IG_0606
Additional Maths

Polynomials Notes

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Polynomials

Introduction:

Let us observe the following algebraic expressions in single variable.

(i) $3x^2-5x+11$

(ii) $ax+b$

(iii) 7

(iv) $4x^3-3x^2+9x+32$

all the above are the examples of polynomials in one variable.

§ Degree of a Polynomial: The highest power of the variable in a polynomial is called it degree.

Example 1.

(i) $3x^2-5x+11$ is a polynomial of degree '2', and we also call it "Quadratic polynomial"

In this $3x^2$, $-5x$ and 11 are called terms and 3 is the coefficient of x^2 in the term $3x^2$.

(ii) $ax+b$ has degree 1, called Linear polynomial. $a, b \in R$

(iii) 7 is constant polynomial and its degree is '0' $(7=7x^0)$

(IV) $4x^3-3x^2+9x+32$ has degree '3' and is called cubic polynomial.

(V) '0' is called zero polynomial and degree is not defined.

§ In General:

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

is a polynomial of degree 'n', $a_n \neq 0$, $a_n, a_{n-1}, \ldots a_0 \in R$

where 'n' is a non-negative integer.

(or $n = 0, 1, 2, 3, \ldots$)

Polynomials in variable x may be denoted by $P(x)$, $Q(x)$, ...

§ Value of a Polynomial $P(x)$ at $x = a$ is $P(a)$ → obtained by replacing x by a in each term.

Example 2. Find the value of each of the following polynomials at the indicated value of the variables:

(i) $P(x) = 5x^2 - 3x + 6$ at $x = 1$

Solution: $P(1) = 5 \times 1^2 - 3 \times 1 + 6 = 8$

(ii) $P(x) = 4t^4 + 5t^2 + 6$ at $x = a$

$P(a) = (4a^4 + 5a^2 + 6)$

§ Zero of a polynomial. It is the value of x for which the value of polynomial becomes zero.

Example 3. (i) Given $P(x) = 2x + 3$, find its zero.

Put $2x + 3 = 0 \Rightarrow x = -3/2$

$\therefore -\frac{3}{2}$ is zero of the polynomial $P(x) = 2x + 3$

(ii) Find the zero's of the polynomial $P(x) = x^2 - 5x + 6$

Let us solve $x^2 - 5x + 6 = 0$

$(x-3)(x-2) = 0$

$x = 3$ or 2

$\therefore x = 3$ and 2 are the zero's of $P(x)$.

Conversely:

Verify: 3 is a zero of $P(x) = x^2 - 5x + 6$

Let us find, $P(3) = 3^2 - 5 \times 3 + 6 = 0$

$\therefore 3$ is a zero of $P(x)$.

Polynomials.

Division of Polynomials:

Divide the polynomial $P(x) = 3x^4 - 4x^3 - 3x - 1$ by $G(x) = (x-1)$

$$
\begin{array}{r|l}
x-1 & 3x^4 - 4x^3 - 3x - 1 \;\big(\, 3x^3 - x^2 - x - 4 \\
 & \underline{-3x^4 \mp 3x^3} \\
 & \quad -x^3 - 3x - 1 \\
 & \quad \underline{\mp x^3 \qquad \pm x^2} \\
 & \quad\; -x^2 - 3x - 1 \\
 & \quad\; \underline{\mp x^2 \pm x} \\
 & \qquad -4x - 1 \\
 & \qquad \underline{\mp 4x \pm 4} \\
 & \qquad\quad -5
\end{array}
$$

We find, that when $P(x)$ is divided by $G(x)$ then quotient $Q(x) = 3x^3 - x^2 - x - 4$ and Remainder $R = -5$

Verify

$(3x^4 - 4x^3 - 3x - 1) = (x-1)(3x^3 - x^2 - x - 4) + (-5)$

$P(x) = G(x) \cdot Q(x) + R$

or (Dividend) = (Division × Qutient) + Remainder.

See for Numbers

$\frac{17}{5} = 3\frac{2}{5}$

or

$\boxed{17 = 5 \times 3 + 2}$

$\boxed{a = bq + r}$

for $\frac{a}{b}$

quotiat $= q$, & Remainder $= r$

Again

when $P(x)$ is divided by $(x-1)$

$P(1) = 3 \times 1^4 - 4 \times 1^3 - 3 \times 1 - 1$

$= 3 - 4 - 3 - 1 = -5 = R.$

Conclusion: when $P(x)$ is divided by $(x-1)$, Then

Remainder $\longrightarrow R = P(1)$

[Note 1 is zero of $(x-1)$]

§ Remainder Theorem:

Let $P(x)$ be any polynomial of degree greater than or equal to one and 'a' be any real number. If $P(x)$ is divided by the linear polynomial $(x-a)$, then the remainder is $P(a)$. or $R = P(a)$

Note (i) When $P(x)$ is divided by $(x+a)$ then $R = P(-a)$

(ii) When $P(x)$ is divided by $(ax+b)$ then $R = P(-\frac{b}{a})$

[zero of $(ax+b)$ is $-\frac{b}{a}$]

Example 4. Find the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $(x-1)$.

Solution: Let $P(x) = x^4+x^3-2x^2+x+1$ and zero of $(x-1)$ is 1

$\therefore$ Remainder $R = P(1) = 1^4+1^3-2\times1^2+1+1 = 2$ ✓

$\therefore$ Remainder $= 2$ ✓

Example 5. Find the remainder when $P(x) = x^3 + 4x^2 - 3x + 10$ is divided by $(x+4)$.

Solution: Given $P(x) = x^3 + 4x^2 - 3x + 10$ and zero of $(x+4) = -4$

$\therefore$ Remainder $R = P(-4)$

$= (-4)^3 + 4(-4)^2 - 3\times(-4) + 10$

$= -64 + 64 + 12 + 10 \quad = 22$ ✓

$\therefore R = 22$ ✓

Example 6:

Find the remainder when $P(x) = x^3 - 6x^2 + 2x - 4$ is divided by $(1-2x)$.

Solution: Given $P(x) = x^3 - 6x^2 + 2x - 4$ and zero of $(1-2x)$ is $\frac{1}{2}$

$\therefore$ Remainder when $P(x)$ is divided by $(1-2x)$

$\left\{ \begin{array}{l} 1-2x=0 \\ x = \frac{1}{2} \end{array} \right.$

$R = P(\frac{1}{2}) = (\frac{1}{2})^3 - 6(\frac{1}{2})^2 + 2\times\frac{1}{2} - 4$

$= \frac{1}{8} - 6\times\frac{1}{4} + 1 - 4$

$= \frac{1-12+8-32}{8} = -\frac{35}{8}$

$\therefore R = -\frac{35}{8}$ ✓

Polynomials

§ **Factor theorem:** If $P(x)$ is a polynomial of degree $n \geq 1$ and 'a' is any real numbers, Then

(i) If $P(a) = 0$, then $P(x) = (x-a)\,Q(x)$, which shows that $(x-a)$ is a factor of $P(x)$ ✓

(ii) If given $(x-a)$ is a factor of $P(x)$, then $P(x) = (x-a)\cdot Q(x)$

$\therefore$ $P(a) = 0$ ✓

Example 7. Examine whether $(x+2)$ is a factor of x^3+3x^2+5x+6.

Solution: Let $P(x) = x^3+3x^2+5x+6$ and zero of $(x+2)$ is -2

Now $P(-2) = (-2)^3+3(-2)^2+5\cdot(-2)+6$

$= -8+12-10+6 = 0$

$\therefore$ Using factor theorem, $(x+2)$ is factor of $P(x)$

Example 8. Find the value of k, if $(x-1)$ is a factor of, $4x^3+3x^2-4x+k$,

Solution: Let $P(x) = 4x^3+3x^2-4x+k$ and zero of $(x-1)$ is 1.

as $(x-1)$ is a factor of $P(x) \Rightarrow P(1) = 0$

or $4\times1^3+3\times1^2-4\times1+k = 0$

$4+3-4+k=0$

$\therefore$ $k = -3$ ✓

Example 9. Factorise, $x^3+13x^2+32x+20$. (using factor theorem)

Solution: Let $P(x) = x^3+13x^2+32x+20$

we know that $(x-a)$ is a factor of $P(x)$ if $P(a)=0$

Try $a=1$, $P(1) = 1^3+13\times1^2+32\times1+20 = 66 \neq 0$

Try $a=-1$, $P(-1) = (-1)^3+13(-1)^2+32\times(-1)+20$

$= -1+13-32+20 = 0$ ✓

$\therefore$ $(x-(-1)) = (x+1)$ is factor $P(x)$ ✓

[Here a is the factor of the constant 20, or 'a' may be $\pm1, \pm2, \pm4, \pm5, \pm10$ or ±20]

$P(x) = x^3+13x^2+32+20 = (x+1)(\cdots)$

$= (x+1)(x^2+12x+20)$

$= (x+1)(x+2)(x+10)$ ✓

$$
\begin{array}{r|l}
x+1 & x^3+13x^2+32x+20 \quad (x^2+12x+20 \\
 & \underline{-x^3 \pm x^2} \\
 & 12x^2+32x \\
 & \underline{-12x^2 \pm 12x} \\
 & 20x+20 \\
 & \underline{-20x \pm 20} \\
 & \times
\end{array}
$$

Polynomials.

Example 10. The polynomial $P(x) = 2x^3 - 3x^2 + qx + 56$ has a factor of $(x-2)$.

(a) Show that $q = -30$ --- [1]

(b) Factorise $p(x)$ completely and hence state all the solutions of $p(x) = 0$. [SP-20/01/Q1] --- [4]

Solution: $p(x) = 2x^3 - 3x^2 + qx + 56$ — ①

(a) $(x-2)$ is a factor of $p(x)$. [∵ zero of $(x-2)$ is 2.

$\therefore \quad P(2) = 0$

fr ① $\quad 2\times2^3 - 3\times2^2 + q\times2 + 56 = 0$

$16 - 12 + 2q + 56 = 0$

$2q = -60$

$q = -30$ ✓

(b) hence

$p(x) = 2x^3 - 3x^2 - 30x + 56$ — ② [$q = -30$)

Now to factorise $p(x)$:

Given, $(x-2)$ is factor of $p(x)$. — ③ (divid $P(x)$ by $(x-2)$)

$$
\begin{array}{r|l}
 & 2x^2 + x - 28 \\
x-2 & 2x^3 - 3x^2 - 30x + 56 \\
 & \underline{2x^3 \mp 4x^2} \\
 & x^2 - 30x \\
 & \underline{-x^2 \mp 2x} \\
 & -28x + 56 \\
 & \underline{-28x + 56} \\
 & \times
\end{array}
$$

$\therefore$ fr ② & ③

$P(x) = (x-2)(2x^2 + x - 28)$

or $P(x) = (x-2)(x+4)(2x-7)$ ✓

∵ $2x^2 + x - 28$
$= 2x^2 + 8x - 7x - 28$
$= 2x(x+4) - 7(x+4)$
$= (2x-7)(x+4)$

$\therefore$ Solutions of $p(x) = 0$

$(x-2)(x+4)(2x-7) = 0$

$\therefore x = 2, -4$ and $\frac{7}{2}$ ✓

or $x = 2, -4$ and 3.5 ✓

Example 11. It is given that $(x+4)$ is a factor of,
$p(x) = 2x^3 + 3x^2 + ax - 12$, When $p(x)$ is divided by $(x-1)$ the remainder is b.

(i) Show that $a = -23$ and find the value of constant b. ---[2]

(ii) Factorise $p(x)$ completely and hence state all the solutions of $p(x) = 0$ [S-18/21/Q4] ---[4]

Solution: Given $p(x) = 2x^3 + 3x^2 + ax - 12$ ——①

(i) and $(x+4)$ is a factor of $P(x) \Rightarrow P(-4) = 0$ (as zero of $(x+4)$ is -4)

$\Rightarrow$ fr① $2(-4)^3 + 3(-4)^2 + a(-4) - 12 = 0$

$\Rightarrow -128 + 48 - 4a - 12 = 0$

or $-4a = +92 \Rightarrow a = \underline{-23}$ ✓

Now from ① $p(x) = 2x^3 + 3x^2 - 23x - 12$ ——②

when $p(x)$ is divided by $(x-1)$ remainder is b

$\Rightarrow P(1) = b$ [∵ zero of $(x-1)$ is 1]

or $b = p(1) = 2\times1^3 + 3\times1^2 - 23\times1 - 12$

$= 2 + 3 - 23 - 12 = -30 \Rightarrow \mathbf{b} = -30$ ✓

(ii) Now $p(x) = 2x^3 + 3x^2 - 23x - 12$

$= (x+4)(2x^2 - 5x - 3)$ (Given $(x+4)$ is a factor of $p(x)$)

$= (x+4)(x-3)(2x+1)$ ✓

∵

$$
\begin{array}{r|l}
 & 2x^2 - 5x - 3 \\
\hline
(x+4) & 2x^3 + 3x^2 - 23x - 12 \\
 & \underline{2x^3 + 8x^2} \\
 & -5x^2 - 23x \\
 & \underline{-5x^2 - 20x} \\
 & -3x - 12 \\
 & \underline{-3x - 12} \\
 & \times
\end{array}
$$

Now factorise

$2x^2 - 5x - 3$

$2x^2 - 6x + x - 3$

$2x(x-3) + 1(x-3)$

$(x-3)(2x+1)$

Now for solution of $p(x) = 0$

$(x+4)(x-3)(2x+1) = 0$

or $x = \underline{-4}, \underline{+3}$ and $\underline{-\frac{1}{2}}$ ✓

Polynomials

Example 12. It is given that $x+3$ is a factor of the polynomial $p(x) = 2x^3 + ax^2 - 24x + b$. The remainder when $p(x)$ is is divided by $x-2$ is -15. Find the remainder when $p(x)$ is divided by $x+1$. [S-18/22/Q3]....[6]

Solution: Given $p(x) = 2x^3 + ax^2 - 24x + b$ —— ①

$(x+3)$ is a factor of $p(x)$.

$\Rightarrow p(-3) = 0$ $\quad$ ($\because$ zero of $(x+3)$ is -3)

frm ① $\quad p(-3) = 2(-3)^3 + a(-3)^2 - 24(-3) + b = 0$

$\Rightarrow 9a + b = -18$ —— ②

also when $p(x)$ is divided by $(x-2)$, Remainder is -15

$\therefore p(2) = -15$ $\quad$ ($\because$ zero of $x-2$ is 2)

frm ①

$p(2) = 2(2)^3 + a(2)^2 - 24 \times 2 + b = -15$

$\Rightarrow 4a + b = 17$ —— ③

Solving ② & ③ we get $a = -7$ and $b = 45$

$\therefore$ frm ① $\quad p(x) = 2x^3 - 7x^2 - 24x + 45$ —— ④

$\therefore$ Remainder when $p(x)$ is divided by $(x+1) = p(-1)$

or Remainder $= p(-1)$ $\quad$ ($\because$ zero of $x+1$ is -1)

frm ④ $\quad = 2(-1)^3 - 7(-1)^2 - 24(-1) + 45$

$= -2 - 7 + 24 + 45$

$\therefore$ **Remainder $= 60$** ✓

Polynomials

Example 13. The cubic equation $x^3+ax^2+bx-36=0$ has a repeated positive integral root.

(i) If the repeated root is $x=3$, find the other positive root, and the value of a and of b. --[4]

(ii) There are other possible values of a and b for which the cubic equation has a repeated positive integral root. In each case state all the three integral roots of the equation. --[4]

[W-17/23/Q11]

Solution: $x^3+ax^2+bx-36=0$ —— ①

(i) If 3 is a repeated root then the third positive root will be 4. as $[36 = 3^2 \times 4$

$\therefore$ 3 is a root of eqn ① $\Rightarrow 3^3+a\times3^2+b\times3-36=0$

$\Rightarrow 3a+b=3$ —— ②

and 4 is a root of eqn ① $\Rightarrow 4^3+a\times4^2+b\times4-36=0$

$\Rightarrow 4a+b=-7$ —— ③

Solving ② and ③ $a=-10$ ✓ and $b=33$ ✓

(ii) Now the constant term of the equation -36 may be written as,

$$36 = 6^2\times 1$$

$\therefore$ $x=6$, $x=6$ and $x=1$ ✓

Also $$36 = 2^2\times 9$$

roots are $x=2$, $x=2$ and $x=9$ ✓

and $$36 = 1^2\times 36$$

$\therefore$ $x=1$, $x=1$ and $x=36$ ✓

Polynomials

Example 14. (i) Solve the equation. $6c^3 - 7c^2 + 1 = 0$ ---[5]

It is given that $y = \tan x + 6\sin x$

(ii) find $\frac{dy}{dx}$ ---[2]

(iii) if $\frac{dy}{dx} = 7$, show that $6\cos^3 x - 7\cos^2 x + 1 = 0$ --[2]

(iv) Hence solve the equation, $\frac{dy}{dx} = 7$ for $0 \le x \le \pi$ rad. --[2]

W-17/22/Q10

Solution (i). To solve $6c^3 - 7c^2 + 1 = 0$ — (1)

Consider a polynomial, $p(c) = 6c^3 - 7c^2 + 1$

[As the constant term is 1 the zeros of $p(c)$ may be ± 1]

Check, $p(1) = 6\times1^3 - 7\times1^2 + 1 = 0$

as $p(1) = 0 \Rightarrow (c-1)$ is a factor of $p(c)$

$\therefore 6c^3 - 7c^2 + 1 = (c-1)(6c^2 - c - 1)$

or $(c-1)(2c-1)(3c+1)$

$$
\begin{array}{r}
6c^2 - c - 1 \\
c-1 \overline{)\, 6c^3 - 7c^2 + 1} \\
-\;6c^3 \mp 6c^2 \\
\hline
-c^2 + 1 \\
\mp c^2 \pm c \\
\hline
-c + 1 \\
\mp c \pm 1 \\
\hline
\times
\end{array}
$$

and

$6c^2 - c - 1$

$6c^2 - 3c + 2c - 1$

$3c(2c-1) + 1(2c-1)$

$(3c+1)(2c-1)$

from (1) $(c-1)(2c-1)(3c+1) = 0$

$\therefore$ Solutions are $c = 1, \frac{1}{2}, -\frac{1}{3}$ ✓

(ii) Given $y = \tan x + 6\sin x$

differentiating $\frac{dy}{dx} = \sec^2 x + 6\cos x$ ✓

(iii) $\frac{dy}{dx} = 7 \Rightarrow \sec^2 x + 6\cos x = 7$

$\Rightarrow \frac{1}{\cos^2 x} + 6\cos x = 7$

$\Rightarrow 6\cos^3 x - 7\cos^2 x + 1 = 0$ ✓

(iv) To solve $\frac{dy}{dx} = 7$

from Part (iii) we get, $6\cos^3 x - 7\cos^2 x + 1 = 0$

Let $\cos x = c \Rightarrow 6c^3 - 7c^2 + 1 = 0$

from Part (i) $c = 1, \frac{1}{2}, -\frac{1}{3}$

or $\cos x = 1, \frac{1}{2}, -\frac{1}{3}$

Case I $\cos x = 1$ $\quad 0 \le x \le \pi$

$= \cos 0$

$\therefore x = 0$ rad. ✓

Case II $\cos x = \frac{1}{2}$

$= \cos\frac{\pi}{3}$

$\therefore x = \frac{\pi}{3}$ or 1.05 rad. ✓

Case III $\cos x = -\frac{1}{3}$ $\quad (\pi - \alpha)$

$= -\cos 1.23$

$\therefore x = (\pi - 1.23) = 1.91$ rad ✓

$\therefore x = 0,\ 1.05$ and 1.91 ✓

Polynomials

Example 15. The polynomial $f(x) = ax^3 - 15x^2 + bx - 2$ has a factor $(2x-1)$ and a remainder of 5 when divided by $x-1$.

(i) Show that $b = 8$ and find the value of a. --[4]

(ii) Using the values of a and b from part (i), express $f(x)$ in the form $(2x-1).g(x)$, where $g(x)$ is a quadratic factor to be found. --[2]

(iii) Show that the equation $f(x) = 0$ has only one real root. --[2]

[S-15/11/Q6]

Solution: Given $f(x) = ax^3 - 15x^2 + bx - 2$ —①

(i) as $(2x-1)$ is a factor of $f(x) \Rightarrow f(\frac{1}{2}) = 0$ (∵ zero of $(2x-1)$ is $\frac{1}{2}$)

$\Rightarrow f(\frac{1}{2}) = a(\frac{1}{2})^3 - 15(\frac{1}{2})^2 + b \times \frac{1}{2} - 2 = 0$

$\Rightarrow \frac{a}{8} - \frac{15}{4} + \frac{b}{2} - 2 = 0 \Rightarrow a + 4b = 46$ —②

Also when $f(x)$ is divided by $(x-1)$, remainder is 5

$\Rightarrow f(1) = 5$ (∵ zero of $(x-1)$ is 1)

$\Rightarrow a \times 1^3 - 15 \times 1^2 + b \times 1 - 2 = 5$

$\Rightarrow a - 15 + b - 2 = 5 \Rightarrow a + b = 22$ —③

Solving ② and ③ we get $\underline{b = 8 \text{ and } a = 14}$ ✓

(ii) $f(x) = 14x^3 - 15x^2 + 8x - 2$ (for $b = 8$ and $a = 14$ in ①) —④

∴ Now as $(2x-1)$ is a factor of $f(x)$

$f(x) = (2x-1)(7x^2 - 4x + 2)$ ✓

$$\begin{array}{r|l} & 7x^2 - 4x + 2 \\ 2x-1 & 14x^3 - 15x^2 + 8x - 2 \\ & -14x^3 \mp 7x^2 \\ \hline & -8x^2 + 8x \\ & \mp 8x^2 \pm 4x \\ \hline & 4x - 2 \\ & 4x - 2 \\ & - \quad + \\ \hline & \times \end{array}$$

(iii) Now $f(x) = 0$

$\Rightarrow (2x-1)(7x^2 - 4x + 2) = 0$

$\Rightarrow 2x - 1 = 0$ or $7x^2 - 4x + 2 = 0$

$x = \frac{1}{2}$ or for this $b^2 - 4ac < 0$

hence No real Solution

for $7x^2 - 4x + 2$

$a = 7, b = -4, c = 2$

∴ $b^2 - 4ac = (-4)^2 - 4 \times 7 \times 2$

$= 16 - 56$

$= -40 < 0$

∴ $f(x) = 0$ has Only one real Solution $x = \frac{1}{2}$ ✓