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IG_0606 Additional Maths

Permutations and Combinations Notes

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Permutations and Combinations

§ Factorial Notation:

The product of first 6 natural numbers:

$$6 \times 5 \times 4 \times 3 \times 2 \times 1$$

can be represented in short by symbol $6!$

$6!$ read as six factorial. or $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

Similarly $3! = 3 \times 2 \times 1 = 6$

In General the product of n natural number is denoted by:

$$\boxed{n! = n(n-1)(n-2)\cdots 3.2.1.}$$

Note: (i) $1! = 1$

(ii) we define $0! = 1$ for use in our later results.

Example: (1) (i) $\frac{5!}{3!} = \frac{5 \times 4 \times \cancel{3 \times 2 \times 1}}{\cancel{3 \times 2 \times 1}} = 5 \times 4 = 20$

or $\frac{5!}{3!} = \frac{5 \times 4 \times \cancel{3!}}{\cancel{3!}} = 5 \times 4 = 20$

(ii) $\frac{8!}{5!} = \frac{8 \times 7 \times 6 \times 5!}{5!} = 8 \times 7 \times 6.$

Example 2 (i) $7 \times 6 \times 5! = 7!$

(ii) $47 \times 46! = 47!$

(iii) $n! = n(n-1).(n-2)!$

(iv) $\frac{n!}{(n-2)!} = \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = n(n-1).$

§ Permutions (or Arrangements):

How many different two letter words (with or without meaning) can be made from the 3 letters of the word 'CAB'. ? (Repetition of letters is not allowed)

AB , AC
BA , BC
CA , CB

∴ Number 2 letter words = 6.

Alternatively we understand that at the first place any of the 3 letters can be taken and then at the second place any one of the remaining two letters.

i.e. $$\underset{\text{Ist Place}}{\underline{3}} \times \underset{\text{IInd Place}}{\underline{2}} = 6$$

Mathematically we say that the permutions of of 3 different letters taken 2 at a time, and is denoted by $^3P_2 = 6$

§ In General:

(i) The permutations of n different objects taken r at a time, denoted by:

$$^nP_r = n(n-1)(n-2) \text{-----} (n-(r-1))$$
$$= \frac{n!}{(n-r)!} \quad : \quad r \leq n$$

(ii) The permutations (arrangements) of n different objects taken all of them at a time is:

$$^nP_n = n!$$

(iii) $^nP_1 = n$ ✓ (iv) $^nP_2 = n(n-1)$ ✓

Example 3. The letters of the word THURSDAY are arranged in a straight line. Find the number of different arrangements of these letters if,

(i) there are no restrictions ---[1]

(ii) the arrangement must start with letter T and end with letter Y. ---[1]

(iii) the second letter in the arrangement must be Y. ---[1]

M-17/12/Q6(a)

Solution: In the word THURSDAY, No of letters = 8

(i) The number of arrangement of 8 letters in any way $= {}^8P_8 = 8!$
$= 40320$ ✓

(ii) Starts with 'T' and ends with letter 'Y' → T _ _ _ _ _ _ Y (1 2 3 4 5 6)

∴ Remaining '6' letters to be arranged $= {}^6P_6 = 6! = 720$ ✓

(iii) Second letter is Y. _ Y _ _ _ _ _ _

∴ Remaining 7 letters are to be arranged $= {}^7P_7 = 7! = 5040$ ✓

Example 4. A 6-digit number is to be formed using the digit 1, 3, 5, 6, 8, 9. Each of these digits may be used only once in any 6-digit number. Find how many different 6-digit numbers can be formed if:

(i) there are no restrictions. [1]

(ii) the number formed is even, [1]

(iii) the number formed is even and greater than 300 000. [3]

W-17/13/Q9(a)

Solution (i) 6 digit numbers (no restrictions) $= {}^6P_6 = 6! = 720$ ✓

(ii) even number * * * * * 2 ways (unit place even digits 6 or 8) (one of them)

and rest of the 5 digits,

Can be arranged in ${}^5P_5 = 5!$ ways.

∴ Total number even numbers $= 5! \times 2$
$= 120 \times 2 = 240$ ✓

(iii) even and greater than 300 000.

1* 1 cannot be → $4 \times$ _ _ $4!$ _ _ $\times 2$ (6 or 8) one of them for even

(iii) continued
$= 4 \times 4! \times 2$
$= 4 \times 24 \times 2 = 192$ ✓

Permutations

Example 5. A 6-character password is to be chosen from the following 9 characters:

letters: A B E F

numbers: 5 8 9

Symbols: * $

Each character may be used only once in any password. Find the number of different 6-character passwords that may be chosen if.

(i) the are no restrictions. ---[1]

(ii) the password consist of 2 letters, 2 numbers and 2 symbols in that order. --[2]

(iii) the password must start and finish with a symbol. ---[2]

[SP-20/02/Q5(a)]

Solution (i) Password with 6 characters out of 9 with no restrictions.

$$= {}^9P_6 = \frac{9!}{(9-6)!} = \frac{9!}{3!} \qquad \left[\because {}^nP_r = \frac{n!}{(n-r)!}\right.$$

$$= 9\times8\times7\times6\times5\times4$$

$$= \underline{60480} \checkmark$$

(ii) with 2 letters (out of 4), 2 numbers (out of 3) and 2 symbols (out of 2), $= {}^4P_2 \times {}^3P_2 \times {}^2P_2$

$$= (4\times3)\times(3\times2)\times(2\times1)$$

$$= \underline{144} \checkmark$$

(iii) Starts and finit with symbols (2 out of 2) = ${}^2P_2$

and the rest of the 4 characters (out of rest 7) = ${}^7P_4$

$\therefore$ Total number of passwords $= {}^2P_2 \times {}^7P_4$

$$= 2\times1 \times 7\times6\times5\times4$$

$$= \underline{1680} \checkmark$$

Scanned with CamScanner

Permutations

Examples 6. Four parts in a play are to be given to four of the girls chosen from the seven girls in a drama class. Find the number of different ways in which this can be done. [S-18/22/Q5(a)] --[2]

Solution: Arrangement 7 taken 4 at a time $= {}^7P_4 = 7\times6\times5\times4$
$= \underline{840}$ ✓

Example 7. A group of five people consists of two women, Alice and Betty and three men Carl, David and Ed.
(i) Three of these five people are chosen at random to be a chairperson, a treasurer and a secretary. Find the number of ways in which this can be done if the chairperson and treasurer are both men. ---[2]
These five people sit in a row of five chairs. Find the number of different possible seating arrangements if,
(ii) David must sit in the middle. ---[1]
(iii) Alice and Carl must sit together. [M-18/22/Q3] ---[2]

Solution (i) 2 men out of three and third may be any one out of remain 3.
$= {}^3P_2 \times {}^3P_1 = (3\times2)\times3 = \underline{18}$ ✓

(ii) — — $\underset{\text{is fixed.}}{\text{David}}$ — — and remaining 4 out of 4 in any way $= {}^4P_4 = 4! = \underline{24}$ ✓

(iii) [Alice & Carl], and remain 3 can be arraged $= {}^4P_4 = 4!$
but Alice & Carl can be rearranged $= {}^2P_2 = 2!$

$\therefore$ Total no of arrangements $= 4! \times 2!$
$= 24\times2$
$= \underline{48}$ ✓

Example 8 (i) Find how many different numbers can be formed using 4 of the digits 1, 2, 3, 4, 5, 6 and 7 if no digit is repeated. --- [1]

Find how many of these 4-digit numbers are..

(ii) odd --- [1]

(iii) odd and less than 3000. [W-13/23/Q2] --- [3]

Solution (i) Given '7' different digits,

∴ 4 digit number (out of 7 digits) if no digit is repeated $= {}^7P_4 = 7\times6\times5\times4$

$= \underline{840}$ ✓

(ii) for number to be odd at unit place one of (1, 3, 5, 7)

$\_\ \_\ \_\ \underline{4}$ (one of 1, 3, 5, 7)

and at the remaining 3 place, any 3 out 6 remaining $= {}^6P_3$

∴ Total odd numbers $= {}^6P_3 \times 4$

$= (6\times5\times4)\times4$

$= \underline{480}$ ✓

(iii) for a number less than 3000 at thousand place we can have 1 or 2,

Case I 1 at thousand place. $\underset{Th}{\underline{1}} \times \underset{H}{\underline{\ \ }}\ \underset{T}{\underline{\ \ }} \times \underset{Unit}{\underline{3}}$ (Remain odd digits 3, 5, 7)

∴ at Remaining 2 Place (Hundred & Ten) 2 digit out of $5 = {}^5P_2$

∴ Total ways in Case I $= 1\times {}^5P_2 \times 3 = \underline{60}$ ✓

Case II. digit 2 at thousand's place

digit 2 → $\underset{Th}{\underline{1}} \times \underset{H}{\underline{\ \ }}\ \underset{T}{\underline{\ \ }} \times \underset{Units}{\underline{4}}$ (1, 3, 5, 7 1 of the 4 odd digits)

for $H \& T = {}^5P_2$

∴ Case II - Numbers $= 1\times {}^5P_2 \times 4 = \underline{80}$ ✓

∴ Total numbers Odd less than 3000 $= 60 + 80 = \underline{140}$ ✓

Permutations

Example 9. How many 5 digit numbers are there that have 5 different digits and are divisible by 5. [S-17/23/Q5(a)] -- [3]

Solutions: There ten different digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9,
for a 5 digit number divisible by:
In the unit place it should have 0 or 5

Case I

0 in the unit place: _ _ _ _ $\times 1$ (unit - 0)
rest of the four 4 places by any of the 9 rest of the digit

$$= {}^9P_4 \times 1$$

$$= 9 \times 8 \times 7 \times 6 \times 1 = 3024 \quad \text{--- (1)}$$

Case II 5 in the unit place:

_ _ _ _ $\times 1$ (unit place - 5)

0 cannot be here (Extreme left Place)

$$(8 \times 8 \times 7 \times 6) \times 1 = 2688 \quad \text{--- (2)}$$

0 maybe here

$\therefore$ Total 5 digit numbers divisible by 5 = 3024 + 2688 ((1)+(2))
= 5712

Example 10. A man and two women are to sit in a row of five empty chairs. Calculate the number of ways, they can be seated of,
(i) the two women must sit next to each other. --- [2]
(ii) all three people must sit next to each other. [S-16/21/Q10(b)] --- [2]

Solution: Let man = 'm'
& 2 women = W_1 & W_2

(i)
$\boxed{W_1 \& W_2}$ 1 2 3 4 5

$\boxed{W_1 \& W_2}$ may be seated = 4 Ways
(1,2), (2,3), (3,4), (4,5)
and the man in remaining = 3 ways
but W_1 & W_2 may interchange = 2!
$\therefore$ Total arrangements = $4 \times 2! \times 3 = 24$

(ii)
$\boxed{1, 2, 3}$ 4, 5
$\boxed{W_1, W_2, M}$ _ _

$\boxed{W_1\ W_2\ m}$ may be seated in 3 positions
$\boxed{1,2,3}$ $\boxed{2,3,4}$ or $\boxed{3,4,5}$
but they can mutually be rearranged in 3! ways.
$\therefore$ Total arrangements = $3 \times 3!$
= 18

§ Combinations (or Selections):

A combination is a selection of some (or all) out of a number of different object.
In a combination, the order of selection of the objects is immaterial.

Example: Given four object A, B, C and D.
How many ways we can select 3 object at a time:

ABC, ABD, ACD, BCD. i,e 4 ways.
It is denoted by $^4C_3 = 4$ ✓ [combinations of 4 diff obj. taken 3 at a time]

But we know the permutation $^4P_3 = \frac{4!}{1!} = 4\times3\times2 = 24$ ✓

Let us see how do we relate the permutations and combinations

Combinations	Permutions $^3P_3 = 3! = 6$
ABC	ABC, ACB; BAC, BCA; CAB, CBA
ABD	ABD, ADB; BAD, BDA; DAB, DBA
ACD	ACD, ADC; CAD, CDA; DAC, DCA
BCD	BCD, BDC; CBD, CDB; DBC, DCB
$^4C_3 = 4$	$^4P_3 = 24$.

Now $^4C_3 \times 3! = {^4P_3}$

or $^4C_3 = \frac{^4P_3}{3!}$

In General. $^nC_r \times r! = {^nP_r} \Rightarrow {^nC_r} = \frac{^nP_r}{r!}$

or $\boxed{^nC_r = \frac{n!}{r!(n-r)!} \quad r \leq n}$

Notes

§ Combination (Selections):

(i) The combination of 'n' distinct objects, taken 'r' at a time, denoted by ^nC_r, is given by:

$$\boxed{^nC_r = \frac{n!}{r!(n-r)!}} \quad \text{for } 0 \leq r \leq n$$

(ii) The combination of 'n' distinct objects taken all of them at a time:

$$^nC_n = 1$$

$$\left[^nC_n = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \cdot 0!} = 1 \right.$$

Example: (i) $^nC_0 = 1$

(ii) $^nC_1 = n$ $\quad \left[\because {^nC_1} = \frac{n!}{1!(n-1)!}\right]$

(iii) $^nC_2 = \frac{n(n-1)}{2!}$ $\quad \left[\because {^nC_2} = \frac{n!}{2!(n-2)!}\right]$

(iv) $^{10}C_3 = \frac{10!}{3!(10-3)!} = \frac{10!}{3!7!} = \frac{10\times9\times8}{3\times2\times1}$

(v) $^{29}C_2 = \frac{29\times28}{2\times1}$ $\left[\because \frac{29!}{2!27!} = \frac{29\times28\times\cancel{27!}}{2!\cancel{27!}}\right.$

(vi) $^{10}C_7 = \frac{10!}{7!3!} = \frac{10\times9\times8\times\cancel{7!}}{\cancel{7!}\times3!} = \frac{10\times9\times8}{3\times2\times1}$

from (iv) and (vi) we note

$$\boxed{^{10}C_3 = {^{10}C_7}}$$

§ In General

$$\boxed{^nC_r = {^nC_{n-r}}}$$

Example 11. A examination consists of a section 'A' containing 10 short questions, and a section B containing 5 long questions. Candidates are required to answer 6 questions from from section 'A' and 3 questions from section B.
Find the number of different selections of questions that can be made if:
(i) there are no restrictions, ---[2]
(ii) Candidate must answer the first 2 questions in sections A, and the first question in section B. [SP-20/02/Q5(b)] --[2]

Solution. Section A – 10 questions ; Section B – 5 question
6 questions to be answered from A & 3 from Section B.

(i) $^{10}C_6 \times {^5C_3} = \frac{10!}{6!\,4!} \times \frac{5!}{3! \times 2!} = \underline{2100}$ ✓
no restrictions.

(ii) First 2 q in section A → 4 more from remaining 8 q in A $= {^8C_4}$
and First 1 q in section B → 2 more from remaining 4 q in B $= {^4C_2}$
∴ Total number of selections $= {^8C_4} \times {^4C_2}$

$$= \frac{8!}{4!\,4!} \times \frac{4!}{2! \times 2!} = 420 ✓$$

Example 12. A team of 3 people is to be selected from 7 women and 6 men. Find the number of different teams that could be selected, if there must be more women than men on the team. [S-16/22/Q 3] ----[3]

Solution

women 7 & men 6

Team of 3 to be selected,
with more women – Case I – 2W & 1m $= {^7C_2} \times {^6C_1} = 126$
Case II – 3W & 0m $= {^7C_3} \times {^6C_0} = 35$

∴ Total $= 126 + 35 = \underline{161}$ ✓

Combinations

Example 13: Three singers are chosen at random from a group of 5 Chinese, 4 Indian and 2 British singers. Find the number of different ways in which this can be done if.

(i) no Chinese singer is chosen. --[1]

(ii) One singer of each nationality is chosen. --[2]

(iii) the three singers chosen are all of the same nationality. --[2]

[S-18/22/Q5(b)]

Solution: Chinese 5, Indian 4, British 2

To be chosen total 3 singers,

(i) No Chinese, So from the remaining (4 Indian & 2 British) that is from 6, → 3 to be chosen $= {}^6C_3 = \frac{6!}{3!3!} = \frac{6\times5\times4}{3\times2\times1} = \underline{20}$ ✓

(ii) One singer of each nationality → One Chinese & one Indian & one British

$$= {}^5C_1 \times {}^4C_1 \times {}^2C_1$$

$$= 5\times4\times2 \qquad [\because {}^nC_1 = n]$$

$$= \underline{40} ✓$$

(iii) all three of the same nationality, so

3 Chinese or 3 Indian (but not British as there are only 2 British)
(out of 5) or (out of 4)

$$= {}^5C_3 + {}^4C_3$$

$$= \frac{5!}{3!\times2!} + \frac{4!}{3!\times1!}$$

$$= 10 + 4 = \underline{14} ✓$$

Example 14. 7 children have to be divided into two groups, one of 4 children and the other 3 children. Given that there are 3 girls and 4 boys, find the number of different ways this can be done if,

(i) there are no restrictions. ---[1]

(ii) all the boys are in one group. ---[1]

(iii) One boy and one girl are twins and must be in the same group. [M-17/12/Q6(b)] ---[3]

Solution: Boys – 4 and Girls – 3. Total 7 children.

To be divided into two groups of 4 & 3.

(i) there are no restrictions

In first group choose any 4 out of total $7 = {}^7C_4$

and in the second group 3 out of remaining $3 = {}^3C_3$

$\therefore$ Total number of groups $= {}^7C_4 \times {}^3C_3$ $[{}^nC_n = 1]$

$= \frac{7!}{4!3!} \times 1 = \underline{35}$ ✓

(ii) All boys in one group, 4 boys in a group of $4 = {}^4C_4$

and 3 girls in a group of $3 = {}^3C_3$

$\therefore$ Total $= {}^4C_4 \times {}^3C_3 = 1 \times 1 = \underline{1}$ ✓

(iii) Case I twins in group of 4 & 2 more from $5 = {}^5C_2 = 10$ ✓

Case II twins in group of 3 & 1 more from $5 = {}^5C_1 = 5$ ✓

$\therefore$ Total number of groups $= 10 + 5$

$= \underline{15}$ ✓

Example 15. A 5 digit number is be to formed from the seven digits

(a) 1, 2, 3, 5, 6, 8 and 9. Each digit can only be used only once in any 5-digit number. Find the number of different 5-digit numbers that can be formed if:

(i) There are no restrictions. [S-17/12/Q8] ---[1]

(ii) The number is divisible by 5. ---[1]

(iii) The number is greater than 60,000. ---[1]

(iv) The number is greater than 60,000 and even. ---[3]

(b) Ranjit has 25 friends of whom 15 are boys and 10 are girls. Ranjit wishes to hold a birthday party, but can invite only 7 friends. Find the number of different ways these 7 friends can be selected if,

(i) There are no restrictions. ---[1]

(ii) Only 2 of the 7 friends are boys. ---[1]

(iii) the 25 friends include a boy and his sister who can be separated. ---[3]

Solution: Given 7 digits 1, 2, 3, 5, 6, 8 and 9. 5 digit number is to formed.

(i) If No restrictions $^7P_5 = 2520$ ✓

(ii) Number divisible by '5' will have digit 5 in Unit place:

$\_ \ \_ \ \_ \ \_ \times \underset{\text{unit}}{1}(5)$

∴ Req. numbers $= {}^6P_4 \times 1 = 360$ ✓

(iii) The number greater than 60,000 will start with (6, 8 or 9) ∴ $3 \times {}^6P_4 = 3 \times 360 = 1080$

(iv) Greater than 60,000, should start with 6, 8, 9 and for even at Unit (2, 6, 8)

Case I Starts with 6 or 8 & even

$\underset{(6 \text{ or } 8)}{2} \times \underbrace{\_ \times \_ \times \_}_{^5P_3} \times \underset{\text{unit } (2,6,8)}{2}$

$= 2 \times (5 \times 4 \times 3) \times 2 = 240$ ✓

Case II Starts 9 → $1 \times \underbrace{(\_ \times \_ \times \_)}_{^5P_3} \times \underset{(2, 6 \text{ or } 8)}{3}$

$= 1 \times (5 \times 4 \times 3) \times 3 = 180$ ✓

∴ Total numbers $= 240 + 180 = 420$ ✓

(b) Boys & Girls — Total

15 & 10 = 25

To be invited = 7

(i) No restrictions $^{25}C_7 = 480700$ ✓

(ii) Only 2 boys of 7 ⇒ 5 Girls

∴ $^{15}C_2 \times {}^{10}C_5 = 26460$ ✓

(iii)

Case I when Brother & Sister are invited

$= {}^{23}C_5 = 33649$ — (1)

Case II without Brother & Sister.

$= {}^{23}C_7 = 245157$ — (2)

∴ Total number of way to invite from (1) & (2)

$= 33649 + 245157$

$= 278806$ ✓

Example16 (a) A team of 5 students is to be chosen from a class of 10 boys and 8 girls. Find the number of different teams that may be chosen if,

(i) there are no restrictions. ---[1]

(ii) the team must contain atleast one boy and one girl. ---[4]

(b) A computer password, which must contain 6 characters, is to be chosen from the following 10 characters:

Symbols: ? ! #

Numbers: 3 5 7

Letters: W X Y Z.

Each character may be used once only in any password.
Find the number of possible passwords that may be chosen if,

(i) there are no restrictions. ---[1]

(ii) Each password must start with a letter and finish with a number --[2]

(iii) Each password must contain atleast one symbol. W-16/13/Q9 . -[3]

Solution (a) 10 boys & 8 girls → Total 18

(i) Team of 5 is to be chosen.

No Restriction → $^{18}C_5 = 8568.$

(ii) at least one boy & one girl.

Boy8 & Girls8

Team of 5, Case I → $4B \& 1G = {}^{10}C_4 \times {}^8C_1 = 1680$

Case II → $3B \& 2G = {}^{10}C_3 \times {}^8C_2 = 3360$

Case III $2B \& 3G = {}^{10}C_2 \times {}^8C_3 = 2520$

Case IV $1B \& 4G = {}^{10}C_1 \times {}^8C_4 = 700.$

∴ Total number of teams:

$= 1680 + 3360 + 2520 + 700$

$= 8260$ ✓

(b) Symbols, Numbers, Letters,
3, 3, 4

Password must contain 6 characters

(i) No restrictions → $^{10}P_6 = 151200$ ✓

(ii) Letter × × × × Number

$= 4 \times (\quad {}^8P_4 \quad) \times 3$

$= 4 \times {}^8P_4 \times 3 = 20160$ ✓

(iii) (No Restriction) − (No Symbol)
Part (i)

$= 151200 - {}^7P_6$

$= 151200 - 5040$

$= 146160$ ✓

Example 17 (a) A lock can be opened using only the number 4351. State whether this is a permutation or a combination of digits, giving a reason for the answer. ---[1]

(b) There are twenty numbered balls in a bag. Two of the balls are numbered 0, six are numbered 1, five are numbered 2 and seven are numbered 3, as shown in the table below.

Number on balls	0	1	2	3
Frequency	2	6	5	7

Four of these balls are chosen at random, without replacement. Calculate the number of ways this can be done so that,

(i) the four balls all have the same number. ---[2]

(ii) the four balls all have different numbers. ---[3]

(iii) the four balls have numbers that total 3. S-15/21/Q5 ---[3]

Solution:

(a) Permutation, because the order matters.

(b) (i) the four balls have the same number–
may 1 or 2 or 3 (but not 0 as only 2 balls nubered 2)

$= {}^6C_4 + {}^5C_4 + {}^7C_4 = 15+5+35 = \underline{55}$ ✓

(ii) the four balls all have different numbers (0, 1, 2, 3)

$= {}^2C_1 \times {}^6C_1 \times {}^5C_1 \times {}^7C_1$

$= 2\times6\times5\times7 = \underline{420}$ ✓

(iii) The four balls have numbers that total 3.

Case I $(0,0,1,2) = {}^2C_2 \times {}^6C_1 \times {}^5C_1 = 1\times6\times5 = 30$ — ①

Case II $(1,1,1,0) = {}^6C_3 \times {}^2C_1 = 20\times2 = 40$ — ②

$\therefore$ Total number of ways $= 30+40$ (from ① & ②)

$= \underline{70}$ ✓