IG 0606 Additional Maths

Integration Notes

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IG- Maths
0606

Integration

(Indefinite Integral)

Date P-1

§ (i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1$

here C is the constant of Integration

(ii) $\int 1 dx = x$ (iii) $\int k dx = kx$

Integration is the reverse process of differentiation

Example:

$\frac{d}{dx} x^5 = 5x^4$

we write

$\int 5x^4 dx = x^5$

$\frac{d}{dx}(x^5+7) = 5x^4$

$\therefore \int 5x^4 dx = x^5 + C$

Example (a) $\frac{dy}{dx} = x^5$ find the equation of curve.

$\Rightarrow y = \int x^5 dx$

$= \frac{x^6}{6} + C$

(b) $\frac{dy}{dx} = \sqrt{x} = x^{1/2}$

$y = \int x^{1/2} dx$

$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{2}{3} x^{3/2} + C.$

Example 1 Find $\int \frac{x^3+x^2+1}{x^2} dx$ ---[3] [S-16/22/Q9(a)]

$= \int (x+1+x^{-2}) dx$

$= \frac{x^2}{2} + x + \frac{x^{-1}}{-1} + C$

$= \frac{x^2}{2} + x - \frac{1}{x} + C$ ✓

Example 2. Find the equation of the curve which passes the point (2,17), and for which $\frac{dy}{dx} = 4x^3+1$. [S-17/21/Q1] --[4]

Solution: $\frac{dy}{dx} = 4x^3+1$

$\therefore y = \int (4x^3+1) dx$

$= \frac{4x^4}{4} + x + C$

$y = x^4 + x + C$ —(1)

Curve passes through (2,17)

from (1) $17 = 2^4 + 2 + C$

$\Rightarrow C = -1$

Hence from (1) The required equation of the curve is,

$y = x^4 + x - 1$ ✓

IG-Maths 0606

classmate Date Page P-2

Integration

Example 3. Find y in terms of x, given that $\frac{d^2y}{dx^2} = 6x + \frac{2}{x^3}$ and that when $x = 1$, $y = 3$ and $\frac{dy}{dx} = 1$. ---[6]

W-17/21/Q7

Solution: $\frac{d^2y}{dx^2} = 6x + \frac{2}{x^3}$

$$\Rightarrow \frac{dy}{dx} = \int (6x + 2x^{-3})dx$$

$$= 3x^2 + \frac{2x^{-2}}{-2} + c$$

or $\frac{dy}{dx} = 3x^2 - \frac{1}{x^2} + c$ — (1)

Given $x = 1$ and $\frac{dy}{dx} = 1$ frm (1)

$$1 = 3 \times 1^2 - \frac{1}{1^2} + c$$

$$\Rightarrow c = -1$$

frm (1) $\frac{dy}{dx} = 3x^2 - x^{-2} - 1$ — (2)

$$\therefore y = \int (3x^2 - x^{-2} - 1)dx$$

$$= \frac{3x^3}{3} - \frac{x^{-1}}{-1} - x + k$$

$$y = x^3 + \frac{1}{x} - x + k$$ — (3)

Now $x = 1$, $y = 3$ frm (3)

$$3 = 1^3 + \frac{1}{1} - 1 + k$$

$$\Rightarrow k = 2$$

fr (3) required value of y

$$y = x^3 + \frac{1}{x} - x + 2$$ ✓

Integration

Example-4. The gradient of the normal to a curve at the point with coordinates (x, y) is given by $\frac{\sqrt{x}}{1-3x}$.

(i) Find the equation of the curve, given that the curve passes through the point $(1, -10)$. ---[5]

(ii) Find, in the form $y = mx + c$, the equation of the tangent to the curve at the point where $x = 4$. $\boxed{W-17/22/Q7}$ ---[4]

Solution: gradient of Normal $-\frac{1}{\frac{dy}{dx}} = \frac{\sqrt{x}}{1-3x}$

$$\Rightarrow \frac{dy}{dx} = \frac{3x-1}{\sqrt{x}} = 3x^{1/2} - x^{-1/2} \quad \text{—(1)}$$

(i) $\therefore y = \int (3x^{1/2} - x^{-1/2})\, dx$

$$= 3 \cdot \frac{x^{3/2}}{3/2} - \frac{x^{1/2}}{1/2} + C$$

$$\therefore y = 2x^{3/2} - 2x^{1/2} + C \quad \text{—(2)}$$

It passes through $(1, -10)$

$$\Rightarrow -10 = 2 \times 1^{3/2} - 2 \times 1^{1/2} + C$$

$$\Rightarrow -10 = \cancel{2} - \cancel{2} + C \Rightarrow C = -10$$

from (2) $\quad y = 2x^{3/2} - 2x^{1/2} - 10 \quad \text{—(3)}$

(ii) at $x = 4$, from (3) $y = 2 \times 4^{3/2} - 2 \times 4^{1/2} - 10$

$$\text{or } y = 16 - 4 - 10 = 2$$

$\therefore$ Point $(4, 2)$ ✓

from (1), now $\left(\frac{dy}{dx}\right)_{x=4} = \frac{3 \times 4 - 1}{\sqrt{4}} = \frac{11}{2} = 5.5$ ✓

$\therefore$ The equation of tangent at $(4, 2)$, $m = 5.5$

$$y - 2 = 5.5(x - 4)$$

$$\Rightarrow y = 5.5x - 20 \checkmark$$

Integration

Example 5. A curve passes through the point $\left(2, -\frac{4}{3}\right)$ and is such that $\frac{dy}{dx} = (3x+10)^{-1/2}$.

(i) Find the equation of the curve ---[4]

The normal to the curve, at the point $x=5$, meets the line $y = -\frac{5}{3}$ at point P.

(ii) Find the x-coordinate of P. [S-16 | 12 | Q9] ---[6]

Solution: $\frac{dy}{dx} = (3x+10)^{-1/2}$ — (1)

Note:

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a\cdot(n+1)}$$

(i) $y = \frac{(3x+10)^{1/2}}{\frac{1}{2}\times 3} + C$

or $y = \frac{2}{3}(3x+10)^{1/2} + C$ — (2)

passes through $\left(2, -\frac{4}{3}\right)$, in (2)

$$-\frac{4}{3} = \frac{2}{3}(3\times 2 + 10)^{1/2} + C \Rightarrow -\frac{4}{3} = \frac{2\times 4}{3} + C \Rightarrow C = -4 ✓$$

from (2) the required equ^n of the curve is

$$y = \frac{2}{3}(3x+10)^{1/2} - 4 ✓ \quad \text{— (3)}$$

(ii) at $x=5$, from (3) $y = \frac{2}{3}\times 5 - 4 = -2/3$, $\therefore$ Point $\left(5, -\frac{2}{3}\right)$

from (1) $\left(\frac{dy}{dx}\right)_{x=5} = \frac{1}{5} \Rightarrow$ gradient of the normal $m = -5$

$\therefore$ The equation of the normal at $\left(5, -\frac{2}{3}\right)$, $m = -5$

$$y + \frac{2}{3} = -5(x-5) \quad \text{— (4)}$$

Now (4) intersects the line $y = -5/3$ at 'P'

from (4) $-\frac{5}{3} + \frac{2}{3} = -5(x-5)$

x-Coord. of P. is $\rightarrow \Rightarrow$ $x = 5.2$ ✓

IG-Maths 0606

Integration

(Indefinite Integral)

§1. Integration of Exponentional Function.

(i) $\int e^x dx = e^x + c$ $\quad \because \frac{d}{dx} e^x = e^x$

(ii) $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$ $\quad \because \frac{d}{dx} e^{ax+b} = e^{ax+b} \times a$

§2. Reverse of differential of logarithmic function:

(i) $\int \frac{1}{x} dx = \ln x + c; x > 0$ $\quad \left\{ \because \frac{d}{dx} \ln x = \frac{1}{x} \; ; x > 0 \right.$

(ii) $\int \frac{1}{(ax+b)} dx = \frac{1}{a} \ln(ax+b) + c \; ; (ax+b) > 0$ $\quad \left\{ \because \frac{d}{dx} \ln(ax+b) = \frac{a \times 1}{(ax+b)} \; ; (ax+b) > 0 \right.$

Example 6. (a) Find $\int e^{2x+1} dx$ --- [2]

(b) (i) Given that $y = \frac{x}{\ln x}$; find $\frac{dy}{dx}$ --- [3]

(ii) Hence, $\int \left(\frac{1}{\ln x} - \frac{1}{(\ln x)^2} + \frac{1}{x^2} \right) dx$ --- [3]

$\boxed{M\text{-}17/22/Q9}$

Solution:

(a) $\int e^{2x+1} dx = \frac{1}{2} e^{2x+1} + c$ ✓

(b)(i) $y = \frac{x}{\ln x}$

$\frac{dy}{dx} = \frac{\ln x \cdot \frac{d}{dx} x - x \frac{d}{dx} \ln x}{(\ln x)^2}$ (Quotient rule)

$= \frac{\ln x \times 1 - x \times \frac{1}{x}}{(\ln x)^2}$

$\frac{dy}{dx} = \frac{\ln x - 1}{(\ln x)^2}$ ✓ — ①

(ii) $\int \left(\frac{1}{\ln x} - \frac{1}{(\ln x)^2} + \frac{1}{x^2} \right) dx$

$= \int \left(\frac{\ln x - 1}{(\ln x)^2} \right) dx + \int \frac{1}{x^2} dx$

$= \int \frac{(\ln x - 1)}{(\ln x)^2} + \int x^{-2} dx$

$= \frac{x}{\ln x} + \frac{x^{-1}}{-1}$ $\quad \left\{ \because \text{from ①} \; \frac{d}{dx} \left(\frac{x}{\ln x} \right) = \left(\frac{\ln x - 1}{(\ln x)^2} \right) \right.$

$= \frac{x}{\ln x} - \frac{1}{x} + c$ ✓

Integration

Example 7. The curve $y = f(x)$ passes through the point $\left(\frac{1}{2}, \frac{7}{2}\right)$ and is such that $f'(x) = e^{2x-1}$.

(i) Find the equation of the curve. ---[4]

(ii) Find the value of x for which $f''(x) = 4$, giving your answer in the form $ax + b\ln\sqrt{2}$, where a and b are constants. --[4]

S-17/12/Q11

Solution (i) $f'(x) = e^{2x-1}$ —— (1)

$$f(x) = \int e^{2x-1}dx$$

or $f(x) = \frac{1}{2}e^{2x-1} + c$ —— (2)

curve (2) passes through a point $\left(\frac{1}{2}, \frac{7}{2}\right)$

$\therefore \frac{7}{2} = \frac{1}{2}e^{2\times\frac{1}{2}-1} + c = \frac{1}{2}e^{0} + c = \frac{1}{2} + c$

or $\frac{7}{2} = \frac{1}{2} + c \Rightarrow c = 3$

from (2) the equation of the curve is:

$$f(x) = \frac{1}{2}e^{2x-1} + 3 \quad \text{—— (2)}$$

(ii) from (1) $f'(x) = e^{2x-1}$

diff. w.r.t x

$f''(x) = 2e^{2x-1} = 4$ given.

$\Rightarrow e^{2x-1} = 2$

$\Rightarrow 2x - 1 = \log_e 2$

$\Rightarrow 2x = \ln 2 + 1$

$\Rightarrow x = \frac{1}{2} + \frac{1}{2}\ln 2$

$\Rightarrow x = \frac{1}{2} + \ln 2^{1/2}$

$\Rightarrow x = \frac{1}{2} + \ln\sqrt{2}$

Integration

Example 8: (i) Show that $\frac{d}{dx}[0.4x^5(0.2-\ln 5x)] = kx^4\ln 5x$, where k is an integer. --[2]

(ii) Express $\ln 125x^3$ in terms of $\ln 5x$ --[1]

(iii) Hence find $\int(x^4 \ln 125x^3)\,dx$ $\boxed{S\text{-}17/22/Q5}$ --[2]

Solution: (i) $\frac{d}{dx}[0.4x^5(0.2-\ln 5x)]$

$= 0.4x^5 \cdot \frac{d}{dx}(0.2-\ln 5x) + (0.2-\ln 5x)\frac{d}{dx}0.4x^5$ (Product rule)

$= 0.4x^5 \times \left(0-\frac{1}{5x}\times 5\right) + (0.2-\ln 5x)\times 0.4\times 5x^4$

$= \cancel{-0.4x^4} + \cancel{0.4x^4} - 2x^4\ln 5x$

$= -2x^4\ln 5x$ ✓ —— ①

(ii) $\ln 125x^3 = \ln(5x)^3$

$= 3\ln 5x$ ✓ —— ②

(iii) $\int(x^4 \ln 125x^3)\,dx$

$= \int(x^4 \times 3\ln 5x)\,dx$ frm ②

$= 3\int x^4 \ln 5x\,dx$

$= -\frac{3}{2}\int -2x^4 \ln 5x\,dx$

$= -\frac{3}{2}[0.4x^5(0.2-\ln 5x)] + c$

$\left[\because \text{from ①}\ \frac{d}{dx}[0.4x^5(0.2-\ln 5x)] = -2x^4\ln 5x.\right]$

Integration

Example 9. (i) Show that $\frac{d}{dx}\left(\frac{\ln x}{x^3}\right) = \frac{1-3\ln x}{x^4}$ ---[3]

(ii) Find the coordinates of the stationary points on the curve $y = \frac{\ln x}{x^3}$ ---[3]

(iii) Use the result from part (i) to find. $\int \frac{\ln x}{x^4}\,dx$ $\boxed{W-17/23/Q9}$ --[4]

Solution (i) $\frac{d}{dx}\left(\frac{\ln x}{x^3}\right) = \frac{x^3 \cdot \frac{d}{dx}\ln x - \ln x \cdot \frac{d}{dx}x^3}{(x^3)^2}$ (Quotient rule)

$$= \frac{x^3 \times \frac{1}{x} - \ln x \cdot 3x^2}{x^6}$$

$$= \frac{x^2(1-3\ln x)}{x^6}$$

$$= \frac{1-3\ln x}{x^4} \checkmark$$

(ii) for stationary points, $\frac{dy}{dx} = 0 \Rightarrow \frac{1-3\ln x}{x^4} = 0$

$$\Rightarrow 1 - 3\ln x = 0$$

$$\Rightarrow 1 - \ln x^3 = 0$$

$$\Rightarrow \ln x^3 = 1$$

$$\Rightarrow x^3 = e^1$$

$$\Rightarrow x = e^{1/3} \checkmark$$

$y = \frac{\ln x}{x^3}$; $y = \frac{\ln e^{1/3}}{e} \Rightarrow y = \frac{\frac{1}{3}\ln e}{e} = \frac{1}{3} \times \frac{1}{e} = \frac{1}{3e}$

$\therefore$ Stationary Point $\left(e^{1/3}, \frac{1}{3e}\right) \checkmark$

(iii) from (i) $\frac{d}{dx}\left(\frac{\ln x}{x^3}\right) = \left(\frac{1}{x^4} - \frac{3\ln x}{x^4}\right)$

$\therefore \int\left(\frac{1}{x^4} - \frac{3\ln x}{x^4}\right)dx = \frac{\ln x}{x^3}$

or $\int x^{-4}\,dx - 3\int \frac{\ln x\,dx}{x^4} = \frac{\ln x}{x^3}$

or $\frac{x^{-3}}{-3} - 3\int \frac{\ln x}{x^4}\,dx = \frac{\ln x}{x^3}$

$\Rightarrow -3\int \frac{\ln x}{x^4}\,dx = \frac{1}{3x^3} + \frac{\ln x}{x^3}$

$\Rightarrow \int \frac{\ln x}{x^4}\,dx = -\frac{1}{9x^3} - \frac{\ln x}{3x^3} + c \checkmark$

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Integration

Example 10. A curve $y=f(x)$ is such that $f'(x)=6x-8e^{2x}$

(i) Given that the curve passes through the points $P(0,-3)$, find the equation of the curve. ---[5]

Normal to the curve $y=f(x)$ at P meets the line $y=2-3x$ at the point Q.

(ii) Find the area of the triangle OPQ, where O is origin. ---[5]

W-16/13/Q10

Solution: $f'(x)=6x-8e^{2x}$ —— (1)

(i) $\therefore f(x)=\int(6x-8e^{2x})dx$

$= 6\times\frac{x^2}{2}-8\frac{e^{2x}}{2}+c$

or $f(x)=3x^2-4e^{2x}+c$ —— (2)

Curve passes through the point $P(0,-3)$

fr (2) $-3=0-4e^0+c \Rightarrow c=1$

fr (2) $f(x)=3x^2-4e^{2x}+1$ —— (3)

(ii) Now from (1) $f'(0)=0-8e^0=-8=$ gradient of the tangent to the curve at $P(0,-3)$.

$\therefore$ Gradient of Normal at $P=\frac{1}{8}$ $\left(m_2=-\frac{1}{m_1}\right)$

$\therefore$ Eqn of normal at $P(0,-3)$, $m_2=1/8$

$y+3=\frac{1}{8}(x-0) \Rightarrow y=\frac{1}{8}x-3$ —— (4)

Now given line $y=2-3x$ —— (5)

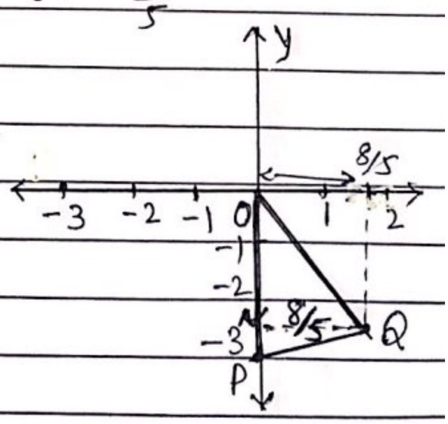
Solving (4) and (5) $\frac{1}{8}x-3=2-3x \Rightarrow x=\frac{8}{5}$ ✓ of $Q\left(\frac{8}{5},-\frac{14}{5}\right)$

$y=-\frac{14}{5}$

Area of triangle $OPQ=\frac{1}{2}\times OP\times QN$

$=\frac{1}{2}\times 3\times\frac{8}{5}$

$=\underline{2.4 \text{ sq units}}$



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Example 11. (i) Find $\frac{d}{dx}(x^3 \ln x)$ --- [2]

(ii) Hence find $\int x^2 . \ln x \, dx$ $\boxed{S\text{-}13/12/Q10(b)}$ --- [3]

Solution (i)

$$\frac{d}{dx}(x^3 . \ln x) = x^3 . \frac{d}{dx} \ln x + \ln x . \frac{d}{dx} x^3 \quad \text{(Product rule)}$$

$$= x^3 . \frac{1}{x} + \ln x . 3x^2$$

$$= x^2 + 3x^2 \ln x \text{ ——— (1)}$$

(ii) To find $\int x^2 \ln x$.

from (1) $\int (x^2 + 3x^2 \ln x) dx = x^3 \ln x$ $\left[\frac{d}{dx} x^3 \ln x = x^2 + 3x^2 \ln x \right.$

or $\int x^2 dx + \int 3x^2 \ln x \, dx = x^3 \ln x$

$$\frac{x^3}{3} + 3 \int x^2 \ln x \, dx = x^3 \ln x$$

$$\Rightarrow 3 . \int x^2 \ln x \, dx = x^3 \ln x - \frac{1}{3} x^3$$

$$\Rightarrow \int x^2 \ln x \, dx = \frac{1}{3}\left(x^3 \ln x - \frac{1}{3} x^3\right) + C$$

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Integration

§ Indefinite Integration of Trigonometric Funtions:

1 (i) $\int \sin x \, dx = -\cos x + c$ $\quad \left\{ \because \frac{d}{dx}\cos x = -\sin x \right.$

(ii) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$ $\quad \because \frac{d}{dx}\cos(ax+b) = -a\sin(ax+b)$

2. (i) $\int \cos x \, dx = \sin x + c$ $\quad \left\{ \because \frac{d}{dx}\sin x = \cos x \right.$

(ii) $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$ $\quad \because \frac{d}{dx}\sin(ax+b) = a\cos(ax+b)$

Example 12. (i) Differentiate $\sin x \cdot \cos x$ w.r.t x, giving your answer in terms of $\sin x$. ---[3]

(ii) Hence find $\int \sin^2 x \, dx$. $\boxed{\text{M-15 / 22 / Q4}}$ ---[3]

Solution: (i) $\frac{d}{dx}(\sin x \cdot \cos x) = \sin x \cdot \frac{d}{dx}\cos x + \cos x \cdot \frac{d}{dx}\sin x$ [Product rule]

$= \sin x(-\sin x) + \cos x \cdot \cos x$

$= -\sin^2 x + \cos^2 x$

$= -\sin^2 x + (1 - \sin^2 x)$

$= 1 - 2\sin^2 x$ ✓ ——— (1)

(ii) for (1)

$\int (1 - 2\sin^2 x)\,dx = \sin x \cos x$

or $\int 1\,dx - 2\int \sin^2 x\,dx = \sin x \cos x$

or $x - 2\int \sin^2 x\,dx = \sin x \cos x$

or $-2\int \sin^2 x\,dx = \sin x \cos x - x$

$\therefore \int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{2}\sin x \cos x + c$ ✓

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Integration

classmate Date Page P-12

Example 13. $\int \sin\frac{x}{6}\,dx$ [SP-20/01/Q8(b)] ---[2]

Solution: $\int \sin\frac{x}{6}\,dx = \dfrac{-\cos\frac{x}{6}}{\frac{1}{6}}$ $\quad \because \int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)$

$= -6\cos\frac{x}{6} + c$ ✓

Example 14(i) Differentiate $1+\tan x/3$ w.r.t x ---[2]

(ii) Hence find $\int \sec^2 x/3\,dx$ ---[2] [M-18/22/Q6]

Solution (i) $\frac{d}{dx}\left(1+\tan x/3\right) = 0 + \sec^2\frac{x}{3} \times \frac{1}{3}$

$= \frac{1}{3}\sec^2 x/3$ ---(1)

(ii) from (1) $\int \frac{1}{3}\sec^2 x/3\,dx = 1+\tan x/3$

or $\int \sec^2 x/3\,dx = 3\tan x/3 + 3 + k$

$= 3\tan x/3 + c$ ✓

Example 15. A curve, showing the relationship between x and y is such that $\frac{d^2y}{dx^2} = 6\cos 3x$. Given that the curve has a gradient of $4\sqrt{3}$ at the point $\left(\frac{\pi}{9}, -\frac{1}{3}\right)$, find the equation of the curve. --[6] [W-15/13/Q9]

Solution: $\frac{d^2y}{dx^2} = 6\cos 3x$

$\therefore \frac{dy}{dx} = \int 6\cos 3x\,dx$

$= 6\cdot\frac{\sin 3x}{3} + c$

or $\frac{dy}{dx} = 2\sin 3x + c$ ---(1)

Given gradient $\frac{dy}{dx} = 4\sqrt{3}$ at $\left(\frac{\pi}{9}, -\frac{1}{3}\right)$

$\therefore$ from (1) $4\sqrt{3} = 2\sin 3\times\frac{\pi}{9} + c$

or $4\sqrt{3} = 2\times\frac{\sqrt{3}}{2} + c \Rightarrow c = 3\sqrt{3}$ ✓

$\therefore$ from (1) $\frac{dy}{dx} = 2\sin 3x + 3\sqrt{3}$ ---(2)

from (2) $\frac{dy}{dx} = 2\sin 3x + 3\sqrt{3}$

$\therefore y = \int(2\sin 3x + 3\sqrt{3})\,dx$

or $y = -\frac{2}{3}\cos 3x + 3\sqrt{3}x + d$ ---(3)

Passes through $\left(\frac{\pi}{9}, -\frac{1}{3}\right)$

$\therefore -\frac{1}{3} = -\frac{2}{3}\cos\frac{\pi}{3} + 3\sqrt{3}\times\frac{\pi}{9} + d$

$\Rightarrow d = -\frac{\sqrt{3}}{3}\pi$ ✓

from (3) Required equation of the curve,

$y = -\frac{2}{3}\cos 3x + 3\sqrt{3}x - \frac{\sqrt{3}}{3}\pi$ ✓

Integration

§ Definite Integration:

$$\int_a^b f(x)dx = [F(x)]_a^b$$
$$= F(b) - F(a)$$

here 'b' is called the upper limit and 'a' the lower limit of the definite integral.

Let the indefinite integral $\int f(x)dx = F(x) + C$ where c is a constant, may take any value.

Example (a) $\int_1^2 x^3 dx = \left[\frac{1}{4}x^4\right]_1^2 = \frac{1}{4}\left[(2)^4 - (1)^4\right] = \frac{15}{4}$ ✓

(b) $\int_0^{\frac{\pi}{2}} \cos x\, dx = [\sin x]_0^{\frac{\pi}{2}} = \left(\sin\frac{\pi}{2} - \sin 0\right) = 1 - 0 = 1$ ✓

(c) $\int_0^1 e^x dx = [e^x]_0^1 = e^1 - e^0 = (e-1)$ ✓

(d) $\int_1^e \frac{1}{x} dx = [\ln x]_1^e = (\ln e - \ln 1) = 1 - 0 = 1$ ✓

Example 15. Giving your answer in its simplest form, find the exact value of. (i) $\int_{0.2}^1 (e^{5x-1})\, dx$ ---[4]

(ii) $\int_1^2 \left(x + \frac{1}{x^2}\right)^2 dx$ [SP-20|01|Q8(a)] --[2]

Solution (i) $\int_{0.2}^1 (e^{5x-1})dx$

$= \left[\frac{1}{5}e^{5x-1}\right]_{0.2}^1$

$= \frac{1}{5}[e^4 - e^0]$

$= \frac{1}{5}(e^4 - 1)$ ✓

(ii) $\int_1^2 \left(x + \frac{1}{x^2}\right)^2 dx$

$= \int_1^2 \left(x^2 + 2 \times \frac{1}{x} + \frac{1}{x^4}\right) dx$

$= \left[\frac{x^3}{3} + 2\ln x + \frac{x^{-3}}{-3}\right]_1^2$

$= \left(\frac{8}{3} + 2\ln 2 - \frac{1}{24}\right) - \left(\frac{1}{3} + 2\ln 1 - \frac{1}{3}\right)$

$= \left(2\ln 2 + \frac{21}{8}\right)$ ✓

IG- Maths 0606

Integration
(Definite Integral)

Example 16. It is given that $y = (10x+2)\ln(5x+1)$

(i) find $\frac{dy}{dx}$ --[4]

(ii) Hence show that: $\int \ln(5x+1)\,dx = \frac{(ax+b)}{5}\ln(5x+1) - x + c$, where a and b are integers and c is a constant of integral. --[3]

(iii) Hence find $\int_0^{1/5} \ln(5x+1)\,dx$, giving your answer in the form, $\frac{d+\ln f}{5}$, where d and f are integers. --[2]

S-17	13	Q10

Solution (i) $y = (10x+2)\ln(5x+1)$

$$\frac{dy}{dx} = (10x+2)\cdot\frac{d}{dx}\ln(5x+1) + \ln(5x+1)\frac{d}{dx}(10x+2) \quad \left[\text{Product rule}\right]$$

$$= (10x+2)\times\frac{1}{(5x+1)}\times 5 + \ln(5x+1)\times 10$$

$$= 2\cancel{(5x+1)}\times\frac{5}{\cancel{(5x+1)}} + 10\ln(5x+1)$$

$$\therefore \frac{dy}{dx} = 10 + 10\ln(5x+1) \quad \text{——— } (1)$$

(ii) $\int(10 + 10\ln(5x+1))\,dx = (10x+2)\ln(5x+1) + c$ from (1)

or $\int 10\,dx + 10\int\ln(5x+1)\,dx = (10x+2)\ln(5x+1) + c$

or $10x + 10\int\ln(5x+1)\,dx = (10x+2)\ln(5x+1) + c$

or $10\int\ln(5x+1)\,dx = (10x+2)\ln(5x+1) - 10x + c$

or $\int\ln(5x+1)\,dx = \frac{(5x+1)}{5}\ln(5x+1) - x + c$ ✓ ——— (2)

(iii) $$\int_0^{1/5}\ln(5x+1)\,dx = \left[\frac{(5x+1)}{5}\ln(5x+1) - x\right]_0^{1/5}$$

$$= \left(\frac{(5\times\frac{1}{5}+1)}{5}\ln(5\times\tfrac{1}{5}+1) - \frac{1}{5}\right) - \left(\frac{1}{5}\ln 1 - 0\right)$$

$$= \left(\frac{2}{5}\ln 2 - \frac{1}{5}\right)$$

$$= \frac{(2\ln 2 - 1)}{5}$$

$$= \frac{(\ln 2^2 - 1)}{5} = \left(\frac{-1+\ln 4}{5}\right) ✓$$

IG-Maths 0606

Integration

classmate Page P-15

Example 17. (i) Find $\frac{d}{dx}\left(\frac{5}{3x+2}\right)$ ---[2]

(ii) Use your answer to part (i) to find $\int \frac{30}{(3x+2)^2}dx$ ---[2]

(iii) Hence evaluate, $\int_1^2 \frac{30}{(3x+2)^2}dx$

W-17/21/Q5

Solution (i) $\frac{d}{dx}\left(\frac{5}{3x+2}\right) = \frac{d}{dx} 5\cdot(3x+2)^{-1}$

$= 5\times(-1)(3x+2)^{-2}\times 3$

$= \frac{-15}{(3x+2)^2}$ —— ①

(ii) from (1) $\int \frac{-15}{(3x+2)^2}dx = \frac{5}{3x+2}$

or $\int \frac{1}{(3x+2)^2}dx = \frac{-1}{15}\times\frac{5}{(3x+2)}$

$\therefore \int \frac{30}{(3x+2)^2}dx = \frac{-1}{3(3x+2)}\times 30 = \frac{-10}{(3x+2)} + C$ —— ②

(iii) $\int_1^2 \frac{30}{(3x+2)^2}dx = \left[\frac{-10}{(3x+2)}\right]_1^2$ from ②

$= -10\left[\frac{1}{8}-\frac{1}{5}\right] = -10\times\frac{-3}{40} = \frac{3}{4}$ ✓

Example 18. (i) Find $\frac{d}{dx}(x\cdot \ln x)$ ---[2]

(ii) Hence find $\int \ln x\, dx$ ---[2]

(iii) Hence, given that $k>0$, show that $\int_k^{2k} \ln x\, dx = k[\ln 4k - 1]$ — 4

W-17/22/Q9

Solution: (i) $\frac{d}{dx}(x\cdot \ln x) = x\cdot\frac{d}{dx}\ln x + \ln x\times\frac{d}{dx}x$

$= x\times\frac{1}{x} + \ln x\times 1$

$= (1+\ln x)$ —— ①

(ii) from ① $\int(1+\ln x)dx = x\ln x + c$

or $\int 1\,dx + \int \ln x\,dx = x\ln x + c$

or $x + \int \ln x\,dx = x\ln x + c$

or $\int \ln x\,dx = x\ln x - x + c$ —— ②

(iii) $\int_k^{2k} \ln x\,dx = \left[x\ln x - x\right]_k^{2k}$ from ②

$= (2k\ln 2k - 2k) - (k\ln k - k)$

$= k(2\ln 2k - \ln k - 1)$

$= k\left[\ln\frac{(2k)^2}{k} - 1\right]$

$= k(\ln 4k - 1)$ ✓

Integration

Example 19. (i) Find $\int (7x-10)^{-3/5}dx$. ---[2]

(ii) Given that $\int_6^a (7x-10)^{-3/5}dx = \frac{25}{14}$ find the exact value of a. --[3]

W-17|13|Q5

Solution: (i) $\int (7x-10)^{-3/5}dx = \frac{1}{7}\left(\frac{(7x-10)^{2/5}}{2/5}\right) = \frac{5}{14}(7x-10)^{2/5}$ ✓ —— ①

(ii) $\int_6^a (7x-10)^{-3/5}dx = \left[\frac{5}{14}(7x-10)^{2/5}\right]_6^a = \frac{25}{4}$ Given and from ①

$$\text{or } \frac{5}{14}\left[(7a-10)^{2/5} - (32)^{2/5}\right] = \frac{25}{14}$$

$$\text{or } (7a-10)^{2/5} - 4 = \frac{25}{14}\times\frac{14}{5}$$

$$(7a-10)^{2/5} = 5+4 = 9$$

$$\therefore 7a-10 = 9^{5/2}$$

$$7a-10 = 243$$

$$7a = 253$$

$$a = \frac{253}{7} = 36\frac{1}{7} \checkmark$$

Example 20 (i) Show that $\frac{d}{dx}\left(\frac{e^{4x}}{4} - xe^{4x}\right) = pxe^{4x}$, where p is an integer to be found. ---[4]

(ii) Hence find the exact value of $\int_0^{\ln 2} xe^{4x}dx$, giving your answer in the form, $a\ln 2 + \frac{b}{c}$, where a, b and c are the integers to be found. --[4]

S-16|11|Q5

Solution: $\frac{d}{dx}\left(\frac{e^{4x}}{4} - xe^{4x}\right) = \frac{e^{4x}}{4}\times 4 - \left(x\frac{d}{dx}e^{4x} + e^{4x}\frac{d}{dx}x\right)$

$$= \cancel{e^{4x}} - x\times 4e^{4x} \cancel{+e^{4x}}\times 1$$

$$= -4xe^{4x} + c. \text{ —— ①}$$

(ii) $\int_0^{\ln 2} x e^{4x}dx = -\frac{1}{4}\left[\frac{e^{4x}}{4} - xe^{4x}\right]_0^{\ln 2}$ from ①

$$= -\frac{1}{4}\left[e^{4x}\left(\frac{1}{4}-x\right)\right]_0^{\ln 2}$$

$$= -\frac{1}{4}\left[e^{4\ln 2}\left(\frac{1}{4}-\ln 2\right) - e^0\left(\frac{1}{4}-0\right)\right]$$

$$= -\frac{1}{4}\left[e^{\ln 2^4}\left(\frac{1}{4}-\ln 2\right) - \frac{1}{4}\right] = -\frac{1}{4}\left[16\left(\frac{1}{4}-\ln 2\right) - \frac{1}{4}\right]$$

$$= \left(4\ln 2 - \frac{15}{16}\right) \checkmark \text{ ——— } = -4\left(\frac{1}{4}-\ln 2\right) + \frac{1}{16}$$

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Integration

Example 21 (i) Find $\int \sin(5x+\pi)\,dx$ ---[2]

(ii) Hence evaluate $\int_{-\frac{\pi}{5}}^{0} \sin(5x+\pi)\,dx$ ---[2]

S-16/22/Q9(b)

Solution:

(i) $\int \sin(5x+\pi)\,dx = -\dfrac{\cos(5x+\pi)}{5} + c$ —— (1)

(ii) $\int_{-\frac{\pi}{5}}^{0} \sin(5x+\pi)\,dx = -\frac{1}{5}\left[\cos(5x+\pi)\right]_{-\pi/5}^{0}$

$= -\frac{1}{5}\left[\cos\pi - \cos\left(5\times -\frac{\pi}{5}+\pi\right)\right]$

$= -\frac{1}{5}\left[\cos\pi - \cos 0\right] = -\frac{1}{5}\left[-1-1\right] = \frac{2}{5} = 0.4$ ✓

Example 22 (i) Given that $\int_0^k \left(2e^{2x} - \frac{5}{2}e^{-2x}\right)dx = \frac{3}{4}$, where k is a constant, show that $4e^{4k} - 12e^{2k} + 5 = 0$ ---[5]

(ii) Using substitution of $y = e^{2k}$, or otherwise, find the possible values of k. W-13/13/Q11 ---[4]

Solution (i) $\int_0^k \left(2e^{2x} - \frac{5}{2}e^{-2x}\right)dx = \left[2\cdot\frac{e^{2x}}{2} - \frac{5}{2}\cdot\frac{e^{-2x}}{-2}\right]_0^k = \frac{3}{4}$ given

or $\left(e^{2k} + \frac{5}{4}e^{-2k}\right) - \left(1+\frac{5}{4}\right) = \frac{3}{4}$

$\Rightarrow 4e^{4k} - 12e^{2k} + 5 = 0$ ✓ —— (1)

(ii) from (1) $4\left(e^{2k}\right)^2 - 12e^{2k} + 5 = 0$ [Put $y = e^{2k}$

$\Rightarrow 4y^2 - 12y + 5 = 0$

$(2y-5)(y-\frac{1}{2}) = 0$

$\Rightarrow y = \frac{5}{2}$ or $\frac{1}{2}$

$\Rightarrow e^{2k} = 2.5$ or $e^{2k} = 0.5$

$\Rightarrow 2k = \ln 2.5$ or $2k = \ln 0.5$

$\Rightarrow k = \frac{1}{2}\ln 2.5$ or $k = \frac{1}{2}\ln 0.5$

$\Rightarrow \underline{k = 0.458}$ or $\underline{k = -0.347}$ ✓

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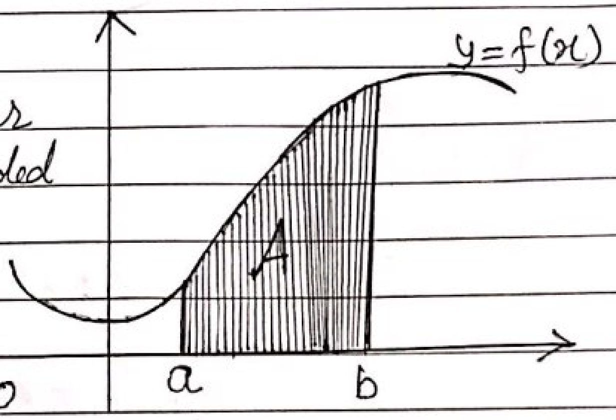
Integration

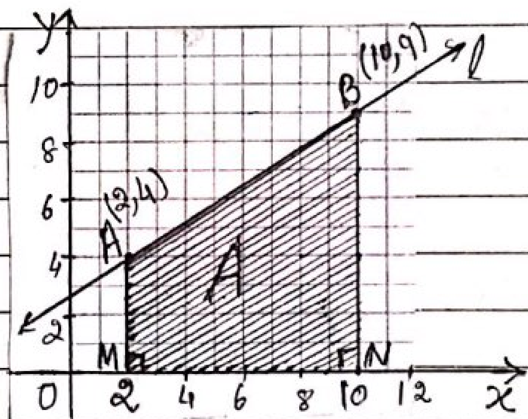
§ Area under a curve:

The area of the region under the curve $y = f(x)$ and bounded by the lines $x = a$ and $x = b$, is given by;

$$\text{Area} = \int_a^b f(x)\,dx \quad ; f(x) \geq 0$$



Example: Find the area bounded by the line $5x - 8y + 22 = 0$ above x-axis and the lines $x = 2$ and $x = 10$.



Solution: Given the equation of line

$5x - 8y + 22 = 0$

or $y = \frac{1}{8}(5x + 22)$ —— ①

∴ Shaded area.

$= \int_2^{10} y\,dx$

$= \int_2^{10} \frac{1}{8}(5x + 22)\,dx$ fr ①

$= \frac{1}{8}\left[\frac{5x^2}{2} + 22x\right]_2^{10}$

$= \frac{1}{8}\left[\left(\frac{5}{2} \times 100 + 22 \times 10\right) - \left(\frac{5}{2} \times 4 + 22 \times 2\right)\right]$

$= \frac{1}{8}[416]$

$= 52$ sq. units ✓

Alternate method:

The shaded region is a Trapezium ABNM

fr ① $x = 2;\ y = 4 = AM$

and $x = 10;\ y = 9 = BN$

Area of Trapezium ABNM

$= \frac{1}{2}(AM + BN) \times MN$

$= \frac{1}{2}(4 + 9) \times (10 - 2)$

$= \frac{1}{2} \times 13 \times 8 = 52$ sq unit ✓

Note: But this method is not applicable for all the curves.

IG-Maths 0606

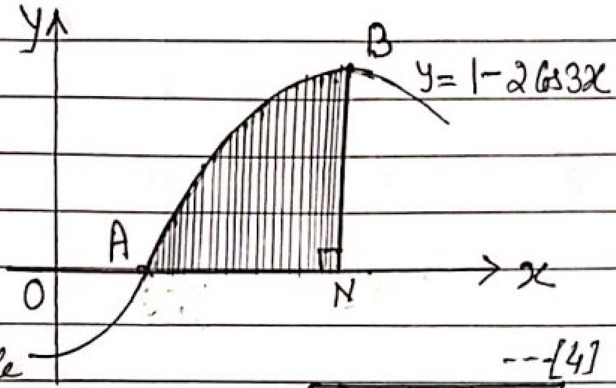
Integration

Example 22. The diagram shows part of the graph of $y = 1 - 2\cos 3x$, which crosses x-axis at the point A and has a maximum at B.

(i) Find the coordinates of A. --- [2]

(ii) Find the coordinates of B. --- [2]

(iii) Show all your working, find the area of the shaded region bounded by the curve, the x-axis, and the perpendicular from B to the x-axis. --- [4]



[S-16/12/Q7]

Solution (i) Given equation of the curve $y = 1 - 2\cos 3x$ — ①

Curve intersects X-axis, when $y = 0 \Rightarrow 1 - \cos 3x = 0$

$\therefore \cos 3x = 1 = \cos\frac{\pi}{3}$

$\Rightarrow 3x = \frac{\pi}{3} \Rightarrow x = \frac{\pi}{9} = 0.35$

$\therefore A(0.35, 0)$ ✓

(ii) There is a maximum (stationary point) at B, $\frac{dy}{dx} = 0$

diff ① $\frac{dy}{dx} = 6\sin 3x = 0$

$\Rightarrow \sin 3x = 0 \Rightarrow 3x = \pi \Rightarrow x = \frac{\pi}{3}$ ✓

from ① $y = 1 - 2\cos\pi = 3$ ✓

$\therefore B\left(\frac{\pi}{3}, 3\right)$ ✓

(iii) Area of shaded region:

$$= \int_{\frac{\pi}{9}}^{3} y\,dx = \int_{\frac{\pi}{9}}^{\frac{\pi}{3}} (1 - 2\cos 3x)\,dx$$

$$= \left[x - \frac{2}{3}\sin 3x\right]_{\frac{\pi}{9}}^{\frac{\pi}{3}}$$

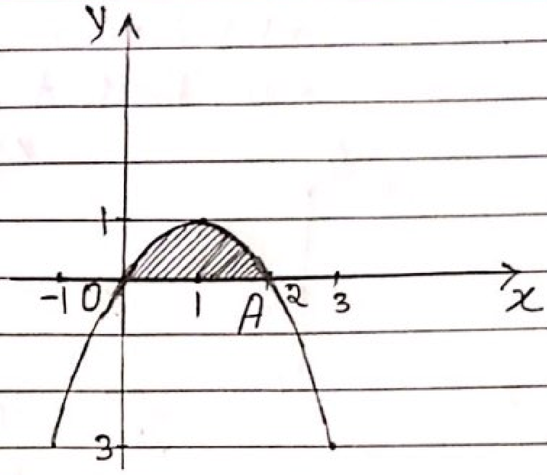
$$= \left[\left(\frac{\pi}{3} - \frac{2}{3}\sin\pi\right) - \left(\frac{\pi}{9} - \frac{2}{3}\sin\frac{\pi}{3}\right)\right]$$

$$= \left[\frac{\pi}{3} - \left(\frac{\pi}{9} - \frac{2}{3} \times \frac{\sqrt{3}}{2}\right)\right]$$

$$= \left(\frac{2\pi}{9} + \frac{\sqrt{3}}{3}\right) = 1.28 ✓$$

Integration

Example 23. Find the area of the region bounded by $y = 2x - x^2$ and x-axis.



Solution: $y = 2x - x^2$ —— (1)

for intersection with x-axis,

$y = 0$

or $2x - x^2 = 0$

$x(2-x) = 0$

or $x = 0,\ x = 2$

Required area of the shaded region

$$= \int_0^2 y\,dx$$

$$= \int_0^2 (2x - x^2)\,dx$$

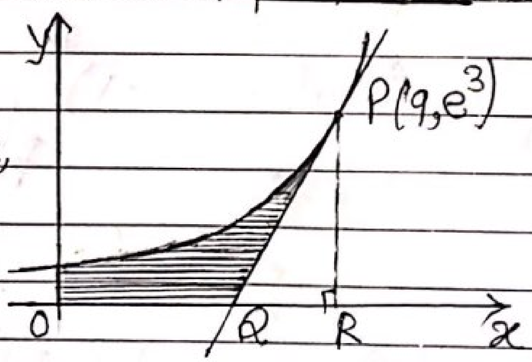
$$= \left[2\cdot\frac{x^2}{2} - \frac{x^3}{3}\right]_0^2$$

$$= \left(4 - \frac{8}{3}\right) - (0-0) = \underline{\frac{4}{3}\text{ sq. unit}}$$

Example 24. The diagram shows part of the curve $y = e^{x/3}$. The tangent to the curve at $P(9, e^3)$ meets the x-axis at Q. [W-13/21/Q11]-[6]

(i) Find the coordinates of Q --- [4]

(ii) Find the area of the shaded region bounded by the curve, the coordinate axes, and the tangent to the curve at P.



Solution: $y = e^{x/3}$ —— (1)

$\frac{dy}{dx} = \frac{1}{3}e^{x/3}$

at P, $\left(\frac{dy}{dx}\right)_{x=9} = \frac{1}{3}e^3$

$\therefore$ Equation of tangent at $P(9, e^3)$

$y - e^3 = \frac{1}{3}e^3(x-9)$ —— (2)

at Q, $y = 0 \Rightarrow x = 6$, $Q(6,0)$

draw $PR \perp x$-axis.

Area of the shaded region:

$= \int_0^9 e^{x/3}dx$ − area of triangle PQR

$= \left[3e^{x/3}\right]_0^9 - \frac{1}{2}(9-6)\times e^3$

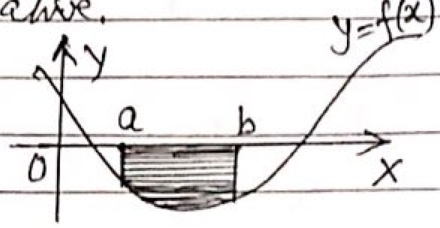
$= (3e^3 - 3) - 1.5e^3$

$= (1.5e^3 - 3)$

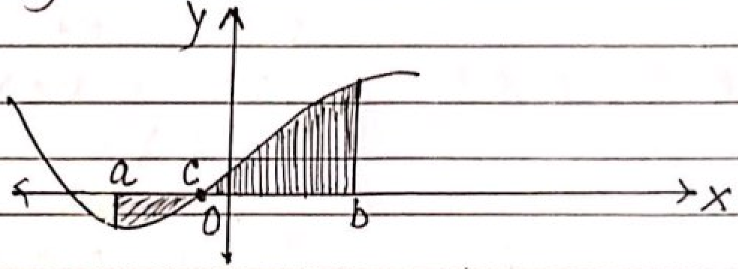
Integration

§ Area of region when $f(x) < 0$ from $x = a$ to $x = b$ is below - x-axis come out to be negative.

Then the required Area is $= \left| \int_a^b f(x)\,dx \right|$



§ But in case some portion of the curve is below x-axis and some above x-axis, the we find the area these two regions separately and add their numerical Values.



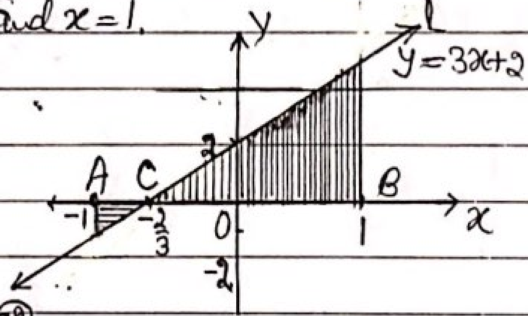
Area of the shaded region $= \left| \int_a^c f(x)\,dx \right| + \int_c^b f(x)\,dx$

Example 25. Find the area of the region bounded by the line $y = 3x + 2$, the x-axis and the line $x = -1$ and $x = 1$.

Solution: Given line $y = 3x + 2$ — (1)

intersection x-axis at $3x + 2 = 0$

or $x = -\frac{2}{3}$



Required Area $= \left| \int_{-1}^{-2/3} y\,dx \right| + \int_{-2/3}^{1} y\,dx$ — (2)

Now $\int_{-1}^{-2/3} y\,dx = \int_{-1}^{-2/3} (3x+2)\,dx$

$= \left[\frac{3}{2}x^2 + 2x \right]_{-1}^{-2/3}$

$= \left[\left(-\frac{2}{3}\right) - \left(\frac{1}{2}\right) \right]$

$= -\frac{1}{6}$ — (3)

and $\int_{-2/3}^{1} (3x+2)\,dx$

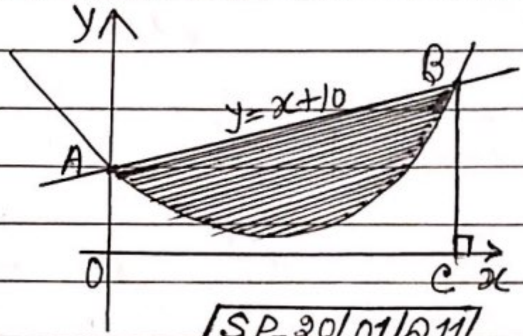
$= \left[\frac{3}{2}x^2 + 2 \right]_{-2/3}^{1}$

$= \left[\left(\frac{7}{2}\right) - \left(-\frac{2}{3}\right) \right] = \frac{7}{2} + \frac{2}{3} = \frac{25}{6}$ — (4)

Hence the required area of the shaded region $= \left| -\frac{1}{6} \right| + \frac{25}{6}$ from (2), (3) & (4)

$= \frac{1}{6} + \frac{25}{6} = \frac{26}{6} = \frac{13}{3}$ ✓

Example 26. The graph of $y = x^2 - 4x + 10$ cuts the y-axis at point A. The graph of $y = x^2 - 4x + 10$ and $y = x + 10$ intersects one another at the points A and B. The line BC is perpendicular to the x-axis. Calculate the area of the shaded region enclosed by the curve and line AB.



SP-20/01/Q11

Solution: Given curve $y = x^2 - 4x + 10$ —— ①

and line $y = x + 10$ —— ②

Solving ① & ② $x = 0$ or $x = 5$; hence the x-coord of B is 5

$A(0, 10)$, $OA = 10$, $B(5, 15)$, $BC = 15$, $OC = 5$

Area of the shaded region = Area of Trapezium OABC − Area under the curve $x = 0$ to $x = 5$

$$= \frac{1}{2}(OA + BC) \times OC - \int_0^5 (x^2 - 4x + 10)\,dx$$

$$= \frac{1}{2}(10 + 15) \times 5 - \left[\frac{x^3}{3} - 2x^2 + 10x\right]_0^5$$

$$= \frac{125}{2} - \left[\left(\frac{125}{3}\right) - (0)\right]$$

$$= \frac{125}{2} - \frac{125}{3}$$

$$= \frac{125}{6} \checkmark$$

Integration

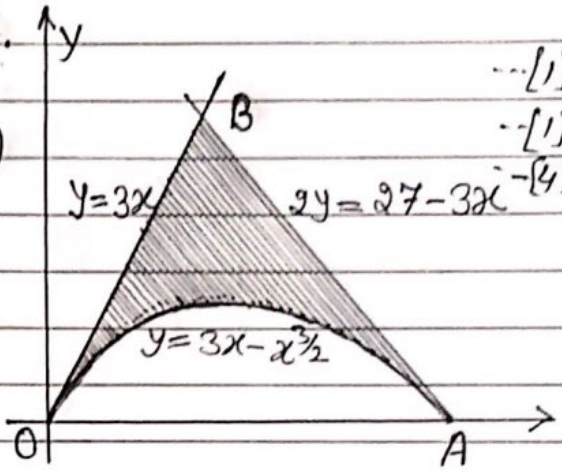
Example 27 (i) Find $\int(3x-x^{3/2})\,dx$ ---[2]

The diagram shows, part of the curve $y=3x-x^{3/2}$ and the lines $y=3x$ and $2y=27-3x$. The curve and the line $y=3x$ meet the x-axis at O and the curve and the line $2y=27-3x$ meets the x-axis at A.

(ii) Find the coordinates of A. ---[1]

(iii) Verify that the coordinates of B are (3,9) --[1]

(iv) Find the area of the shaded area. --[4]



S-16/21/Q11

Solution (i) $\int(3x-x^{3/2})\,dx$

$$=\frac{3x^2}{2}-\frac{2}{5}x^{5/2}+c \checkmark \quad ①$$

(ii) Given curve $y=3x-x^{3/2}$ ——— ②

and line $2y=27-3x$ ——— ③

Solving ② & ③ $6x-2x^{3/2}=27-3x$

or $2x^{3/2}-9x+27=0$

$2y^3-9y^2+27=0$ (Let $\sqrt{x}=y$)

by hit and trial $y=3$ is a solution

or $y=\sqrt{x}=3 \Rightarrow x=9\checkmark, y=0$

$\therefore$ $A(9,0)\checkmark$ (or $OA=9$)

(iii) Given the two lines $2y=27-3x$ — ③

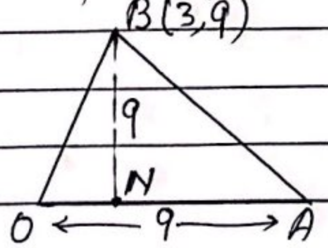
and $y=3x$ — ④

(3,9) satisfy both ③ & ④

$\therefore B(3,9). \checkmark$

(iv) Required shaded area is

Area of triangle OAB − area under curve. — ⑤



Now area of Δ OAB

$= \frac{1}{2}\times OA\times BN$

$= \frac{1}{2}\times 9\times 9 = \frac{81}{2}\checkmark$ — ⑥

Area under the curve

$$=\int_0^9 (3x-x^{3/2})\,dx$$

$$=\left[\frac{3}{2}x^2-\frac{2}{5}x^{5/2}\right]_0^9$$

$$=\left[\left(\frac{243}{2}-\frac{486}{5}\right)-0\right]$$

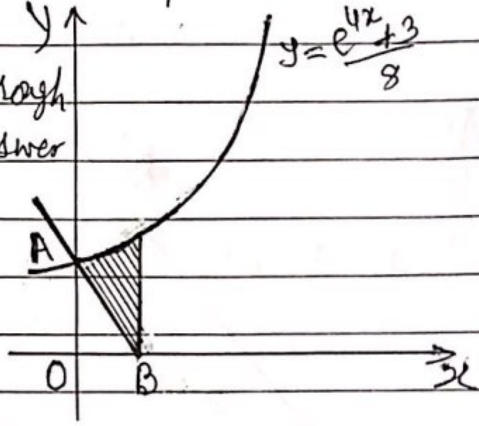
$$=24.3 \quad ⑦$$

$\therefore$ Area of Shaded region

$= \frac{81}{2} - 24.3 = 40.5-24.3$

$= 16.2\checkmark$

Example 28: The diagram shows the graph of the curve $y = \frac{e^{4x}+3}{8}$. The curve meets the y-axis at the point A. The normal to the curve at A meets the x-axis at at the point B. Find the area of the shaded region enclosed by the curve, the line AB and the line through B, parallel to the y-axis. Give your answer in the form $\frac{e}{a}$, where a is a constant.



[S-18/12/Q 11/---[10]

Solution: Given equation of curve,

$$y = \frac{e^{4x}+3}{8} \quad \text{—— ①}$$

the curve cuts y-axis at $x=0$ in ① $y=\frac{1}{2}$ $A(0,\frac{1}{2})$.

diff ① $\frac{dy}{dx} = \frac{1}{8}.e^{4x}.4 = \frac{1}{2}e^{4x}$

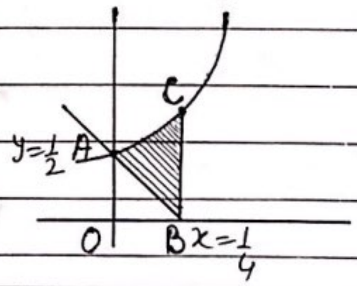
$\left(\frac{dy}{dx}\right)_{A(x=0)} = \frac{1}{2}e^{0} = \frac{1}{2}$ is the gradient of tangent at A.

$\therefore$ Gradient of normal at A $= -2$ $(m_1m_2=-1)$

$\therefore$ Equation of normal at $A(0,\frac{1}{2})$ and $m=-2$

$y-\frac{1}{2} = -2(x-0)$

or $y = -2x+\frac{1}{2}$ —— ②



Normal intersects x-axis at

$y=0$ in ② $x=\frac{1}{4}$ $B(\frac{1}{4},0)$

The required shaded area

$= \text{area } OACB - \text{area of triangle } OAB$

$= \int_0^{1/4} \frac{e^{4x}+3}{8}dx - \frac{1}{2}\times OA\times OB$

$= \frac{1}{8}\left[\frac{e^{4x}}{4}+3x\right]_0^{\frac{1}{4}} - \frac{1}{2}\times\frac{1}{2}\times\frac{1}{4}$

$= \frac{1}{8}\left[\left(\frac{e}{4}+\frac{3}{4}\right)-\left(\frac{1}{4}+0\right)\right] - \frac{1}{16}$

$= \frac{e}{32} + \frac{3}{32} - \frac{1}{32} - \frac{1}{16}$

$= \frac{e}{32}$ ✓