

and $x = \frac{\pi}{3}$; $y = 4 + 5\sin 3\times\frac{\pi}{3}$
$= 4 + 5\sin\pi = 4 + 5\times 0$ $(\sin\pi = 0)$
$= 4$

∴ The equation of tangent at $(\frac{\pi}{3}, 4)$ ✓ and $m = -15$

$y - 4 = -15(x - \frac{\pi}{3})$

or $y = -15x + 5\pi + 4$

or $y = -15x + 19.7$ ✓

Example 9. Differentiate w.r.t x,

(i) $(1+4x)^{10}\cos x$ ---[4]

(ii) $\frac{e^{4x-5}}{\tan x}$ ---[4]

$\boxed{S\text{-}17/23/Q7}$

Solution: (i) $y = (1+4x)^{10}\cdot\cos x$

$\frac{dy}{dx} = (1+4x)^{10}\cdot\frac{d}{dx}\cos x + \cos x\frac{d}{dx}(1+4x)^{10}$

$= (1+4x)^{10}\times(-\sin x) + \cos x\times 10(1+4x)^{9}\times 4$

$= -\sin(1+4x)^{10} + \cos x(1+4x)^{9}$

(ii) $\frac{d}{dx}\frac{e^{4x-5}}{\tan x}$

$= \frac{\tan x\cdot\frac{d}{dx}e^{4x-5} - e^{4x-5}\cdot\frac{d}{dx}\tan x}{\tan^2 x}$

$= \frac{\tan x\times e^{4x-5}\times 4 - e^{4x-5}\cdot\sec^2 x}{\tan^2 x}$

$= \frac{e^{4x-5}[4\tan x - \sec^2 x]}{\tan^2 x}$ ✓

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Differentiation - 2

Example 10 (i) Differentiate $(\cos x)^{-1}$ w.r.t x. ---[2]

(ii) Hence find $\frac{dy}{dx}$; given that $y = \tan x + 4(\cos x)^{-1}$ ---[2]

(iii) Using your answer to part (ii) find the value of x, in the range $0 \le x \le 2\pi$ such that $\frac{dy}{dx} = 4$ $\boxed{W\text{-}17/21/Q12}$ --[6]

Solution: (i) $\frac{d}{dx}(\cos x)^{-1} = -1(\cos x)^{-2} \times \frac{d}{dx}\cos x$ (chain rule)

$$= -\frac{1}{\cos^2 x} \times -\sin x$$

$$= \frac{\sin x}{\cos^2 x} \quad \text{——— (1)}$$

(ii) $y = \tan x + 4(\cos x)^{-1}$

$$\frac{dy}{dx} = \sec^2 x + 4 \times \frac{\sin x}{\cos^2 x} \qquad \text{fm (1)}$$

$$= \frac{1}{\cos^2 x} + \frac{4\sin x}{\cos^2 x}$$

$$= \frac{1 + 4\sin x}{\cos^2 x} \quad \text{——— (2)}$$

(iii) Given $\frac{dy}{dx} = 4$

To solve,

or $\frac{1 + 4\sin x}{\cos^2 x} = 4$ $\qquad 0 \le x \le 2\pi$

or $1 + 4\sin x = 4\cos^2 x$

$\Rightarrow 4\cos^2 x - 4\sin x - 1 = 0$

$\Rightarrow 4(1 - \sin^2 x) - 4\sin x - 1 = 0$

$\Rightarrow 4\sin^2 x + 4\sin x - 3 = 0$

$\Rightarrow (2\sin x - 1)(2\sin x + 3) = 0$

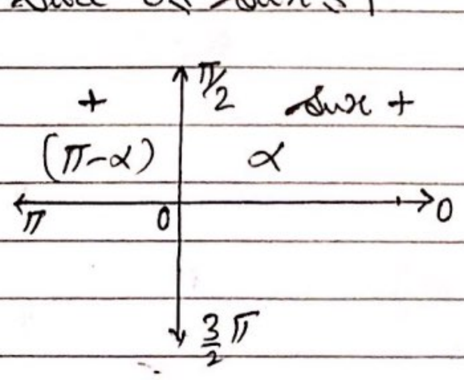
$\sin x = \frac{1}{2}$ or $-\frac{3}{2}$ ✗

$= \sin\frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}$ or $\left(\pi - \frac{\pi}{6}\right)$

$= \frac{\pi}{6}$ or $\frac{5\pi}{6}$ ✓

Since $0 \le \sin x \le 1$



Differentiation - 2

Example 11 (i) Given that $y = \frac{\tan 2x}{x}$, find $\frac{dy}{dx}$ ---[3]

(ii) Hence find the equation of normal to the curve, $y = \frac{\tan 2x}{x}$ at the point where $x = \frac{\pi}{8}$. [M-15/12/Q6] ---[3]

Solution: $y = \frac{\tan 2x}{x}$ ——— (1)

(i) diff w.r.t x

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}\tan 2x - \tan 2x \cdot \frac{d}{dx}x}{x^2} \quad \text{(Quotient Rule)}$$

$$= \frac{x(2\sec^2 2x) - \tan 2x \times 1}{x^2}$$

$$= \frac{2x\sec^2 2x - \tan 2x}{x^2} \checkmark \text{ ——— (2)}$$

(ii) $x = \frac{\pi}{8}$, frm (1) $y = \frac{\tan\left(2\times\frac{\pi}{8}\right)}{\pi/8} = \frac{\tan\frac{\pi}{4}}{\frac{\pi}{8}} = \frac{8}{\pi}\times 1 = \frac{8}{\pi}$

$\therefore$ Point $\left(\frac{\pi}{8}, \frac{8}{\pi}\right)$

frm (2) $\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{8}} = \frac{2\times\frac{\pi}{8}\cdot\sec^2\frac{\pi}{4} - \tan\frac{\pi}{4}}{\left(\frac{\pi}{8}\right)^2} = \frac{\frac{\pi}{4}\times 2 - 1}{\frac{\pi^2}{64}}$

$$m_1 = \left(\frac{32}{\pi} - \frac{64}{\pi^2}\right) \text{ ——— (3)}$$

$\therefore$ gradient of normal: $m_2 = -\frac{1}{m_1} = \frac{-\pi^2}{32(\pi-2)}$ frm (3)

$\therefore$ Equation of normal at $\left(\frac{\pi}{8}, \frac{8}{\pi}\right)$, $m_2 = \frac{-\pi^2}{32(\pi-2)}$

$$y - \frac{8}{\pi} = -\frac{\pi^2}{32(\pi-2)}\left(x - \frac{\pi}{8}\right)$$

or $y = -0.27x + 2.65$ ✓

Differentiation-2

Example 12. Variables x and y are related by the equation, $y = 10 - 4\sin^2 x$, where $0 \leq x \leq \frac{\pi}{2}$. Given that x increasing at a rate of 0.2 radians per seconds, find the corresponding rate of change of y when $y = 8$. [S-13/21/Q3] ---[6]

Solution: $y = 10 - 4\sin^2 x$ —— (1)

$\frac{dy}{dx} = -4 \times 2\sin x \frac{d}{dx}\sin x$ chain rule

$\frac{dy}{dx} = -8\sin x \cdot \cos x$ —— (2)

when $y = 8$ fr (1) $10 - 4\sin^2 x = 8$

$\Rightarrow 4\sin^2 x = 2$

$\sin^2 x = \frac{1}{2}$

$\sin x = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$ × $\quad 0 \leq x \leq \frac{\pi}{2}$

$x = \frac{\pi}{4}$ ✓

Now $\left(\frac{dy}{dx}\right)_{\frac{\pi}{4}} = -8 \sin\frac{\pi}{4} \cos\frac{\pi}{4}$ fm (2)

$= -8 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = -4$ ✓ —— (3)

Now $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

given $\frac{dx}{dt} = 0.2$

fr (3) $\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = -4$

when $y = 8$, $x = \frac{\pi}{4}$

$\frac{dy}{dt} = -4 \times 0.2$

$= -0.8$

∴ y is decreasing at the rate of 0.8 radians per sec.

Differentiation – 2

Example 13. A curve has equation $y = 7 + \tan x$ find,

(i) the equation of the tangent to the curve at the point where $x = \frac{\pi}{4}$, ---[4]

(ii) the value of x between 0 and π radians for which $\frac{dy}{dx} = y$. ---[4]

W-16/23/Q6

Solution: $y = 7 + \tan x$ —— (1)

(i) Diff. $\frac{dy}{dx} = \sec^2 x$ —— (2)

$$\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = \sec^2\frac{\pi}{4} = 2$$

at $x = \frac{\pi}{4}$

from (1) $y = 7 + \tan\frac{\pi}{4} = 7 + 1 = 8$

$\therefore$ Equation of tangent at Point $(\frac{\pi}{4}, 8)$, gradient $m = 2$

$$y - 8 = 2\left(x - \frac{\pi}{4}\right)$$

or $y = 2x + 6.429$ —— (3)

(ii) To solve:

$$\frac{dy}{dx} = y$$

or $\sec^2 x = 7 + \tan x$ from (1) & (2)

or $1 + \tan^2 x = 7 + \tan x$ [$\because \sec^2 x = 1 + \tan^2 x$

or $\tan^2 x - \tan x - 6 = 0$

$(\tan x - 3)(\tan x + 2) = 0$

or $\tan x = 3, -2$ $\quad 0 \leq x \leq \pi$

when $\tan x = 3$

$x = \tan^{-1} 3 = 1.25$ rad

$\tan x = -2$

$x = \pi - \tan^{-1} 2$

$= \pi - 1.1071$

$= 2.03$ rad

$x = 1.25$ rad or 2.03 rad

Example 14. Find $\frac{dy}{dx}$, when

(i) $y = \cos 2x \cdot \sin\left(\frac{x}{3}\right)$ ---[4]

(ii) $y = \frac{\tan x}{1+\ln x}$ ---[4]

$\boxed{S\text{-}14/21/Q10}$

Solution:

(i) $y = \cos 2x \cdot \sin x/3$

$\frac{dy}{dx} = \cos 2x \cdot \frac{d}{dx}\sin x/3 + \sin\frac{x}{3} \cdot \frac{d}{dx}\cos 2x$ (Product rule)

$= \cos 2x \cdot \cos\frac{x}{3} \times \frac{1}{3} + \sin\frac{x}{3} \times (-\sin 2x) \times 2$

$= \frac{1}{3}\cos x/3 \cdot \cos 2x - 2\sin 2x \cdot \sin x/3$ ✓

(ii) $y = \frac{\tan x}{(1+\ln x)}$

$\frac{dy}{dx} = \frac{(1+\ln x)\frac{d}{dx}\tan x - \tan x \cdot \frac{d}{dx}(1+\ln x)}{(1+\ln x)^2}$ (Quotient Rule)

$= \frac{(1+\ln x)\sec^2 x - \tan x \times \frac{1}{x}}{(1+\ln x)^2}$

$= \frac{x\sec^2 x(1+\ln x) - \tan x}{x(1+\ln x)^2}$ ✓

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Differentiation - 2

Example 15. Variables x, y and t are such that:

$$y = 4\cos\left(x+\frac{\pi}{3}\right) + 2\sqrt{3}\sin\left(x+\frac{\pi}{3}\right) \text{ and } \frac{dy}{dt} = 10$$

(i) Find the value of $\frac{dy}{dx}$ when $x = \frac{\pi}{2}$ --- [3]

(ii) Find the value of $\frac{dx}{dt}$ when $x = \frac{\pi}{2}$ --- [2]

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Solution:

$$y = 4\cos\left(x+\frac{\pi}{3}\right) + 2\sqrt{3}\sin\left(x+\frac{\pi}{3}\right)$$

diff. w.r.t x

$$\frac{dy}{dx} = -4\sin\left(x+\frac{\pi}{3}\right) + 2\sqrt{2}\cos\left(x+\frac{\pi}{3}\right)$$

$$\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}} = -4\sin\left(\frac{\pi}{2}+\frac{\pi}{3}\right) + 2\sqrt{2}\cos\left(\frac{\pi}{2}+\frac{\pi}{3}\right)$$

$$= -4\sin\left(\frac{5\pi}{6}\right) + 2\sqrt{3}\cos\frac{5\pi}{3}$$

$$= -4\times\frac{1}{2} + 2\sqrt{3}\times\left(-\frac{\sqrt{3}}{2}\right)$$

$$= -2-3$$

$$= -5 \checkmark \quad \text{—— (1)}$$

$\sin\frac{5\pi}{6}$

$= \sin\left(\pi-\frac{\pi}{6}\right)$

$= \sin\frac{\pi}{6} = \frac{1}{2} \checkmark$

and

$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

(ii) $\frac{dx}{dt} = ?$ at $x = \frac{\pi}{2}$

$$\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}} \times \left(\frac{dx}{dt}\right)_{x=\frac{\pi}{2}} = \frac{dy}{dt}$$

$$-5 \times \left(\frac{dx}{dt}\right)_{x=\frac{\pi}{2}} = 10$$

$\because$ Given $\frac{dy}{dt} = 10$

$$\Rightarrow \left(\frac{dx}{dt}\right)_{x=\frac{\pi}{2}} = \frac{10}{-5} = \underline{-2} \checkmark$$