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IG_0606 Additional Maths

Logarithmic and Exponential Functions Notes

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Logarithmic and Exponential functions

§ Exponential Function:

$y = a^x \quad , a \neq 1, a > 0, \quad x \in R$

$m, n \in R$

Laws of Exponents:

1. $a^m \times a^n = a^{m+n}$
2. $\dfrac{a^m}{a^n} = a^{m-n}$ or $a^m \div a^n = a^{m-n}$
3. $(a^m)^n = a^{m.n}$
4. $a^m . b^m = (ab)^m \qquad a>0, b>0, a\neq 1, b \neq 1$
5. $\dfrac{a^m}{b^m} = \left(\dfrac{a}{b}\right)^m$
6. $a^0 = 1$
7. $a^{-m} = \dfrac{1}{a^m}$
8. $a^{\frac{1}{m}} = \sqrt[m]{a}$
9. $a^{m/n} = (\sqrt[n]{a})^m$ or $\sqrt[n]{a^m}$

§ Logarithmic Function:

Def: $\log_a x = y \quad$ if $\quad a^y = x$

$a>0, a\neq 1, x>0$

we say log x to the base a is y.

Example:

$\log_{10} 100 = 2 \quad$ as $\quad 10^2 = 100$

Note: log to base 10 is specially written as lg.

or $lg_{10} 100 = 2$.

(i) $2^5 = 32$

may be written as

$\log_2 32 = 5$

(ii) $3^4 = 81$

$\log_3 81 = 4$

(iii) $\log_5 5 = 1$ as $5^1 = 1$

(iv) $\log_3 1 = 0$ as $3^0 = 1$

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Logarithmic and exponential functions.

§ Laws of Logarithms: $a>0, a\neq 1, m,n \in R, m>0, n>0$

1. $\log_a mn = \log_a m + \log_a n$
2. $\log_a \frac{m}{n} = \log_a m - \log_a n$
3. $\log_a m^n = n \cdot \log_a m$
4. $\log_a a = 1 \quad (\because a^1 = a)$
5. $\log_a 1 = 0 \quad (\because a^0 = 1)$
6. $\log_a \frac{1}{m} = -\log_a m$
7. $\log_a a^m = m$
8. $\log_a m = \frac{\log_c m}{\log_c a}$ (Change of base)
9. $\log_c a = \frac{1}{\log_a c}$
10. $a^{\log_a x} = x$

Proof of these laws is given on Page-21

Example 1 (i) Simplify, $\log_a \sqrt{2} + \log_a 8 + \log_a \frac{1}{2}$, giving your answer in the form $p\log_a 2$, when p is a constant --[2]

(ii) Solve the equation $\log_3 x - \log_9 4x = 1$ --[4]

SP-20/02/Q10(b)(c)

Solution (i) $\log_a \sqrt{2} + \log_a 8 + \log_a \frac{1}{2}$

$= \log_a 2^{\frac{1}{2}} + \log_a 2^3 + \log_a 2^{-1}$

$= \frac{1}{2}\log_a 2 + 3\log_a 2 - \log_a 2$

$= \frac{5}{2}\log_a 2$ ✓

(ii) Solve

$\log_3 x - \log_9 4x = 1$ —— (1)

$\left[\log_9 4x = \frac{\log_3 4x}{\log_3 9} = \frac{\log_3 4x}{2}\right]$

or $\log_3 x - \frac{1}{2}\log_3 4x = 1$

or $\log_3 x - \frac{1}{2}\left[\log_3 4 + \log_3 x\right] = 1$

$\log_3 x - \frac{1}{2}\log_3 x - \frac{1}{2}\log_3 4 = 1$

or $\frac{1}{2}\log_3 x - \frac{1}{2}\log_3 4 = 1$

$\Rightarrow \log_3 x - \log_3 4 = 2$

$\Rightarrow \log_3 \frac{x}{4} = 2 \Rightarrow \frac{x}{4} = 3^2 \Rightarrow x = 36$ ✓

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Logarithmic function

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Example 2 (a) Solve the equation $\lg(2y-7)+\lg y = 2\lg 3$ ---[4]

(b) Write: $\dfrac{\log_2 p - \log_2 q}{(\log_2 r)(\log_r 2)}$ as a single logarithm to base 2. --[2]

$\boxed{M-18/22/Q8}$

Solution (a) Solve, $\lg(2y-7)+\lg y = 2\lg 3$

$\Rightarrow \lg(2y-7)\times y = \lg 3^2$ $\quad [\log_a m + \log_a n = \log_a mn]$

$\Rightarrow 2y^2-7y = 9 \Rightarrow 2y^2-7y-9=0$

$(2y-9)(y+1)=0$

$\Rightarrow y=\frac{9}{2}$ or $y=-1$ ✗ as $y>0$

$\therefore \underline{y=4.5}$ ✓

(b) Simplify. $\dfrac{\log_2 p - \log_2 q}{\log_2 r \times \log_r 2} = \dfrac{\log_2 p/q}{\cancel{\log_2 r} \times \frac{1}{\cancel{\log_2 r}}} = \log_2 p/q$ ✓ $\quad \left[\because \log_a b = \frac{1}{\log_b a}\right]$

Example 3. It is given that $y=\log_a(ax)+2\log_a(4x-3)-1$, where 'a' is a positive integer.

(i) Explain why x must be greater than 0.75. ---[1]

(ii) Show that y can be written as $\log_a(16x^3-24x^2+9x)$ ---[3]

(iii) Find the value of x for which $y=\log_a(9x)$ $\boxed{M-17/22/Q6(b)}$ ---[2]

Solution (i) for $\log_a(4x-3)$ be defined; $4x-3>0 \Rightarrow x>0.75$ ✓

(ii) $y=\log_a(ax)+2\log_a(4x-3)-1$

or $y=\log_a ax+\log_a(4x-3)^2-\log_a a$

$=\log_a \dfrac{\cancel{a}x\times(4x-3)^2}{\cancel{a}}$

$=\log_a(16x^3-24x^2+9x)$ ✓

(iii) for $\log_a(16x^3-24x^2+9x)=\log_a 9x$

$\Rightarrow 16x^3-24x^2+9x=9x$

$\Rightarrow 16x^3-24x^2=0$

$\Rightarrow 8x^2(2x-3)=0$

$\Rightarrow x=3/2$ or $x=0$ ✗

$\therefore x=3/2$ ✓

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Logarithmic functions

Example 4(a) Given that $a^7 = b$, where a and b are positive constants,
find (i) $\log_a b$ ---[1]
(ii) $\log_b a$ ---[1]

(b) Solve the equation. $\log_{81} y = -\frac{1}{4}$ ---[2]

(c) Solve the equation. $\frac{32^{x^2-1}}{4^{x^2}} = 16$ ---[3]

S-17/22/Q7

Solution: (a) (i) Given $a^7 = b$

$\therefore \log_a b = 7$ ✓ $[\because a^x = y \Rightarrow \log_a y = x]$

(ii) $\log_b a = \frac{1}{\log_a b} = \frac{1}{7}$ ✓ fm(i)

(b) Solve, $\log_{81} y = -\frac{1}{4}$

$\Rightarrow y = (81)^{-\frac{1}{4}}$ $[\because \log_a y = x \Rightarrow a^x = y]$

$= (3^4)^{-\frac{1}{4}}$

$= 3^{-1}$

or $y = \frac{1}{3}$ ✓ $= 0.333$ ✓

(c) Solve, $\frac{32^{x^2-1}}{4^{x^2}} = 16$

$\Rightarrow \frac{(2^5)^{x^2-1}}{(2^2)^{x^2}} = 2^4$

$\Rightarrow \frac{2^{5(x^2-1)}}{2^{2x^2}} = 2^4$

$\Rightarrow 2^{(5x^2-5-2x^2)} = 2^4$

$\Rightarrow 3x^2 - 5 = 4$

$\Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$ ✓

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Logarithmic functions

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Example 5. Solve the equation. $\log_5(10x+5) = 2 + \log_5(x-7)$ ---[4]

[W-17/21/Q3]

Solution: Solve, $\log_5(10x+5) = 2 + \log_5(x-7)$

$\Rightarrow \log_5(10x+5) - \log_5(x-7) = 2$

$\Rightarrow \log_5 \frac{(10x+5)}{(x-7)} = 2$

$\Rightarrow \frac{10x+5}{(x-7)} = 5^2$

$\Rightarrow 10x+5 = 25(x-7)$

$\Rightarrow \underline{x = 12}$ ✓

Example 6. Solve $\log_5\sqrt{x} + \log_{25} x = 3$ ---[3]

[M-16/12/Q3]

Solution: Solve $\log_5\sqrt{x} + \log_{25} x = 3$

$\left[\because \log_{25} x = \frac{\log_5 x}{\log_5 25} = \frac{\log_5 x}{\log_5 5^2} = \frac{1}{2}\log_5 x\right.$ (Chang of base)

or $\log_5 x^{1/2} + \frac{1}{2}\log_5 x = 3$

or $\log_5 x^{1/2} + \log_5 x^{1/2} = 3$

$\Rightarrow \log_5 x^{1/2} \times x^{1/2} = 3$

$= \log_5 x = 3$

$\Rightarrow \underline{x = 5^3 = 125}$ ✓

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Logarithmic functions

Example 7 (i) Find the value of $-\log_p p^2$ ---[1]

(ii) Find $\lg\left(\frac{1}{10^n}\right)$ ---[1]

(iii) show that $\frac{\lg 20 - \lg 4}{\log_5 10} = (\lg y)^2$ where y is a constant to be found. ---[2]

(iv) Solve $\log_r 2x + \log_r 3x = \log_r 600$ ---[2]

[S-16/21/Q3]

Solution (i) $-\log_p p^2 = -2\log_p p$ $\left\{ \because \log_a m^n = n\cdot\log_a m \right.$ and $\log_a a = 1$

$= -2\times 1 = -2$ ✓

(ii) $\lg\left(\frac{1}{10^n}\right) = -\lg 10^n$ $\left\{ \because \log_a \frac{1}{x} = -\log_a x \right.$ and $\lg x = \log_{10} x$

$= -n\log_{10} 10$ $\lg 10 = \log_{10} 10 = 1$

$= -n\times 1 = -n$ ✓

(iii) Consider $\frac{\lg 20 - \lg 4}{\log_5 10} = \frac{\lg \frac{20}{4}}{\frac{1}{\log_{10} 5}}$ $\left\{ \log_a b = \frac{1}{\log_b a} \right.$

$= \lg 5 \times \log_{10} 5$

$= \lg 5 \times \lg 5 = \underline{(\lg 5)^2}$ ✓

(iv) Solve $\log_r 2x + \log_r 3x = \log_r 600$

or $\log_r 2x\times 3x = \log_r 600$ $[\because \log_a m + \log_a n = \log_a mn]$

$\Rightarrow 6x^2 = 600$

$\Rightarrow x^2 = 100$

$x = +10$ or -10 ✗ $\because x > 0$

$\therefore \underline{x = 10}$ ✓

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Logarithmic functions

Example 8: By using substitution $y = \log_3 x$ or otherwise, find the value of x for which,

$$3(\log_3 x)^2 + \log_3 x^5 - \log_3 9 = 0$$ --[6]

|W-16/11/Q3|

Solution: $3(\log_3 x)^2 + \log_3 x^5 - \log_3 9 = 0$

or $3(\log_3 x)^2 + 5\log_3 x - \log_3 3^2 = 0$ [Put $y = \log_3 x$

$3 \cdot y^2 + 5y - 2\log_3 3 = 0$ [$\log_3 3 = 1$

or $3y^2 + 5y - 2 = 0$

or $(3y - 1)(y+2) = 0$

$y = 1/3$ or $y = -2$ [$\because y = \log_3 x$

or $\log_3 x = \frac{1}{3}$ or $\log_3 x = -2$

or $x = 3^{1/3}$ or $x = 3^{-2}$

$x = 1.44$ or $x = 1/9$ ✓

Example 9: Solve the equation $2\lg x - \lg\left(\frac{x+10}{2}\right) = 1$ --[5]

|W-16/21/Q3|

Solution: $2\lg x - \log\left(\frac{x+10}{2}\right) = 1$

or $\lg x^2 - \lg\left(\frac{x+10}{2}\right) = \lg 10$ [$\because \lg 10 = \log_{10} 10 = 1$]

or $\lg\left(\frac{x^2}{\frac{x+10}{2}}\right) = \lg 10$

or $\frac{2x^2}{x+10} = 10 \Rightarrow 2x^2 = 10x + 100$

$\Rightarrow 2x^2 - 10x - 100 = 0$

$\Rightarrow 2(x+5)(x-10) = 0$

$\Rightarrow x = 10$ ✓ or $x = -5$ ✗ ($\because x > 0$)

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Example 10. (i) Given that $\log_9 xy = \frac{5}{2}$ show that $\log_3 x + \log_3 y = 5$ ---[3]

(ii) Hence solve solve the equations $\log_9 xy = \frac{5}{2}$

and $\log_3 x \times \log_3 y = -6$ ---[5]

W-16/13/Q5

Solution: Given. $\log_9 xy = \frac{5}{2}$

$\left[\log_a x = \frac{\log_b x}{\log_b a} \right.$ Chage of base Property

or $\frac{\log_3 xy}{\log_3 9} = \frac{5}{2}$

or $\frac{1}{2} \log_3 xy = \frac{5}{2}$

$\left\{ \because \log_3 9 = \log_3 3^2 = 2\log_3 3 = 2 \times 1 = 2 \right.$

$\Rightarrow \log_3 xy = 5$

$\Rightarrow \log_3 x + \log_3 y = 5$ —— (1)

(ii) To Solve $\log_9 xy = 5/2$ —— (2)

and $\log_3 x \cdot \log_3 y = -6$ —— (3)

fr (2) $\log_9 xy = \frac{5}{2} \Rightarrow \log_3 x + \log_3 y = 5$ frm (1)

or $\log_3 y = (5 - \log_3 x)$ —— (4)

frm (3) & (4)

$\log_3 x \times (5 - \log_3 x) = -6$

$\Rightarrow (\log_3 x)^2 - 5\log_3 x - 6 = 0$

$\Rightarrow z^2 - 5z - 6 = 0$ [Put $\log_3 x = z$ —— (5)

$(z-6)(z+1) = 0$

$z = 6$ or $z = -1$

or $\log_3 x = 6$ or $\log_3 x = -1$ fr (5)

$\Rightarrow x = 3^6$ or $x = 3^{-1}$

$x = 729$ or $x = \frac{1}{3}$

$\begin{cases} x = 729 \\ y = 1/3 \end{cases}$ ✓ $\begin{cases} x = 1/3 \\ y = 729 \end{cases}$ ✓

frm (4)

Logarithmic Functions

Example 11 (a) Write $\log_{27} x$ as a logarith to base 3. --- [2]

(b) Given that,

$\log_a y = 3(\log_a 15 - \log_a 3) + 1$, expression y in terms of a. --- [3]

[S-15/21/Q1]

Solution: (a)

$$\log_{27} x = \frac{\log_3 x}{\log_3 27} \qquad \left(\because \log_a x = \frac{\log_b x}{\log_b a} \text{ Chage of base property} \right)$$

$$= \frac{\log_3 x}{3} = \frac{1}{3}\log_3 x \checkmark$$

(b)

$$\log_a y = 3(\log_a 15 - \log_a 3) + 1$$

$$= 3\left[\log_a \frac{15}{3}\right] + \log_a a \qquad \left[\because \log_a m - \log_a n = \log_a \frac{m}{n} \text{ and } \log_a a = 1 \right]$$

$$= \log_a 5^3 + \log_a a$$

or $\log_a y = \log_a 125a$

$$\therefore \quad y = 125a \checkmark$$

Example 12. Given that $\log_p X = 5$ and $\log_p Y = 2$ find.

[W-13/21/Q4]

(i) $\log_p X^2$ ---- [1]

(ii) $\log_p \frac{1}{X}$ --- [1]

(iii) $\log_{XY} p$ ---- [2]

Solution (i) $\log_p X^2 = 2\log_p X = 2 \times 5 = 10 \checkmark$

(ii) $\log_p \frac{1}{X} = -\log_p X = -5 \checkmark$

(iii) $\log_{XY} p = \frac{1}{\log_p XY}$

$$= \frac{1}{\log_p X + \log_p X}$$

$$= \frac{1}{5+2}$$

$$= \frac{1}{7} \checkmark$$

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Logarithmic function

Example 13 (a)(i) State the value of u for which $\lg u = 0$ --- [1]

(ii) Hence solve $\lg|2x+3| = 0$ --- [2]

(b) Express $2\log_3 15 - (\log_a 5)(\log_3 a)$ where $a > 1$, as a single logarithm to base 3, --- [4]

[S-14/22/Q6]

Solution (a)(i) $\lg u = 0 \Rightarrow \underline{u = 1}$ ✓ $(\because \log_a 1 = 0)$

(ii) $\lg|2x+3| = 0$

$\Rightarrow |2x+3| = 1$ frm (i)

$\Rightarrow 2x+3 = \pm 1$

$\Rightarrow 2x+3 = 1$ or $2x+3 = -1$

$\Rightarrow \underline{x = -1}$ or $\underline{x = -2}$ ✓

(b) $2\log_3 15 - \left(\log_a 5\right)\left(\log_3 a\right)$

$= \log_3 15^2 - \dfrac{\log_3 5}{\cancel{\log_3 a}} \times \cancel{\log_3 a}$

$= \log_3 225 - \log_3 5$

$= \log_3 \dfrac{225}{5}$

$= \underline{\log_3 45}$ ✓

Example 14 (i) Given that $\log_4 x = \frac{1}{2}$, find the value of x. --- [1]

(ii) Solve $2\log_4 y - \log_4(5y-12) = \frac{1}{2}$ --- [4]

[S-13/11/Q4]

Solution (i) $\log_4 x = \frac{1}{2} \Rightarrow x = 4^{1/2} = \sqrt{4} = 2$ ✓

(ii) $2\log_4 y - \log_4(5y-12) = \frac{1}{2}$

$\Rightarrow \log_4 y^2 - \log_4(5y-12) = \log_4 2$ frm (i)

$\Rightarrow \log_4\left(\dfrac{y^2}{(5y-12)}\right) = \log_4 2$

$\Rightarrow \dfrac{y^2}{5y-12} = 2$ ✓

$\Rightarrow y^2 - 10y + 24 = 0 \Rightarrow \underline{y = 4, 6}$

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Example 15. The variable x and y are connected by the equation, $y = 10^{2x} - 2(10^x)$, Using substitution $u = 10^x$ or otherwise, find the exact Value of x when $y = 24$ --[3]

[S-16/22/Q10(b)]

Solution: $y = 10^{2x} - 2(10^x)$

for $y = 24 \Rightarrow 10^{2x} - 2(10^x) = 24$

$$\Rightarrow (10^x)^2 - 2(10^x) - 24 = 0$$

$$\Rightarrow u^2 - 2u - 24 = 0 \qquad \text{Put } 10^x = u$$

$$\Rightarrow (u-6)(u+4) = 0$$

$$\Rightarrow u = 6 \text{ or } u = -4$$

$$\Rightarrow 10^x = 6 \text{ or } 10^x = -4^{\times} \text{ false.}$$

$$\Rightarrow \lg 10^x = \lg 6 \Rightarrow \underline{x = \lg 6.}\checkmark$$

Example 16 Solve the equation, $7^{2x+5} = 2.5$ --[3]

[S-17/23/Q1(a)]

Solution: Given. $7^{2x+5} = 2.5$

$$\Rightarrow \log_7 2.5 = 2x+5$$

$$\Rightarrow 2x+5 = \frac{\lg 2.5}{\lg 7} = \frac{0.3979}{0.8451} = 0.47$$

$$\text{or } 2x = 0.47 - 5 = -4.53$$

$$\therefore x = \frac{-4.53}{2} = \underline{-2.26}\checkmark$$

Example 17. Solve $\frac{32^{x^2-1}}{4^{x^2}} = 16$ [S-17/22/Q7] --- [3]

$$\text{or } \frac{(2^5)^{x^2-1}}{(2^2)^{x^2}} = 2^4$$

$$\text{or } \frac{2^{5(x^2-1)}}{2^{2x^2}} = 2^4$$

$$\text{or } 2^{5(x^2-1) - 2x^2} = 2^4$$

$$\Rightarrow 3x^2 - 5 = 4$$

$$\Rightarrow x^2 = 3$$

$$\Rightarrow x = \pm\sqrt{3} \checkmark$$

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Exponential Equations

Example 18. Solve $6^{x-2} = \frac{1}{4}$ $\boxed{S\text{-}15/22/Q6}$ ---[2]

Solution. $6^{x-2} = \frac{1}{4}$ $\left(\because a^x = y \Rightarrow \log_a y = x\right.$

$\Rightarrow \log_6 \frac{1}{4} = x-2$

or $x-2 = \dfrac{\lg \frac{1}{4}}{\lg 6} = \dfrac{-0.602}{0.7781} = -0.7737$

$\therefore x = 2 - 0.7737 = 1.23$ ✓

$x = 1.23$ (2dp) ✓

Example 19 Solve: $2^{x^2-5x} = \frac{1}{64}$ $\boxed{S\text{-}14/21/Q11(a)}$

Solution. $2^{(x^2-5x)} = \frac{1}{2^6}$

or $2^{x^2-5x} = 2^{-6}$ $\because x^{-n} = \frac{1}{x^n}$

$\therefore x^2 - 5x = -6$

or $x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0$

$\therefore x = 2$ or $x = 3$ ✓

Exponential function to base 'e' and Natural Logarithms:

§ Leonhard Eular (1701-1783) proved $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ (constant) Irrational number

or $\underset{h\to 0}{Lt}\,(1+h)^{1/h} = e = 2.71828\ldots$ (approx).

[Use your calculator to check, $(1+0.000001)^{1000000} = 2.718281\ldots$

§ **e is is called natural base to exponentional function e^x.**

§ Natural Logarithms: $\log_e x$ in short $\ln x$ is the logarithm to the natural base.

Example 20. Solve $5e^{3x+4} = 14$ [M-18/22/Q8a(i)] --[2]

Solution: $5e^{3x+4} = 14$

or $e^{3x+4} = \frac{14}{5}$

$\Rightarrow \log_e\left(\frac{14}{5}\right) = 3x+4$

or $3x+4 = \ln 2.8$ $\left[\because \frac{14}{5} = 2.8\right.$

$3x+4 = 1.0296$

$3x = 1.0296 - 4$

$x = \frac{-2.97}{3} = -0.99$

$\therefore x = -0.99$ ✓

Example 21. Solve the equation $e^{3x} = 6e^x$ [W-16/23/Q2] --[3]

Solution: $e^{3x} = 6e^x$

or $\frac{e^{3x}}{e^x} = 6$

$e^{3x-x} = 6$

$e^{2x} = 6$

$e^{2x} = 6$

or $\log_e 6 = 2x$

or $2x = \ln 6 = 1.791$

or $x = \frac{1.791}{2} = 0.896$ ✓

Logarithmic Function

Example 22: Solve $2\lg y - \lg(5y+60) = 1$ [W-13/13/Q2] --- [5]

Solution:

$2\lg y - \lg(5y+60) = 1$

or $\lg y^2 - \ln(5y+60) = 1$

or $\lg\left(\frac{y^2}{5y+60}\right) = \lg 10$

$\Rightarrow \frac{y^2}{5y+60} = 10$

$\Rightarrow y^2 - 50y - 600 = 0$

$(y-60)(y+10) = 0$

$\Rightarrow \underline{y = 60}$ ✓ or $y = -10$ ✗ (as $y > 0$ for $\lg y$ to be defined)

Example 23 (a) Write $(\log_2 p)(\log_3 2) + \log_3 q$ as a single logarithm to base 3. ---[3]

(b) Given that $(\log_a 5)^2 - 4(\log_a 5) + 3 = 0$, find the possible values of a. ---[3]

[S-18/11/Q6]

Solution (a)

$(\log_2 p)(\log_3 2) + \log_3 q$

$= \frac{\log_3 p}{\cancel{\log_3 2}} \times \cancel{\log_3 2} + \log_3 q$

$= \log_3 p + \log_3 q$

$= \underline{\log_3 pq}$ ✓

(b) $(\log_a 5)^2 - 4(\log_a 5) + 3 = 0$

or $(\log_a 5 - 1)(\log_a 5 - 3) = 0$

or $\log_a 5 = 1$ and $\log_a 5 = 3$

or $a^1 = 5$ or $a^3 = 5$

$a = 5$ or $a = \sqrt[3]{5} = 1.71$

$\therefore a = 5$ or 1.71 ✓

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Example 24. The value, V dollars, of a car aged t years is given by,
$$V = 12000\,e^{-0.2t}$$
(i) Write down the value of the car when it was new. --[1]
(ii) Find the time it takes for the value to decrease to 2/3 of the value when it was new. $\boxed{M-17/22/Q2}$ --[2]

Solution: $V = 12000\,e^{-0.2t}$

(i) for new car $t=0$, $V = 12000\,e^{0} = 12000 \times 1$
$= \$\ 12,000$ ✓

(ii) $\frac{2}{3}$ of New Car Value $= \frac{2}{3} \times 12000 = 8000$ — ①

$\therefore\ V = 12000\,e^{-0.2t} = 8000$ (given in ①)

or $e^{-0.2t} = \frac{8000}{12000}$

or $e^{-0.2t} = \frac{2}{3}$

$\Rightarrow \log_e \frac{2}{3} = -0.2t$

or $-0.2t = \ln(2/3)$ — $[2/3 = 0.666666\ldots$

$t = \frac{-0.40546}{-0.2} = 2.0273$ ✓ years.

Example 25. The number of bacteria, N, present in a culture can be modelled by the equation $N = 7000 + 2000\,e^{-0.05t}$ where t is measured in days. Find.
(i) the number of bacteria when $t=10$ --[1]
(ii) the value of t, when the number of bacteria reaches 7500. --[3]
(iii) the rate at which the number of bacteria is decreasing after 8 days. --[3]

Solution (i) at $t=10$, $N = 7000 + 2000\,e^{-0.05\times 10}$
$= 7000 + 2000 \times e^{-0.5}$
$= 7000 + 2000 \times 0.6065$
$= 8213$ ✓

(ii) $7500 = 7000 + 2000\,e^{-0.05t} \Rightarrow e^{-0.05t} = \frac{500}{2000} = 0.25$
$\Rightarrow \ln 0.25 = -0.05t \Rightarrow t = \ln 0.25 / -0.05$
days $= 27.7$ ✓

(iii) $\frac{dN}{dt} = -100\,e^{-0.05t}$
$t=8$, $\frac{dN}{dt} = -100\,e^{-0.05\times 8}$
$= -100\,e^{-0.4}$
$= -100 \times 0.67$
$= -67$ ✓

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Example 26. The profit $P made by a company in its n_{th} year is modelled by. $P = 1000\,e^{an+b}$

In the first year the company made $2000 profit.

(i) Show that $a+b = \ln 2$ ---[1]

In the second year company made $ 3297 profit.

(ii) Find another linear equation connecting a and b. ---[2]

(iii) Solve the two equations from part (i) and part (ii) to find the value of a & b. ---[2]

(iv) Using your values of a and b, find the profit in the 10_{th} year. ---[2]

W-14/23/Q4

Solution (i) $P = 1000\,e^{an+b}$ —— ①

for $n=1$, $P = 2000 \Rightarrow 2000 = 1000\,e^{a+b}$ fm ①

$\Rightarrow a+b = \ln 2$ —— ②

(ii) for $n=2$, $P = 3297$

fm ① $3297 = 1000\,e^{2a+b}$

$\Rightarrow 2a+b = \ln 3.297$ —— ③

(iii) fm ② $a+b = 0.6931$ —— ④ ($\because \ln 2 = 0.6931$)

fm ③ $2a+b = 1.1930$ —— ⑤ ($\because \ln 3.297 = 1.1930$)

Solving ④ & ⑤ $a = 0.5$ ✓ and $b = 0.193$ ✓

(iv) for 10_{th} year, fm ① $P = 1000\,e^{0.5\times 10 + 0.193}$

$= 1000\,e^{5.193}$

$= 1000 \times 180$

$= \underline{180{,}000}$ ✓

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§ Graph of Exponential Function:

$f(x) = b^x \quad , b \neq 1, \; b > 0$

$x \in R$

Case I When $b > 1$

Example: Let $b = 2$ or $f(x) = 2^x$

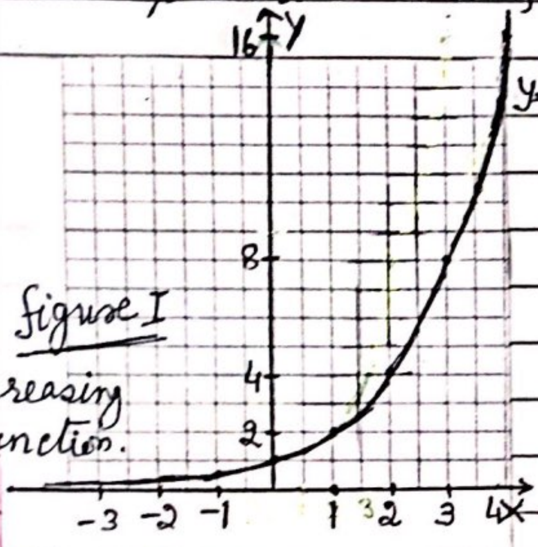


figure I

Increasing function.

$y = 2^x$; $\quad x \to \infty, f(x) \to \infty$

$x = 0, \; f(0) = 1$

$x \to -\infty, f(x) \to 0$

Case II when $0 < b < 1$

Let $b = \frac{1}{2}$, $f(x) = \left(\frac{1}{2}\right)^x$

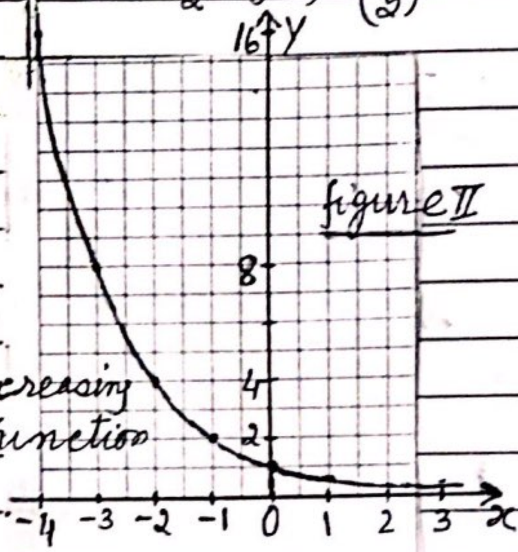


figure II

Decreasing Function

$y = \left(\frac{1}{2}\right)^x$, $\quad x \to \infty, f(x) \to 0$

$x = 0, \; f(0) = 1$

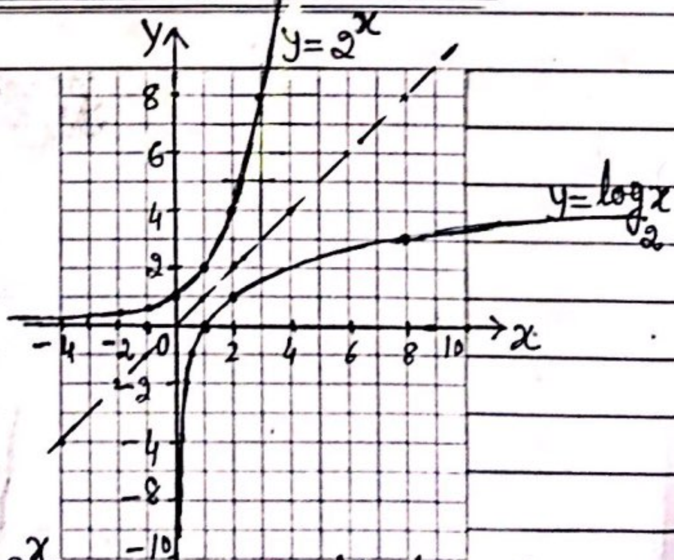
$x \to -\infty, \; f(x) \to \infty$

The graph of $y = \left(\frac{1}{2}\right)^x$ is the reflection of $y = 2^x$ in y-axis.

§ Graphs of $y = 2^x$ and $y = \log_2 x$

x	-3	-2	-1	0	1	2	3	4
$y = 2^x$	0.125	0.25	0.5	1	2	4	8	16

x	$1/8$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
$y = \log_2 x$	-3	-2	-1	0	1	2	3	4



The graphs of $y = \log_2 x$ and $y = 2^x$ are the reflection of each other in the line $y = x$

$\therefore$ are inverse functions.

Graph of Exponential Functions

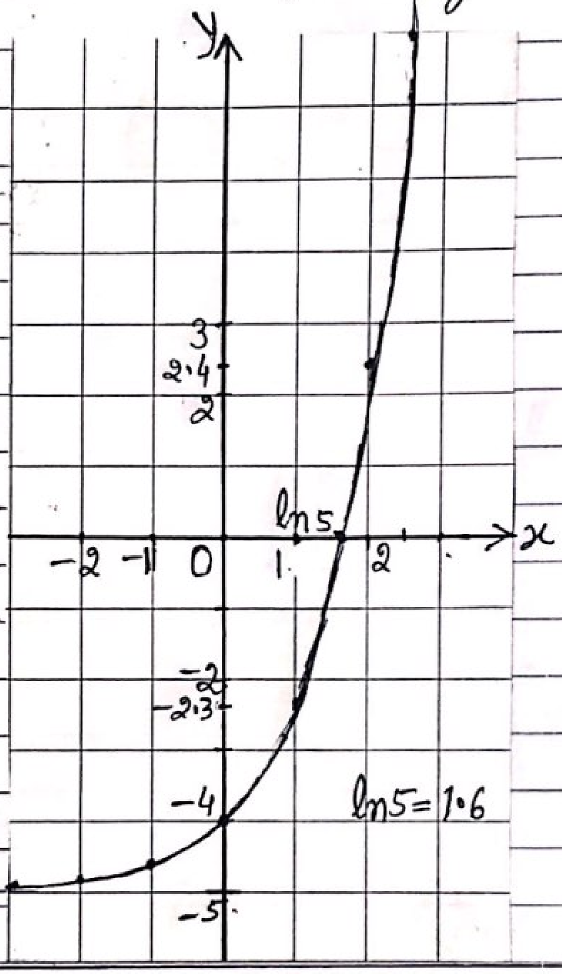
$\boxed{y = ke^{nx} + a}$

Example 27. Sketch the graph of $y = e^{x} - 5$ on the axes, showing the

(i) exact coordinates of any points, where the graph cuts the coordinate axes. ---[3]

(ii) Find the range of value of k for which $e^{x} - 5 = k$ has no solution.

$\boxed{SP\text{-}20/02/Q10(a)}$ ---[1]



Solution: (i) $y = e^{x} - 5$

x	-3	-2	-1	0	1	$\ln 5 = 1.6$	2	3
y	-4.9	-4.86	-4.6	-4	-2.3	0	2.4	15

$x \to -\infty$

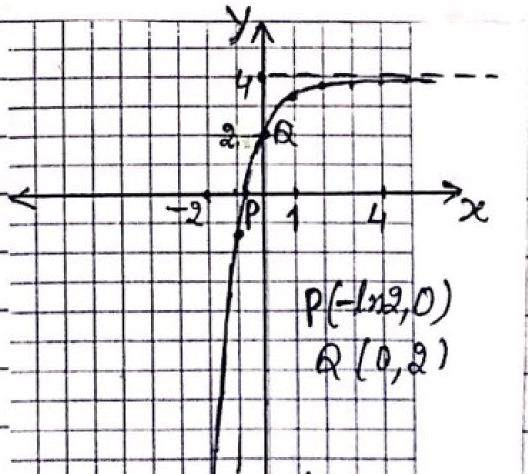
$y \to -5^{+}$ $(-4.999\ldots)$

(ii) $y = e^{x} - 5 = k$

has no solution when

$k \leq -5$ ✓

Example 28: Sketch the graph of $y = -2e^{-x} + 4$ on the axis, showing exact coordinates of any points where the graph cuts the coordinate axes.



Solution: $y = -2e^{-x} + 4$

x	—	-2	-1	$-\ln 2$ $-.69$	0	1	2	3	4	5
y	—	-10.7	-1.4	0	2	3.26	3.7	3.9	3.96	3.99

cuts x-axis at $(-\ln 2, 0)$

cut y-axis at $(0, 2)$

$x \to \infty$, $y \to 4$ asymptote ✓

$x \to -\infty$, $y \to -\infty$

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Graph of Logarithmic function. $y = k\ln(ax+b)$

Example: Sketch the graph of $y = 4\ln(2x-4)$ on the axes, showing the exact coordinates of any point, where the graph cuts the coordinate axes.

Solution: $y = 4\ln(2x-4)$ —— ①

for x-Int, $y=0 \Rightarrow 4\ln(2x-4)=0$

$\Rightarrow 2x-4=1 \quad (\because \ln 1 = 0)$

$\Rightarrow x = 2.5$ ✓

$\therefore$ graph intersects x-axis at $(2.5, 0)$

for Y-Int, put $x=0$ in ① $y = 4\ln(0-4)$ ✗

but $\ln(-4)$ is not defined.

$\therefore$ Curve does not intersects Y-axis ✓

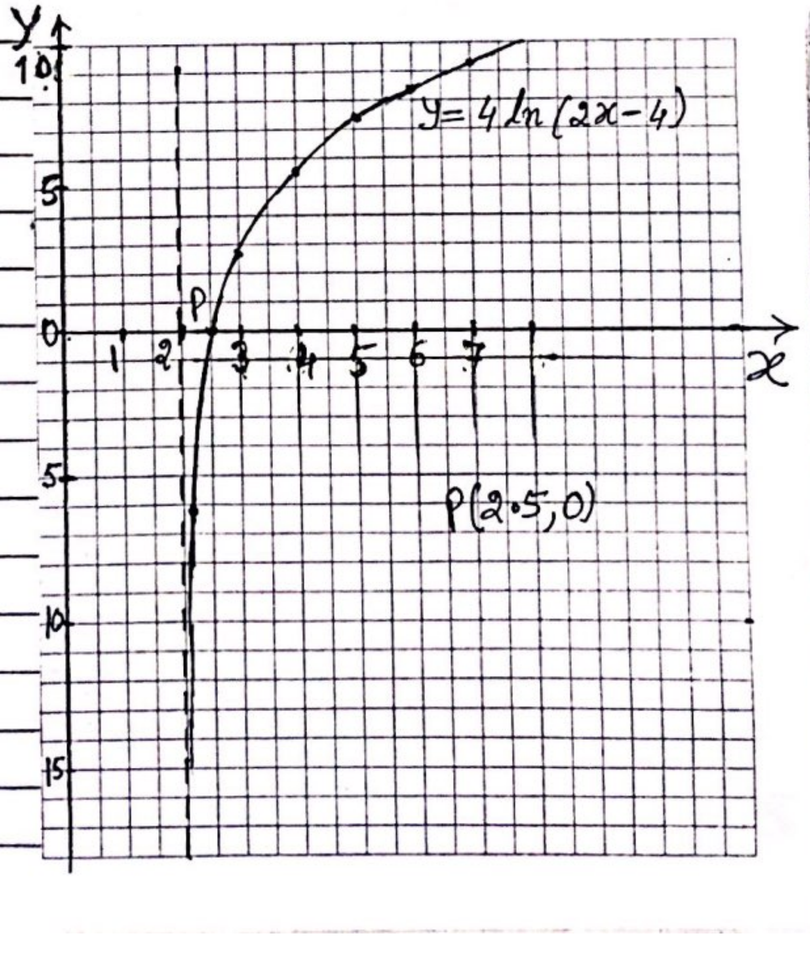
Now $\ln(2x-4)$ is def when $2x-4>0$

$x>2$

$x\to\infty,\ y\to\infty$ and $x\to 2^+,\ y=-\infty$

x	2.01	2.1	2.5	3	4	5	6	7
y	−15.6	−6.4	0	2.7	5.5	7.2	8.3	9.2

$x=2$ line is the asymptote to curve.



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Graph of Logarithmic Function. $y = k\ln(ax+b)$

Example 29. Sketch the graph of $y = 3\ln(2x+2)$ on the axes, showing exact coordinates of any points, where the graph cuts the coordinate axes.

Solution: $y = 3\ln(2x+2)$ —— ①

for x-intercept, $y=0 \Rightarrow 2x+2 = 1$

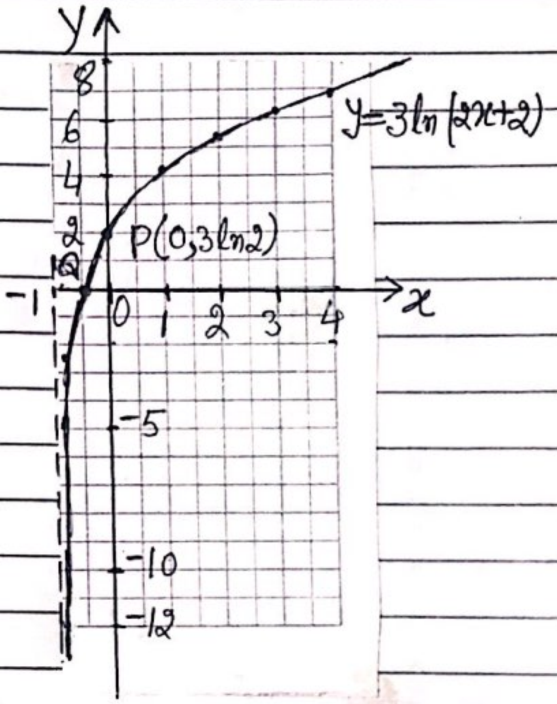
$\Rightarrow x = -0.5$

$\therefore$ Point $(-0.5, 0)$ ✓P

graph cuts y-axis when $x = 0$

$\Rightarrow y = 3\ln 2$ or 2.07

$\therefore$ Point $(0, 3\ln 2)$ ✓Q



For $\ln(2x+2)$ be meaningful

$(2x+2) > 0$

or $x > -1$

$x \to -1,\ y \to -\infty$ } Asymptote is $x=-1$

$x \to \infty,\ y \to \infty$

x	$-.99$	$-.9$	-0.7	-0.5	0	1	2	3	4
y	-11.7	-4.8	-2.5	0	2	4.2	5.4	6.2	6.9

§ To find Inverse of Exponential fn"

Example 30. Given $g(x) = 4e^x - 2$ for $x \in \mathbb{R}$ ---[2]

find $g^{-1}(x)$ | W-13/11/Q12(b)(i) |

Solution: $g(x) = 4e^x - 2$

Let $y = 4e^x - 2$

Interchange x and y

$x = 4e^y - 2$

or $e^y = \dfrac{x+2}{4}$

$\Rightarrow$

$y = \ln\left(\dfrac{x+2}{4}\right)$

$\therefore$ Required. $g^{-1}(x) = \ln\left(\dfrac{x+2}{4}\right)$ ✓

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IG-Maths 0606

Logarithmic Functions

Laws of Logariths:

1. $\log_a mn = \log_a m + \log_a n$

Proof: Let $\log_a m = x$ & $\log_a n = y$ — ①

$\Rightarrow m = a^x$ and $n = a^y$

$\therefore mn = a^x \cdot a^y$

or $mn = a^{x+y}$

$\Rightarrow \log_a mn = x+y$

$= \log_a m + \log_a n$ fr ①

2. $\log_a \frac{m}{n} = \log_a m - \log_a n$

Proof: $\frac{m}{n} = \frac{a^x}{a^y}$ fr ①

or $\frac{m}{n} = a^{x-y}$

$\Rightarrow \log_a \frac{m}{n} = x - y$

$= \log_a m - \log_a n$ fr ①

3. $\log_a m^n = n \log_a m$

Proof: $m^n = (a^x)^n$ fr ①

or $m^n = a^{n \cdot x}$

or $\log_a m^n = n \cdot x$

$= n \cdot \log_a m$ fr ①

8. Chage of base Property:

$$\log_a m = \frac{\log_c m}{\log_c a}$$

Proof:

Let $\log_a m = x$ — ②

$\Rightarrow a^x = m$

$\Rightarrow \log_c a^x = \log_c m$; $c > 0$ New base

$\Rightarrow x \log_c a = \log_c m$

$$x = \frac{\log_c m}{\log_c a}$$

or fr ①

$$\log_a m = \frac{\log_c m}{\log_c a} \quad — ③$$

9. $\log_c a = \frac{1}{\log_a c}$

Proof: Let $m = c$ in ③

$$\log_a c = \frac{\log_c c}{\log_c a}$$

$$\log_a c = \frac{1}{\log_c a} \quad [\because \log_c c = 1$$

or $\log_c a = \frac{1}{\log_a c}$