IG_0606 Additional Maths

Diff. and Integral's Application to Kinematics Notes

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Kinematics

Application of differentiation and integration

distance 'x' is function of time, $x = f(t)$

velocity $v = \dfrac{dx}{dt}$

and acceleration $a = \dfrac{dv}{dt}$

Conversely:

Given $acc = f(t)$

$\Rightarrow v = \int f(t)\,dt$ ✓

and $v = g(t)$

$\Rightarrow x = \int g(t)\,dt$ ✓

Example 1. A particle P is projected from the origin O, so that it moves in a straight line. At time t seconds after projection, the velocity of the particle, $v\,ms^{-1}$, is given by,

$$v = 9t^2 - 63t + 90$$

[M-16/22/Q12]

(i) Show that P comes to first comes to instantaneous rest when $t = 2$ ---[2]

(ii) Find the acceleration of P when $t = 3.5$ ---[2]

(iii) Find an expression for the displacement of P from O at time t seconds. ---[3]

(iv) Find the distance travelled by P

(a) in the first 2 seconds, ---[2]

⊗ (b) in the first 3 seconds. ---[2]

Solution (i) For instantaneous rest $v = 0$

or $9t^2 - 63t + 90 = 0$

$(9t - 18)(t - 5) = 0 \Rightarrow t = 2, t = 5$ ✓

$\therefore t = 2$ is smaller value of t.

(ii) acceleration $a = \dfrac{dv}{dt} = \dfrac{d}{dt}(9t^2 - 63t + 90)$

$= 18t - 63$

acc. at $t = 3.5$, $\left(\dfrac{dv}{dt}\right)_{t=3.5} = 18 \times 3.5 - 63$

$= 0$

(iii) displacement $x = \int v\,dt$

or $x = \int (9t^2 - 63t + 90)\,dt$

$x = \dfrac{9t^3}{3} - \dfrac{63t^2}{2} + 90t$ —— ①

(iv)(a) from ① distance at $t = 2$

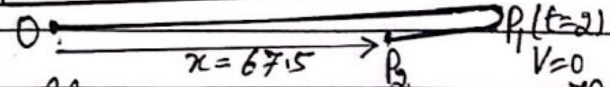
$x = \dfrac{9(2)^3}{3} - \dfrac{63(2)^2}{2} + 90 \times 2$

$= 78m$ —— ②

(b) at $t = 3$ $\boxed{\text{Imp.}}$ ⊗

$x = \dfrac{9(3)^3}{3} - \dfrac{63(3)^2}{2} + 90 \times 3 = 67.5m$ —— ③

$\boxed{\text{How is it that distance travelled in 3 sec. is less than 2 sec. ⊛}}$



$P_1P_2 = 78 - 67.5 = 10.5$

$\therefore$ when $t = 3$ distance travell

$= OP_1 + P_1P_2 = 78 + 10.5 = \underline{\underline{88.5m}}$ ✓

Example 2. A particle P travels in a straight line such that, t s after passing through a fixed point O, its velocity v ms^{-1} is given by $v = (e^{\frac{t^2}{8}} - 4)^3$.

(a) Find the speed of P at O. ---[1]

(b) Find the value of t for which P is instantaneously at rest. ---[2]

(c) Find the acceleration of P when t=1. [SP-2020|02|Q6] ---[4]

Solution (a) $v = (e^{t^2/8} - 4)^3$ ——— ①

when P at O, $t = 0 \Rightarrow v = (e^0 - 4)^3 = (1-4)^3 = -27$

neglect the -ve sign, speed $v = 27\ ms^{-1}$ ✓

(b) P is at instantaneous rest when $v = 0$

or from ① $v = (e^{t^2/8} - 4)^3 = 0$

or $e^{t^2/8} - 4 = 0$

$e^{t^2/8} = 4$

or $\frac{t^2}{8} = \ln 4$

or $t^2 = 8\ln 4 = 11.09$

$t = \sqrt{11.09} = 3.33\ s$ ✓

(c) acceleration of P, $a = \frac{dv}{dt}$

$= \frac{d}{dt}(e^{t^2/8} - 4)^3 = 3 \times \frac{2t}{8}(e^{t^2/8} - 4)^2 \times e^{t^2/8}$

Now at t=1, $a = \frac{3 \times 2}{8} \times e^{1/8}(e^{1/8} - 4)^2$

$= 0.75 \times 1.133(1.133 - 4)^2$

$= 6.98\ ms^{-2}$ ✓

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Diff and Integration's Application to Kinematics

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Example 3. A particle moves in a straight line so that, t seconds after passing through a fixed point O, its displacement, s m, from O, is given by:

$$s = 1 + 3t - \cos 5t$$

(i) Find the distance between the particle's first two positions of instantaneous rest. ---[7]

(ii) Find the acceleration when $t = \pi$. [S-17/21/Q12] ---[2]

Solution: Given $s = 1 + 3t - \cos 5t$ —— ①

speed, $v = \frac{ds}{dt} = 3 + 5\sin 5t$ —— ②

for instantaneous rest $v = \frac{ds}{dt} = 0$

or $3 + 5\sin 5t = 0$

$\Rightarrow \sin 5t = -3/5$

or $5t = 3.785$ or 5.6397

$t = 0.76$ s or 1.13 s

distance at $(t = 0.76) = 4.07$. from ①

distance at $(t = 1.13) = 3.58$

$\therefore$ distance between the two positions $= 4.07 - 3.58$

$= \underline{0.49 \text{ m}}$ ✓

(ii) Now acceleration $a = \frac{dv}{dt} = \frac{d}{dt}(3 + 5\sin 5t)$

$a = 25\cos 5t$

$\therefore$ acc. at $t = \pi$ $= 25\cos 5\pi$

$= 25 \times -1$

$= \underline{-25 \text{ ms}^{-2}}$ ✓

Example 4. A particle P, moving in a straight line, passes through a fixed point O at time $t=0$ s. At t s after leaving O, the displacement of the particle is x m and its velocity is $v\ \text{ms}^{-1}$, where

$$v = 12e^{2t} - 48t, \quad t \geqslant 0$$

(i) Find x in terms of t. ---[4]

(ii) Find the value of t, when acceleration of P is zero. ---[3]

(iii) Find the velocity of P when acceleration is zero. ---[2]

M-18/12/Q8

Solution Given $v = 12e^{2t} - 48t$ —— ①

(i) displacement $x = \int v\,dt = \int (12e^{2t} - 48t)\,dt$

$$= \frac{12}{2}e^{2t} - 48\frac{t^2}{2}$$

$$\text{or } x = 6e^{2t} - 24t^2 + c \text{ —— ②}$$

Now when $t=0$, $x=0$ from ② $0 = 6\times1 - 24\times0 + c \Rightarrow c = -6$

from $x = 6e^{2t} - 24t^2 - 6$ ✓

(ii)

Acceleration $a = \frac{dv}{dt} = \frac{d}{dt}(12e^{2t} - 48t)$

$$= 12\times2e^{2t} - 48$$

When Acc $= 0 \Rightarrow 24e^{2t} - 48 = 0 \Rightarrow e^{2t} = 2 \Rightarrow 2t = \ln 2$

$$\Rightarrow t = \tfrac{1}{2}\ln 2 = \ln\sqrt{2} \checkmark$$

$$\text{or } t = \underline{0.347} \checkmark$$

(iii) Put $t = \ln\sqrt{2}$ in ① for when acc $= 0$

$$v = 12e^{(\ln 2)} - 48\ln\sqrt{2}$$

$$= 24 - 48\times0.3465$$

$$= \underline{7.36} \checkmark$$

Example 5. The velocity $v\,ms^{-1}$ of a particle travelling in a straight line, t seconds after passing through a fixed point O, is given by, $v = \frac{10}{(2+t)^2}$

(i) Find the acceleration of the particle when $t = 3$ ---[3]

(ii) Explain why the particle never comes to rest. ---[1]

(iii) Find an expression for the displacement of the particle from O after time t s.

(iv) Find the distance travelled by the particle between $t = 3$ and $t = 8$.

W-15/23/Q7

Solution: Given $v = 10(2+t)^{-2}$ — ①

(i) acceleration $a = \frac{dv}{dt} = -20(t+2)^{-3} = \frac{-20}{(t+2)^3}$ — ②

$\therefore$ acc. at $t=3$, $= \left(\frac{dv}{dt}\right)_{t=3} = \frac{-20}{(3+2)^3} = \frac{-20}{125} = -0.16\,ms^{-2}$ ✓

(ii) for particle to be on rest $v=0 \Rightarrow \frac{10}{(2+t)^2} = 0$ which is not true for any value of t. ✓

(iii) displacement 's' from O at any time,

from ① $v = \frac{ds}{dt} = 10(2+t)^{-2}$

$\therefore s = \int 10(2+t)^{-2}\,dt = \frac{10(2+t)^{-1}}{-1} + C$

for $t=0$, $s=0$ $\Rightarrow 0 = \frac{-10}{2} + C \Rightarrow C = 5$

$\therefore$ displacement $s = \frac{-10}{(2+t)} + 5$ — ③ ✓

(iv) distance from $t=3$ to $t=8$ is $= \left[\frac{-10}{t+2}\right]_3^8 = -1+2 = 1\,m$ ✓

Kinematics

Example 6. A particle moving in a straight line has a velocity of $v\,ms^{-1}$ such that, t s after leaving a fixed point,
$v = 4t^2 - 8t + 3$.

(i) Find the acceleration of the particle when $t = 3$. ---[2]

(ii) Find the value of t for which the particle momentarily at rest. ---[2]

⊗ (iii) Find the total distance the particle has travelled when $t = 1.5$. ---[5]

W-16/13/Q11

Solution: $v = 4t^2 - 8t + 3$ —— ①

(i) acceleration $a = \frac{dv}{dt} = \frac{d}{dt}(4t^2 - 8t + 3) = 8t - 8$

accele. at $t = 3$, $\therefore \left(\frac{dv}{dt}\right)_{t=3} = 8 \times 3 - 8 = 16$ ✓

(ii) Particle is at rest when $v = 0 \Rightarrow 4t^2 - 8t + 3 = 0$ fr ①

$(2t - 3)(2t - 1) = 0$

$\Rightarrow t = \frac{1}{2}$ or $t = \frac{3}{2}$

(iii) $v = \frac{ds}{dt} = 4t^2 - 8t + 3$ fr ①

$s = \int (4t^2 - 8t + 3)\,dt$

or $s = \frac{4}{3}t^3 - 4t^2 + 3t + c$ —— ②

Now $s = 0$ when $t = 0$

fr ② $0 = 0 + c \Rightarrow c = 0$

fr ② $s = \frac{4}{3}t^3 - 4t^2 + 3t$ —— ③

when $t = \frac{1}{2}$, $s = \frac{4}{3} \cdot \left(\frac{1}{2}\right)^3 - 4\left(\frac{1}{2}\right)^2 + 3 \times \frac{1}{2} = \frac{2}{3}$ ✓

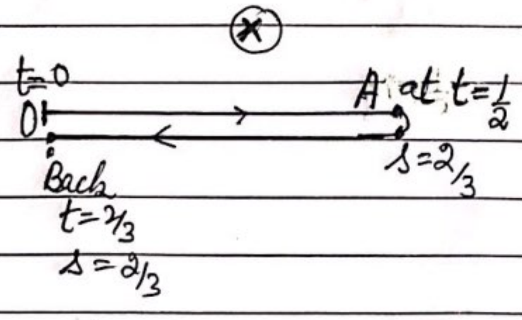
when $t = \frac{3}{2}$, $s = \frac{4}{3}\left(\frac{3}{2}\right)^3 - 4\left(\frac{3}{2}\right)^2 + 3 \times \frac{3}{2} = 0$ or Particles is back at O.

⊗ Total distance travelled

$= OA + AO$

$= \frac{2}{3} + \frac{2}{3}$

$= \frac{4}{3}$ ✓



Kinematics

Example 7. A particle moves in a straight line such that, t s, after passing throught a fixed point O, its velocity $v\ ms^{-1}$, is given by. $v = 5 - 4e^{-2t}$

(i) Find the velocity of the particle at O. --- [1]

(ii) Find the value of t when the acceleration of the particle is $6ms^{-2}$ --- [3]

(iii) Find the distance of the particle from O when $t = 1.5$ --- [5]

(iv) Explain why the particle does not return at O. -- [1]

Solution: $v = 5 - 4e^{-2t}$ —— ①

(i) Velocity at O is when $t = 0$, from ① $5 - 4e^{-2\times 0} = 5 - 4e^{0} = 5 - 4 = 1$ ✓

(ii) diff ① $a = \frac{dv}{dt} = \frac{d}{dt}(5 - 4e^{-2t}) = 8e^{-2t}$

$\therefore$ acceleration $8e^{-2t} = 6$

$\Rightarrow e^{-2t} = \frac{3}{4}$

$\Rightarrow -2t = \ln \frac{3}{4}$

$\Rightarrow t = -\frac{1}{2}\ln\frac{3}{4} = -\frac{1}{2}\times(-0.288)$

$\therefore t = 0.144$ ✓

(iii) from ① $v = \frac{ds}{dt} = 5 - 4e^{-2t}$

or $s = \int(5 - 4e^{-2t})dt$

or $s = 5t + 2e^{-2t} + c$ —— ②

$s = 0$ when $t = 0 \Rightarrow 0 = 0 + 2e^{0} + c \Rightarrow c = -2$

from ② $s = 5t + 2e^{-2t} - 2$ —— ③

Now when $t = 1.5$, $s = 5\times 1.5 + 2e^{-3} - 2 = 7.5 + 0.1 - 2 = 5.60$ ✓

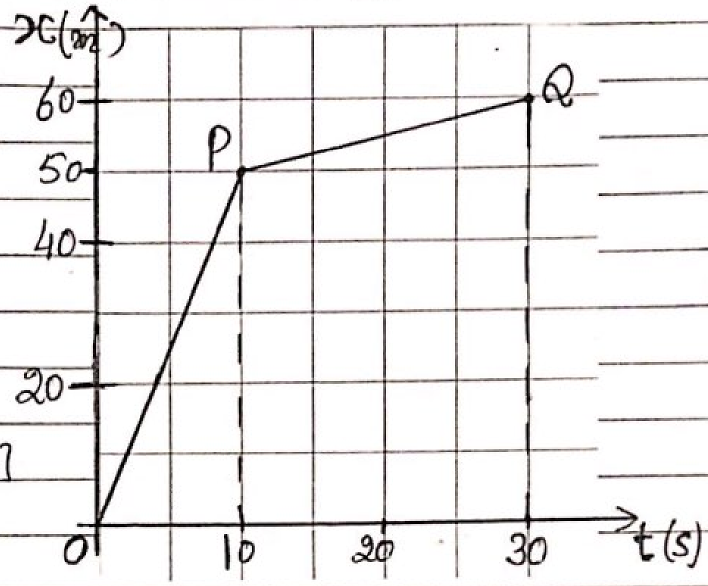
(iv) Velocity is always positive, so no change in the direction.

Kinematics

Distance-time and Velocity-time Graphs.

Example 8. The diagram shows the displacement-time graph of a particle P which moves in a straight line such that, t s after leaving a fixed point O, Its displacement from O is x m. On the axes below, draw the velocity-time graph of P. ---[3]

W-17/12/Q9



Solution: The motion of the particle along line OP shows that the speed is constant.

Velocity = $\frac{\text{Displacement}}{\text{Time}}$

Speed along OP $= \frac{50}{10} = 5\,ms^{-1}$

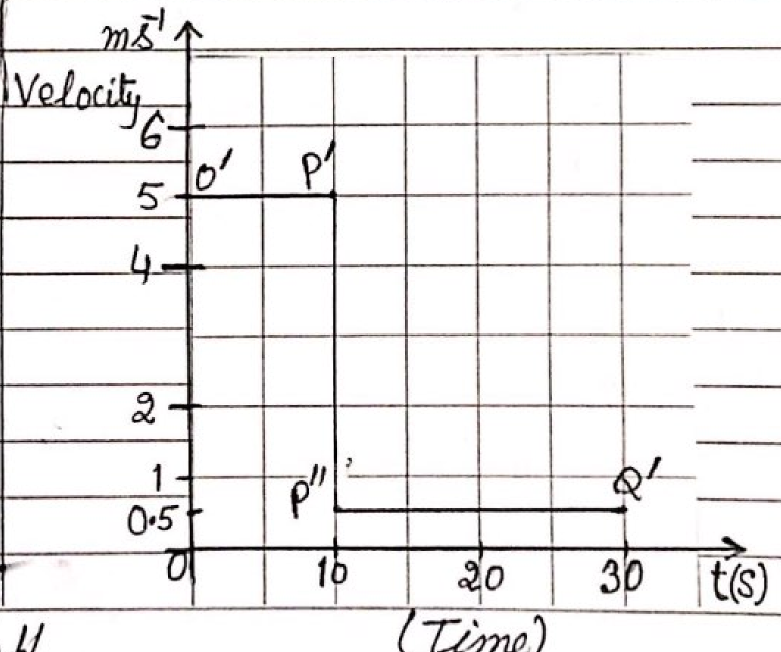
Represented by O'P' on the graph.

and Motion along line PQ

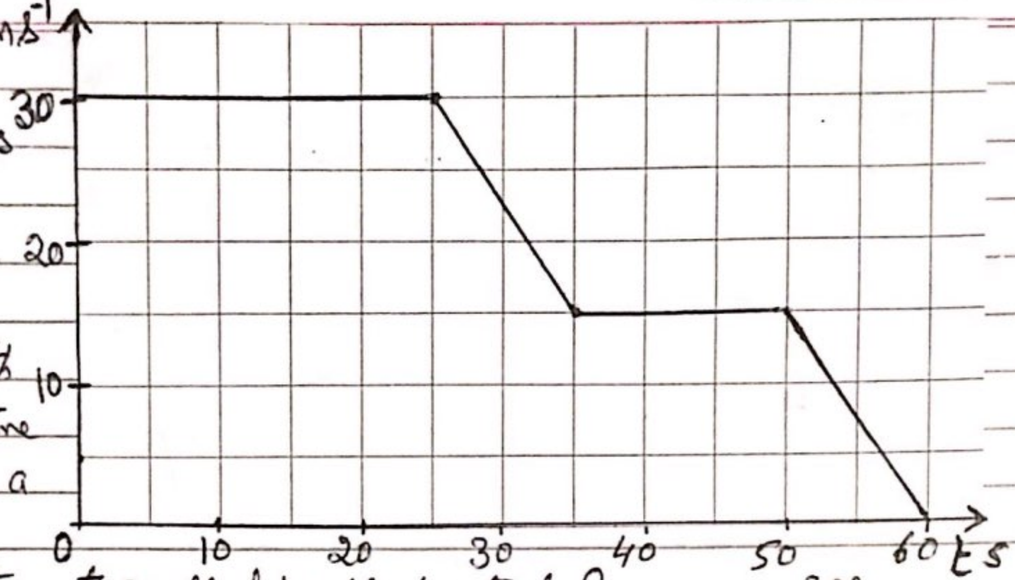
Vel $= \frac{\text{displacement}}{\text{time}}$

$= \frac{10}{20} = 0.5\,ms^{-1}$

Shown along P''Q' on the Velocity-time graph.



Example-9.

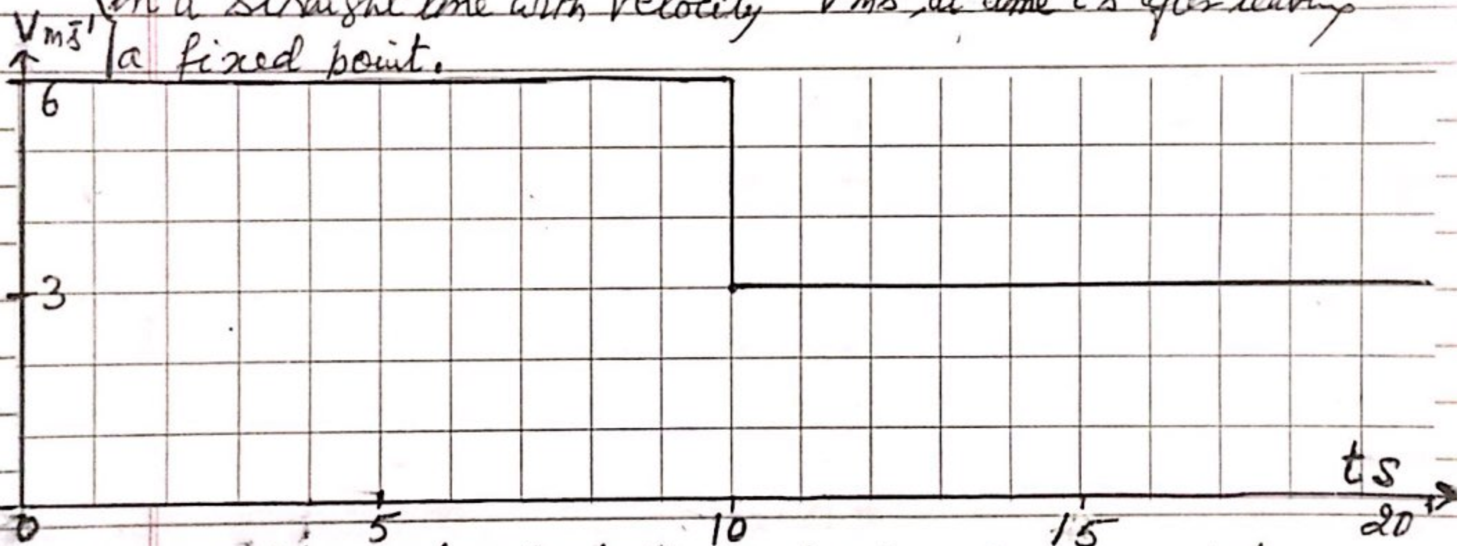


(a) The diagram shows the velocity-time graph of a particle 'P' moving in a straight line with velocity m s^{-1} at time t s after leaving a fixed point.

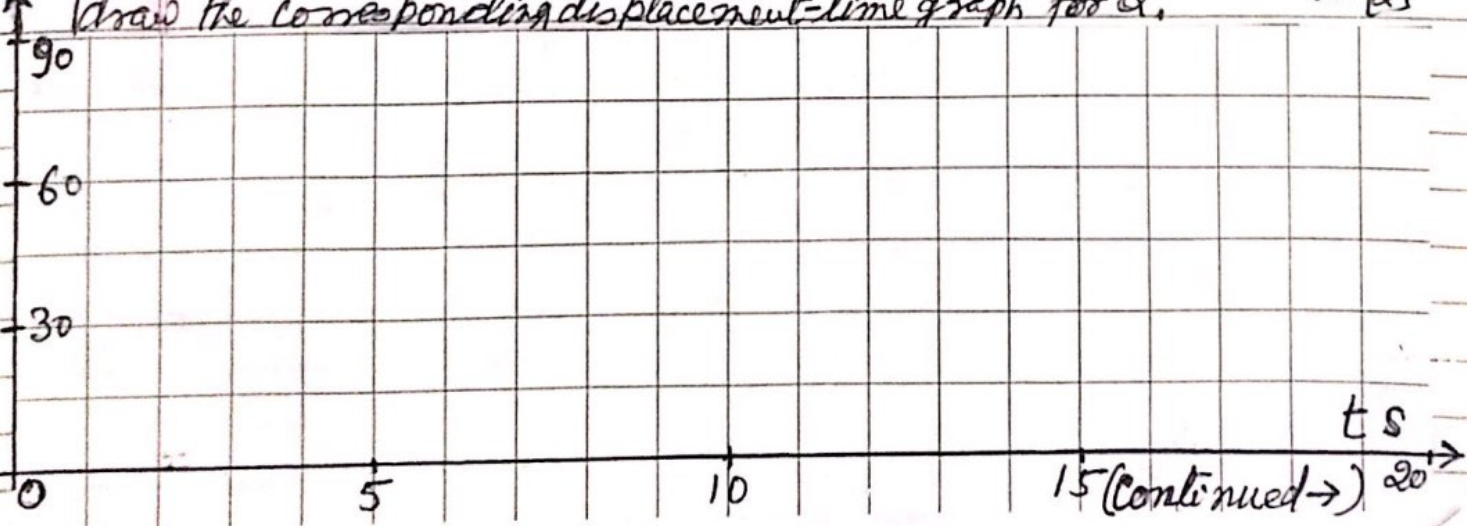
(i) Find the distance travelled by the particle P. ---- [2]

(ii) Write down the deceleration of the particle when $t=30$. --- [1]

(b) The diagram shows a velocity-time graph of a particle 'Q' moving in a straight line with velocity v m s^{-1} at time t s after leaving a fixed point.



The displacement of Q at time t s is s m. On the axes below, draw the corresponding displacement-time graph for Q. ---[2]



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Kinematics

Continued

Example 9 (a) Note: In time - speed Graph, the area between
Solution: the lines $t=a$ and $t=b$, under the graph and the t-axis gives the distance travelled.

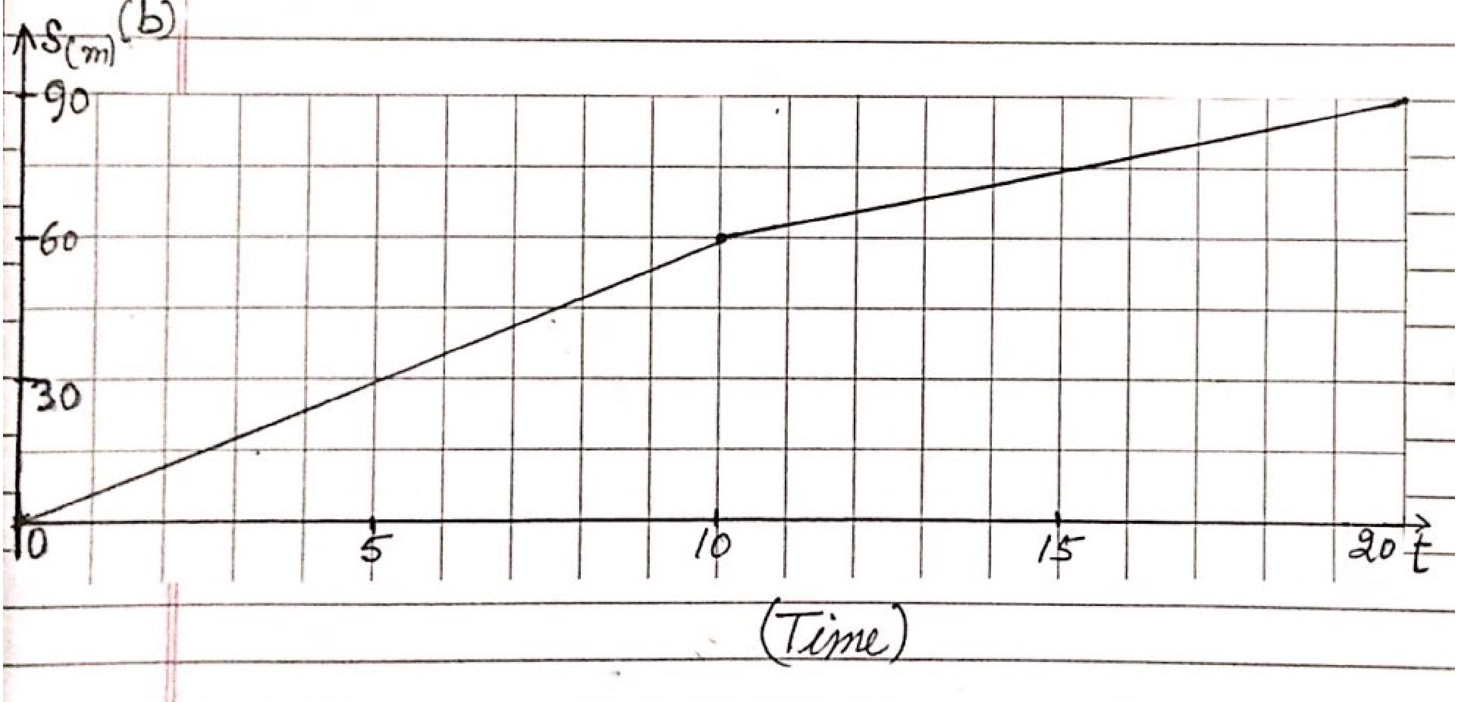
(a)(i) distance travelled by the Particle P,

$$= 25\times30+\frac{1}{2}(30+15)\times10+15\times15+\frac{1}{2}\times10\times15$$

$$= 750 + 225 + 225+75 = 1275\ m^2$$

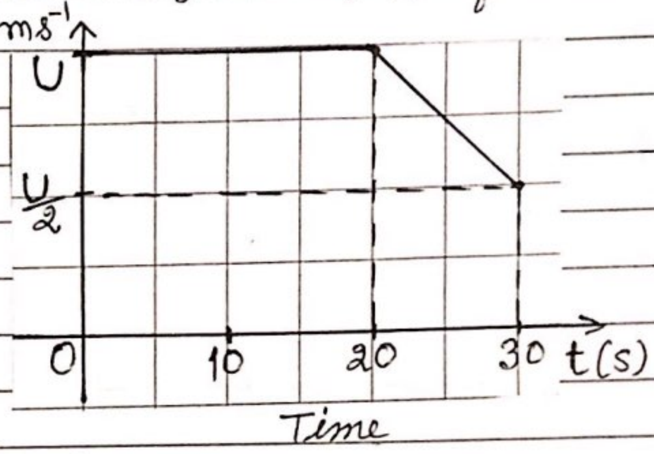
(ii) deceleration at $t=30$ (between $t=25$ to $t=35$) $= \frac{15}{10} = 1.5\ ms^{-2}$

(b)



Kinematics

Example 10. The diagram shows part of the velocity-time graph for a particle, moving at $V\,ms^{-1}$ in a straight line, t s, after passing through a fixed point. The particle travels at $U\,ms^{-1}$ for 20 s and then decelerates uniformly for 10 s, to a velocity of $\frac{U}{2}\,ms^{-1}$. In this 30 s interval, the particle travels 165 m.



(i) Find the value of U. ---[3]

(ii) Find the acceleration of the particle between $t = 20$ and $t = 30$. ---[2]

W-16/11/Q10

Solution (i) Distance travelled = area under the vel-time graph

or $20U + \frac{1}{2}\left(U + \frac{U}{2}\right) \times 10 = 165$ given

or $\frac{55}{2}u = 165$

$\Rightarrow u = 6\,ms^{-1}$ ✓ —— (1)

(ii) Acceleration between $t = 20$ to $t = 30$

gradient of the graph $= -\frac{u/2}{10} = -\frac{1}{20}u$

$= -\frac{1}{20} \times 6$ [from (1) $u = 6$

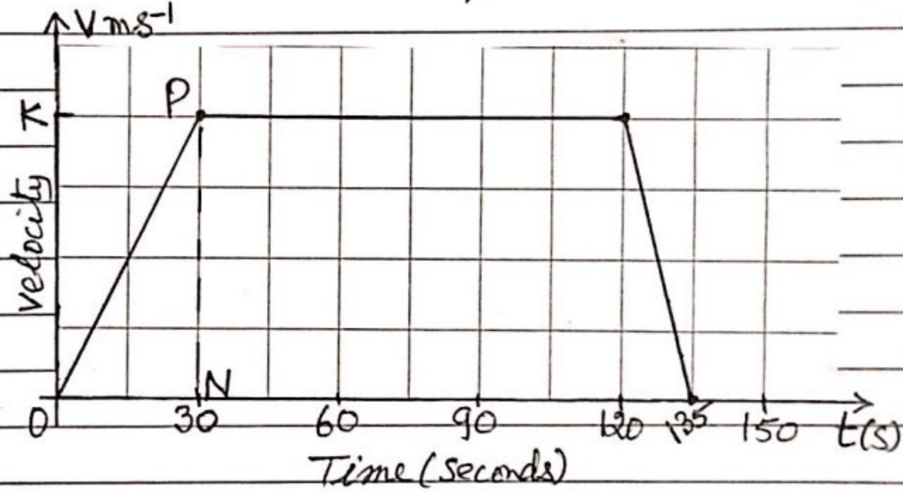
$\therefore$ acceleration $= -0.3\,ms^{-2}$ ✓

Example 11. A particle moves in a straight line. Starting from rest, P moves with constant acceleration for 30 seconds, after which it moves with constant velocity, k ms^{-1} for 90 seconds, P then moves with constant deceleration until it comes to rest, the magnitude of the deceleration is twice the magnitude of the initial acceleration.

(i) Use the information to complete the velocity time graph. ---[2]

(ii) Given that the particle travels 450 metres while it is accelerating, Find the value of k and acceleration of the particle.

(i)



(ii) Distance travelled during acceleration.

area under OP, $\frac{1}{2} \times PN \times ON = \frac{1}{2} \times k \times 30 = 450$ given

$\Rightarrow k = 30 \ ms^{-1}$

$\therefore$ acceleration $= \frac{\text{Speed}}{\text{Time}} = \frac{30}{30} = 1 \ ms^{-2}$ ✓

(Gradient OP)

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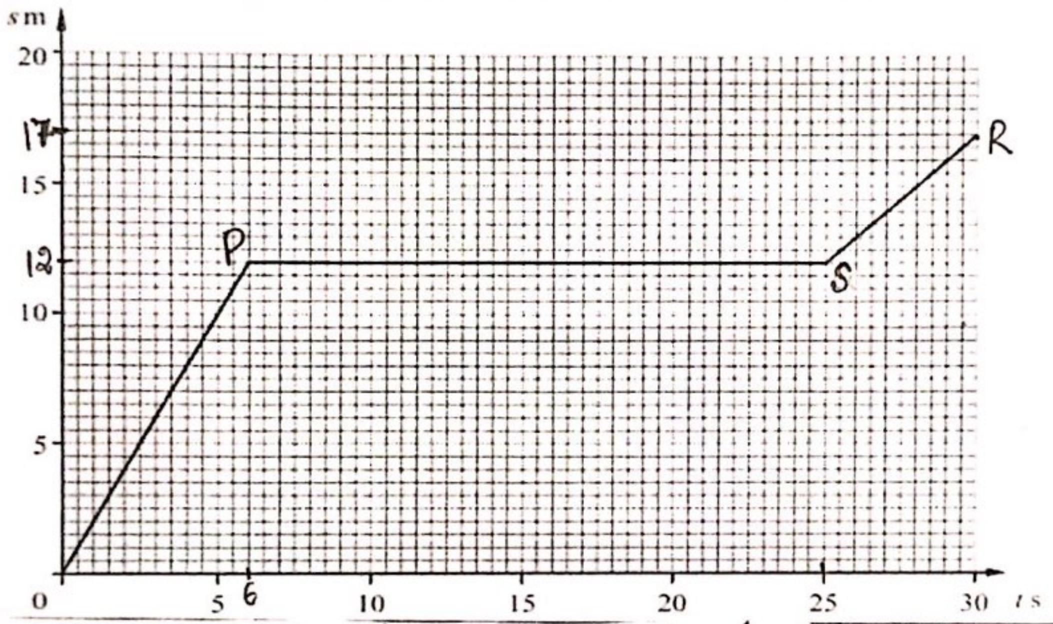
Kinematics

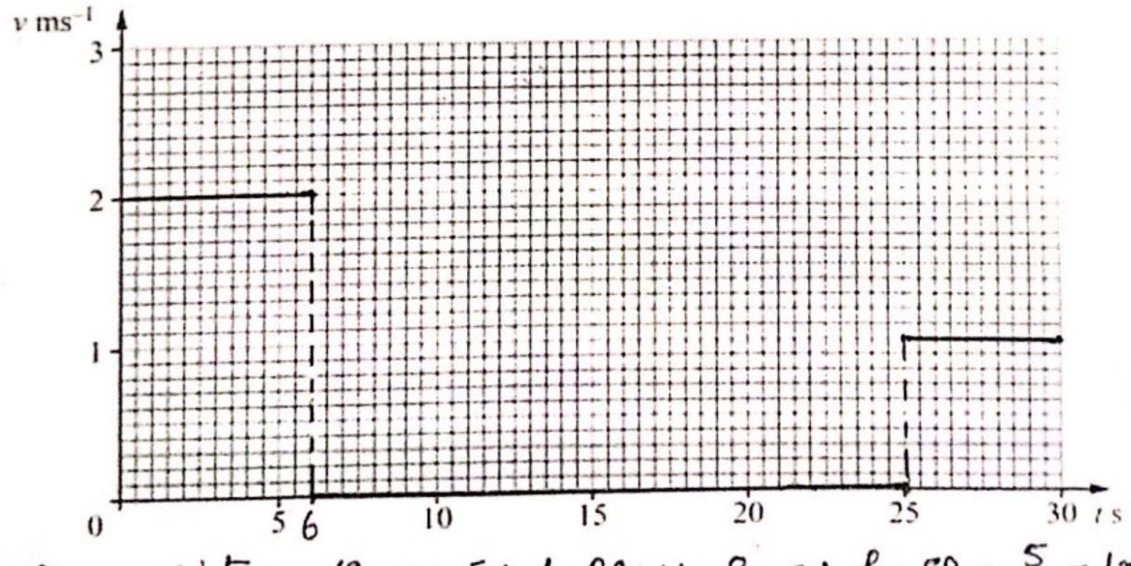
Example 12. The diagram shows the displacement-time graph of a particle Q moving in a straight line with displacement s m from a fixed point at t s.



On the axes below draw the corresponding velocity-time graph for the particle Q. [S-14/11/Q9(b)]

---[3]

Solution:



for OP, $v = \frac{\text{distance}}{\text{Time}} = \frac{12}{6} = 2\,ms^{-1}$; for PS, $V = \frac{0}{19} = 0$; for SR $= \frac{5}{5} = 1\,ms^{-1}$