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M-2

Mechanics

Circular Motion Notes

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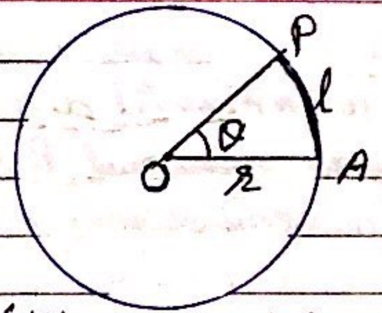
Circular Motion

Notes. P-1

Length of arc:

- Given a circle with centre at O and radius r and the arc $\widehat{AP}$ of the circle of length 'l' subtends an angle 'θ' radians at the centre O, then.



Length of arc $\quad l = r\theta \quad$ ('θ' is measured in radians.) (π rad $= 180°$)

- **Angular Velocity:**

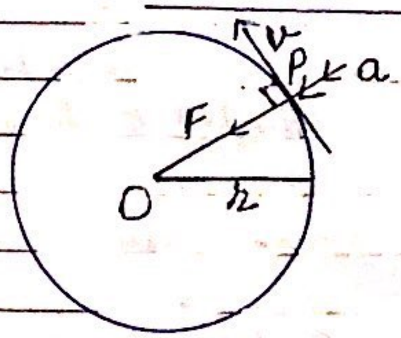
If a particle 'P' is moving round the circle with speed 'v' and the angle 'θ' is changing with speed, ω, called angular speed of P, then

$$v = r\omega$$

$\left\{ \begin{array}{l} r \text{ is constant} \\ \omega = \dfrac{d\theta}{dt} \\ V = \dfrac{dl}{dt} \end{array} \right.$

- **Accelation towards centre:**

A resultant centripetal force 'F' is required for a particle to move in circular motion:
By Newton's second law of motion.



$$F = ma = m \cdot r\omega^2 \quad \text{—(i)}$$

$$\Rightarrow \quad a = r\omega^2 \quad \text{—(ii)}$$

$$\text{or} \quad a = r \cdot \left(\frac{V}{r}\right)^2$$

$\left(\begin{array}{l} V = \omega \cdot r \\ \Rightarrow \omega = \dfrac{V}{r} \end{array} \right.$

$$\Rightarrow \quad a = \frac{V^2}{r} \quad \text{—(iii)}$$

Example 1. A particle P of mass 0·6 kg is on the rough surface of a horizontal disc with centre O. The distance OP is 0·4 m. The disc and P rotate with angular speed 3 rad s^{-1} about a vertical axis which passes through O. Find the magnitude of the frictional force which the disc exerts on the particle, and state the direction of this force. [S-15/52/Q1] ---[3]

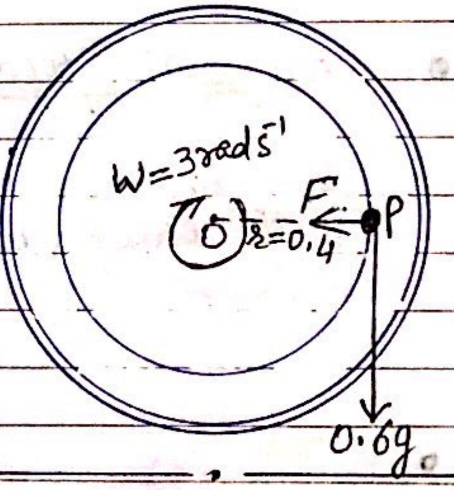
Solution: Force of friction $F = ma$

$= m \cdot w^2 \cdot r$

$= 0{\cdot}6 \times 3^2 \times 0{\cdot}4$

$F = \underline{2{\cdot}16\,N}$ ✓

Direction – radial, P to O or $\overrightarrow{PO}$ ✓



Example 2. One end of a light inextensible string of length 0·5 m is attached to a fixed point A. A particle P of mass 0·2 kg is attached to the other end of the string. P moves with constant speed in a horizontal circle with centre O, which is 0·4 m vertically below A.

(i) Show that the tension in the string is 2·5 N. ---[2]

(ii) Find the speed of P. [W-15/53/Q2] -[3]

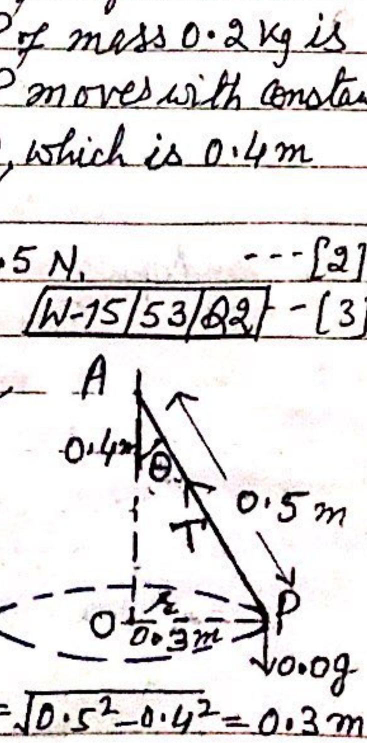
Solution: (i) Vertically; $T\cos\theta = 0{\cdot}2g$

$\Rightarrow T \times \frac{0{\cdot}4}{0{\cdot}5} = 2 \Rightarrow \underline{T = 2{\cdot}5N}$ ✓

(ii) Horizontally. $T\sin\theta = m\frac{V^2}{r}$

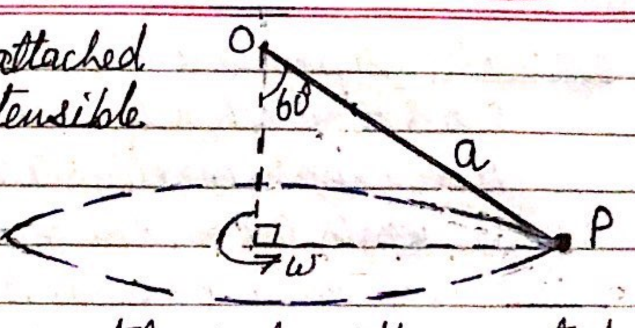
$\Rightarrow 2{\cdot}5 \times \frac{0{\cdot}3}{0{\cdot}5} = 0{\cdot}2 \times \frac{V^2}{0{\cdot}3}$

$\Rightarrow V = \underline{1{\cdot}5\,m s^{-1}}$ ✓



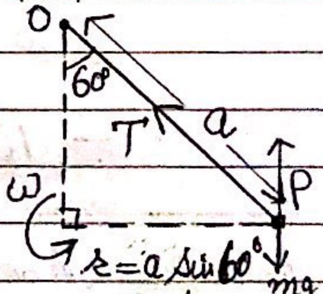
[$r = \sqrt{0{\cdot}5^2 - 0{\cdot}4^2} = 0{\cdot}3\,m$

Example 3: A particle P of mass m attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O.



(a) The particle P moves in a horizontal circle with a constant angular speed ω with string inclined at 60° to the downward vertical through O. Show that $\omega^2 = 2g/a$ --[4]

(b) The particle now hangs at rest a distance a vertically below O. It is then projected horizontally so that it begins to move in a vertical circle with centre O, when the string makes an angle of 60° with the downward vertical through O, the angular speed of P is $\sqrt{2g/a}$. The string first goes slack when OP makes an angle θ with the upward vertical through O. Find the value of $\cos\theta$.
FM-9231/SP-20/03/Q5 ---[6]



Solution: (i) Vertically $T\cos 60 = mg \Rightarrow \frac{1}{2}T = mg$ —— ①

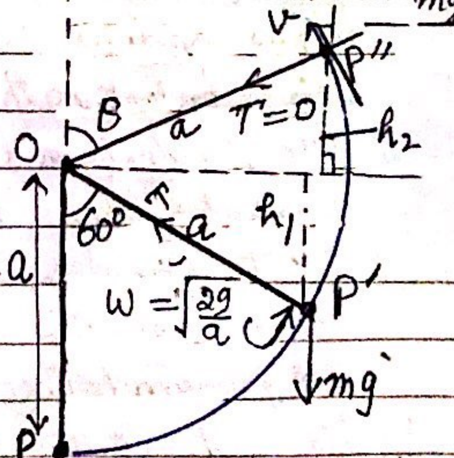
Horizontally $T\sin 60 = m\omega^2 r$

$\Rightarrow T\sin 60 = m\,(a\sin 60)\,\omega^2$

$\Rightarrow T = ma\omega^2$ —— ②

$① \div ② \Rightarrow \frac{1}{2} = \frac{g}{a\omega^2} \Rightarrow \omega^2 = \frac{2g}{a}$ ✓

(ii) Energy from 60° with downward vertical to the point when string goes slack,



Change K.E from P' to P'' = gain in PE P' to P''

$\frac{1}{2}m\left(a\sqrt{\frac{2g}{a}}\right)^2 - \frac{1}{2}mv^2$

$\Rightarrow \frac{1}{2}m \cdot 2ga - \frac{1}{2}mv^2 = mg(h_1 + h_2)$

$\Rightarrow mga - \frac{1}{2}mv^2 = mg(a\cos 60 + a\cos\theta)$

$\Rightarrow v^2 = ga - 2ga\cos\theta$ —— ③

when the string goes slack at P'',

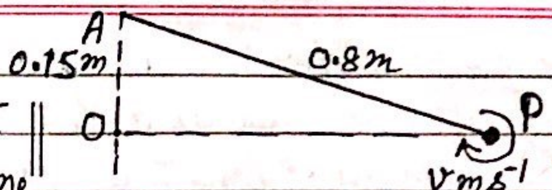
Tension is zero $\Rightarrow mg\cos\theta = \frac{mv^2}{a}$ (components along OP)

$\Rightarrow v^2 = ag\cos\theta$ —— ④

from ③ & ④

$\cos\theta = \frac{1}{3}$ ✓

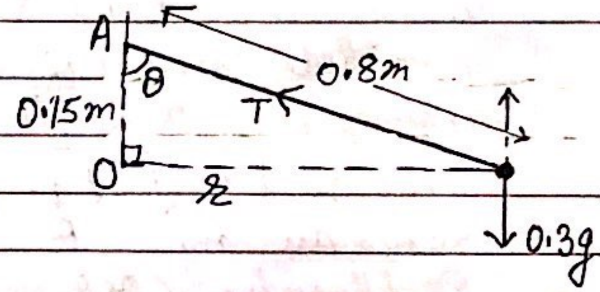
Example 4: A particle P of mass 0.3 kg is attached to a fixed point A by a light inextensible string of length 0.8 m. The fixed point O is 0.15 m vertically below A. The particle P moves with constant speed $v\ ms^{-1}$ in a horizontal circle with centre O.



(i) Show that the tension in the string is 16 N. ---[2]

(ii) Find the value of v [S-19/51/Q1] ---[3]

Solution; Vertically



(i) $T\cos\theta = mg$

$T \times \frac{0.15}{0.8} = 0.3g$

$\Rightarrow T = \underline{16\,N}$ ✓

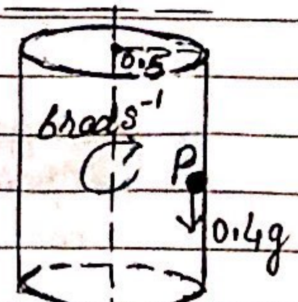
(ii) Horizontally:

$T \cdot \sin\theta = \frac{mv^2}{r}$

$[r = \sqrt{0.8^2 - 0.15^2} = 0.78581$

$\Rightarrow 16 \times \frac{0.78581}{0.8} = \frac{0.3 \times v^2}{0.78581} \Rightarrow \underline{v = 6.416\,ms^{-1}}$

Example 5: A hollow cylinder with a rough inner surface has radius 0.5 m. A particle P of mass 0.4 kg is in contact with the inner-surface of the cylinder. The particle and cylinder rotate together with angular speed $6\,rad\,s^{-1}$ about the vertical axis of the cylinder, so that the particle moves in a horizontal circle. Given that P is about to slip downwards, find the coefficient ---[4] of friction between P and the surface of the cylinder. [W-17/51/Q3]



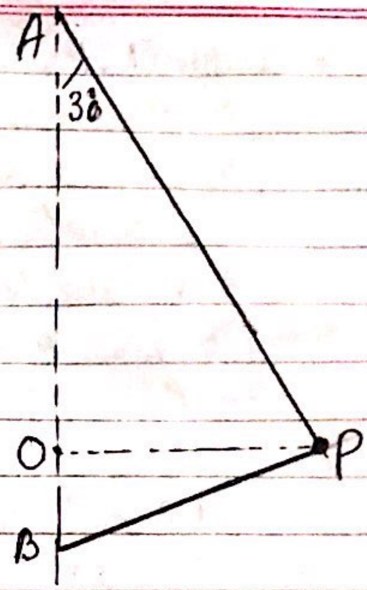
Solution: $R = ma = m\omega^2 r$

$\Rightarrow R = 0.4 \times 6^2 \times 0.5 = 7.2N.$

$F = mg = 0.4 \times 10 = 4N.$

$F = \mu R \Rightarrow \mu = \frac{F}{R} = \frac{4}{7.2} = \underline{0.566}$ ✓

Example 6: A and B are two fixed points on a vertical axis with A above B. A particle P of mass 0.4 kg is attached to A by a light inextensible string of length 0.5 m. The particle P is attached to B by another light inextensible string. P moves with constant speed in horizontal circle with centre O, between A and B. Angle BAP = 30° and angle ABP = 70°.



(i) Given that the tensions in the two strings are equal, find the speed of P. -- [5]

(ii) Given instead that the angular speed of P is 12 rad s^{-1}, find the tensions in the strings. W-19 | 52 | Q5 --[5]

Solutions: $r = 0.5 \sin 30 = 0.25\,m$

(i) Vertically:

$T\cos 30 - T\cos 70^\circ = 0.4g$

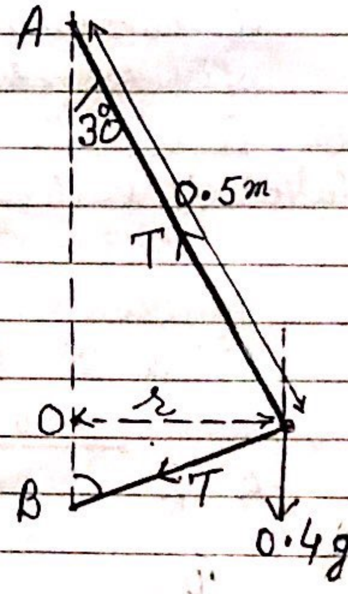
$\Rightarrow T = 7.6335$ —— ①

horizontally:

$T\sin 30^\circ + T\sin 70^\circ = m\frac{v^2}{r}$

$\Rightarrow T\sin 30^\circ + T\sin 70^\circ = 0.4 \times \frac{v^2}{0.25}$ —— ②

from ① & ② $\underline{V = 2.62\,ms^{-1}}$



(ii) Vertically.

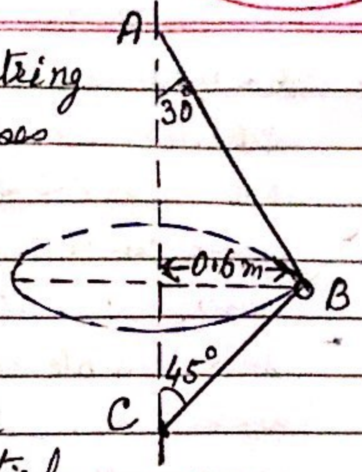
$T_A\cos 30^\circ - T_B\cos 70^\circ = 0.4g$ —— ③

$T_A\sin 30^\circ + T_B\sin 70^\circ = ma = mr\omega^2$

$= 0.4 \times 0.25 \times 12^2$ —— ④

from ③ & ④ $T_A = 8.82\,N$

$T_B = 10.6\,N.$



Example 7: One end of a light inextensible string is attached to fixed point A. The string passes through a smooth bead B of mass 0.3kg and the other end of the string is attached to a fixed point C vertically below A. The bead B moves with constant speed in a horizontal circle of radius 0.6m. The string makes an angle of 30° with the vertical at A and an angle 45° with the vertical at C.

(i) Calculate the speed of B. ---[5]

The lower end of string is detached from C, and B is now attached to this end of the string. The other end of the string remains attached to A. The bead set in motion, so that it moves with angular speed 3 rad s^{-1} in a horizontal circle, which has its centre vertically below A.

(ii) Calculate the tension in the string. W-15 / 51 / Q4 ---[3]

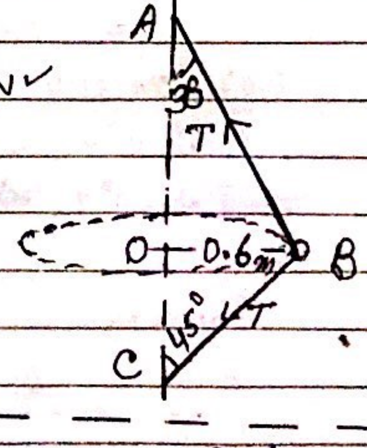
Solution: (i) Vertically:

$T\cos 30 - T\sin 45 = 0.3g \Rightarrow T = 18.9\,N$ ✓

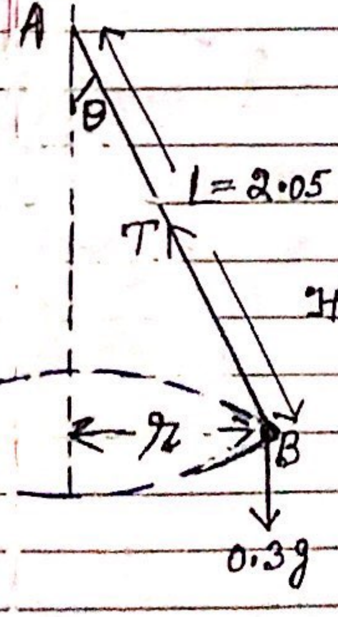
Horizontally:

$T\sin 30° + T\sin 45° = 0.3\dfrac{V^2}{0.6}$

$\Rightarrow \underline{V = 6.75\ ms^{-1}}$



(ii)



length of string AB, $L = \dfrac{0.6}{\sin 30°} + \dfrac{0.6}{\sin 45} = 2.05\,m$

and $r = (2.05\sin\theta)$

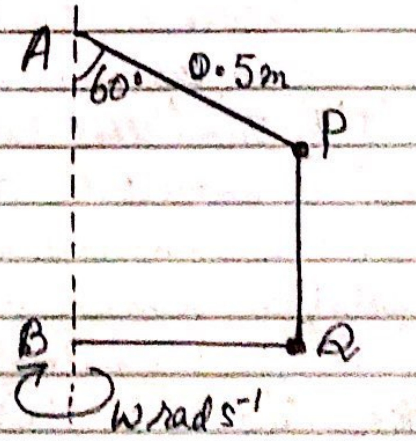
Horizontally:

$T\sin\theta = m\omega^2 r$

$\Rightarrow T\sin\theta = 0.3 \times 3^2 \times 2.05\sin\theta \quad (\omega = 3)$

$\Rightarrow T = \underline{5.53}\ N$

Example 8. Two particles P and Q have masses 0.4 kg and m kg respectively. P is attached to a fixed point A by a light inextensible string of length 0.5 m, which is inclined at an angle 60° to the vertical. P and Q are joined to each other by a light inextensible vertical string. Q is attached to a fixed point B, which is vertically below A, by a light inextensible string. The string BQ is taut and horizontal. The particles rotate in horizontal circles about an axis through A and B with constant angular speed ω rad s⁻¹. The tension in the string joining P and Q is 1.5 N.

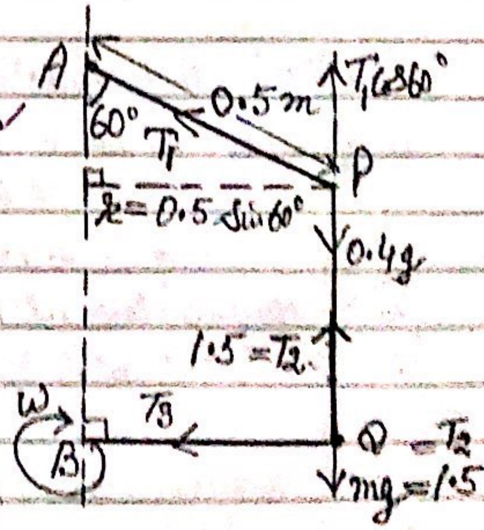


(i) Find the tension in the string AP and the value of ω. ---[4]

(ii) Find m and the tension in the string BQ. ---[3]

[M-17/52/Q5]

Solution: Vertically at P.



(i) $T_1 \cos 60 = mg + 0.4g = 1.5 + 4$ $\Rightarrow T_1 = 11\text{N}$

Horizontally at P.

$T_1 \sin 60° = m\omega^2 r$

$\Rightarrow 11 \times \sin 60 = 0.4 \times \omega^2 \times 0.5 \sin 60°$

$\Rightarrow W = \sqrt{55} = 7.42$

(ii) Vertically at Q:

$mg = T_2 = 1.5 \Rightarrow m = 0.15$

Tension in BQ $= T_3 = m\omega^2 \times r$

$= 0.15 \times 7.42^2 \times 0.5 \sin 60$

or $T_3 = 3.57\text{N}$

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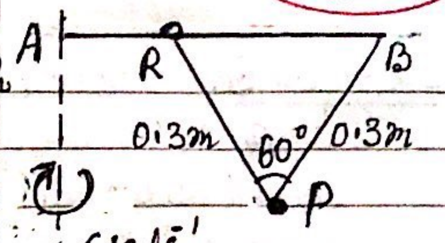
Example 9. A rough horizontal rod AB of length 0.45 m rotates with constant velocity 6 rad s^{-1} about a vertical axis through A. A small ring R of mass 0.2 kg can slide on the rod. A particle P of mass 0.1 kg is attached to the mid point of a light inextensible string of length 0.6 m. One end of the string is attached to R and the other end of the string is attached to B, with angle RPB = 60°. R and P move in horizontal circles as the system rotates, R is in equilibrium.



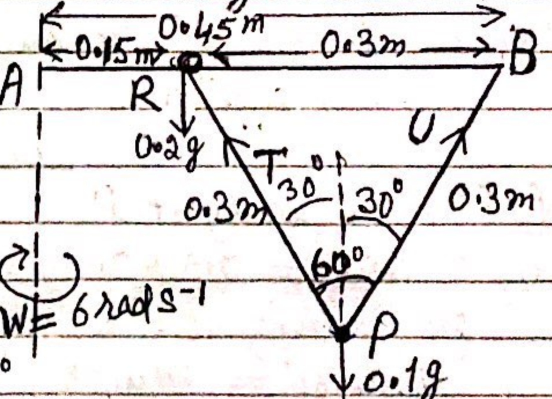
(i). Show that the tension in the portion PR of the string is 1.66 N, correct to 3 s.f. W-18 | 52 | Q7 ---[5]

(ii) Find the coefficient of friction between the ring and the rod; --[5]

Solution: Vertically at P.

$T\cos 30° + U\cos 30° = 0.1g = (1)$ — ①

Horizontally $T\cos 60 - U\cos 60 = 0.1 \times 6^2 \times 0.3$
$= 1.08$ — ②



multiplying ① by $\cos 60°$ & ② by $\cos 30°$ and add, $2T\cos 30°\cos 60°$
$= 1.08\cos 30 + 1 \times \cos 60°$

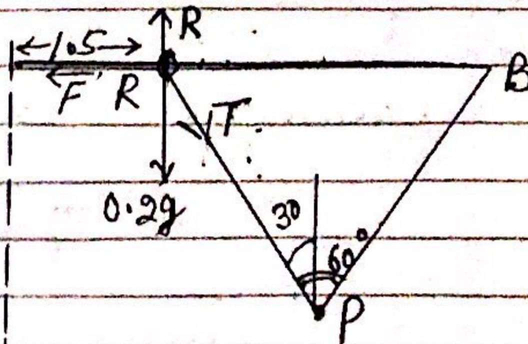
$\Rightarrow \underline{T} = 1.65735 = \underline{1.66\,N}$ ✓

(ii) Horizontally at R,

$F - T\cos 60 = m\omega^2 r = 0.2 \times 6^2 \times 1.5$

$\Rightarrow F = 1.9086 = 1.91\,N$ — ③

Vertically at R.



Normal Reaction $R = 0.2g + T\cos 30 = 3.4353$

$\Rightarrow R = 3.44\,N$ — ④

$F = \mu R$ from ③ & ④

$\Rightarrow \mu = \dfrac{F}{R} = \dfrac{1.9086}{3.4353}$
$= 0.556$

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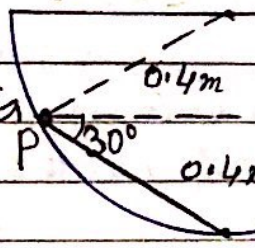
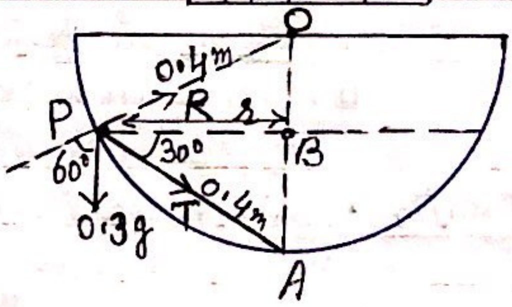


Example 10. One end of a light inextensible string of length 0.4m is attached to the lowest point of a hemisphere of radius 0.4m fixed with its axis vertical. A particle P of mass 0.3kg is attached to the other end of the string. The string is straight and makes an angle of 30° with the horizontal. P moves on the smooth inner surface of the hemisphere in a horizontal circle.

(i) Calculate the smallest possible angular speed of P. ---[4]

(ii) Given that the greatest possible tension in the string is 5N, Calculate the greatest possible speed of P. [M-18|52|Q5] --[4]

Solution:



(i) $r = BP = 0.4\cos 30$

Now at P, Vertically,

$R\cos 60° = 0.3g \Rightarrow R = 6N$ —①

Horizontally,

$F = ma = m \cdot r\omega^2$

$R\cos 30 = 0.3 \times 0.4\cos 30° \times \omega^2$

$\Rightarrow 6\cos 30 = 0.12\cos 30° \times \omega^2 \Rightarrow \omega^2 = 50 \Rightarrow \omega = 5\sqrt{2}$

$\Rightarrow \underline{\omega = 7.07 \text{ rad s}^{-1}}$ ✓

(ii) When $T = 5$, Vertically at P

$R\cos 60° = 0.3g + 5\sin 30°$

$\Rightarrow R = 11N$

Horizontally

$R\cos 30 + T\cos 30 = m \cdot \frac{V^2}{r}$

$\Rightarrow 11\cos 30° + 5\cos 30 = \frac{0.3 \times V^2}{0.4\cos 30°}$

$\Rightarrow \underline{V = 4 \text{ ms}^{-1}}$ ✓

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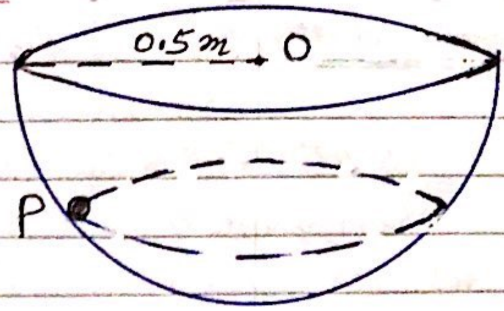
Example 11: A particle P of mass 0.4kg moves with constant speed in a horizontal circle on the smooth inner surface of a fixed hollow hemisphere with centre O and radius 0.5 m.



(i) Given that the speed of the particle is $4ms^{-1}$ and its angular speed is $10 rads^{-1}$ Calculate the angle between OP and the vertical. ---[2]

(ii) Given instead the magnitude of the force exerted on P by the hemisphere is 6N calculate:

(a) the angle between OP and the vertical. ---[2]

(b) the angular speed of B. [W-15/53/Q4] ---[3]

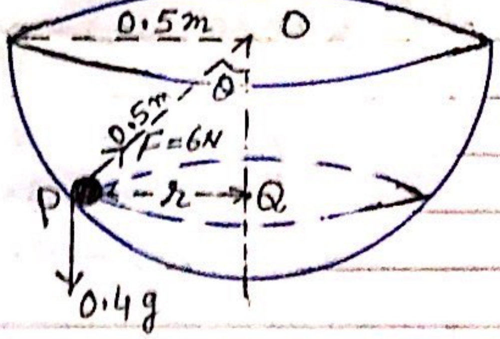
Solution: $\omega = 10 \, rads^{-1}$, $v = 4 \, ms^{-1}$

$v = r\omega \Rightarrow r = \frac{v}{\omega} = \frac{4}{10} = 0.4m$

In rt $\triangle OPQ$

$\sin\theta = \frac{PQ}{OP} = \frac{0.4}{0.5} = 0.8$

$\Rightarrow \theta = \underline{53.1^\circ}$



(ii) (a) Vertically at P,

$F\cos\theta = mg \Rightarrow 6\cos\theta = 0.4\times10$

$\Rightarrow \cos\theta = \frac{4}{6} \Rightarrow \underline{\theta = 48.2^\circ}$ ✓

(b) Horizontally at P.

$F\sin\theta = m\omega^2 r$

$\Rightarrow 6\sin\theta = 0.4\times\omega^2\times0.5\sin\theta$

$\Rightarrow \omega = \underline{5.48 \, rads^{-1}}$

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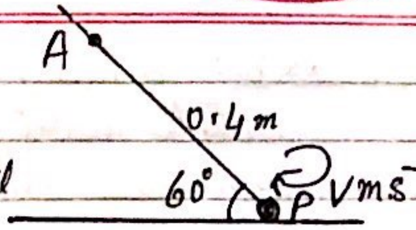
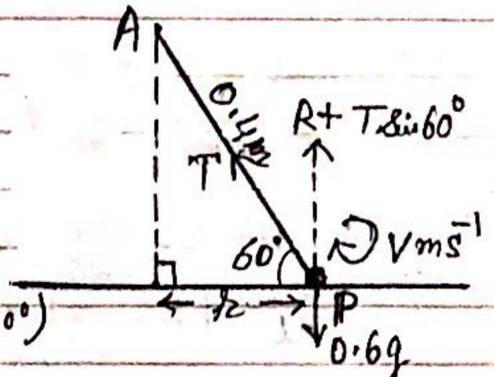
Example 12. One end of a light inextensible string of length 0·4 m is attached to a fixed point A which is above a horizontal surface. A particle P of mass 0·64 kg is attached to the other end of the string. P moves in a circle on the surface with constant speed v ms^{-1}, with the string taut and making an angle of 60° with the horizontal.



(i) Given that $v = 0·5$, calculate the magnitude of the force that the surface exerts on P. ---[4]

(ii) Find the greatest possible value of v for which P remains in contact with the surface. --[3] W-17/51/Q3

Solution: Let the force exerted on P = R



Vertically at P,

(i) $T \sin 60° + R = 0·6g$ —— ①

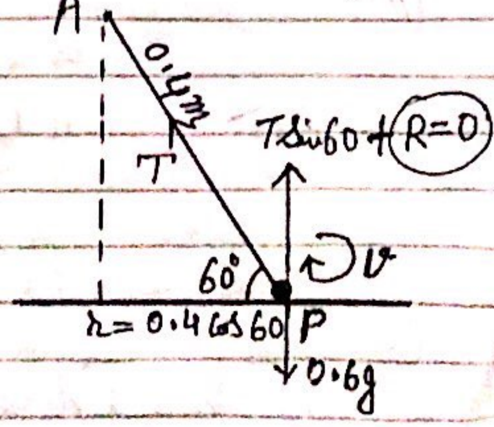
Horizontally at P

$T \cos 60 = m\frac{v^2}{r}$ $\left[r = 0·4 \cos 60°\right)$

$\Rightarrow$ $T \cos 60 = \frac{0·6 \times 0·5^2}{0·4 \times \cos 60} \Rightarrow T = 1·5\,N$

from ① $1·5 \sin 60 + R = 6 \Rightarrow \underline{R = 4·7N}$ ✓

(ii) Vertically at P, when P is on the surface in limiting position $\Rightarrow R = 0$



$T \sin 60 = 0·6g$

$\Rightarrow$ $T = 6·9282$ ✓

Horizontally at P,

$T \cos 60 = m\frac{v^2}{r}$

$\Rightarrow 6·9282 \times \cos 60 = 0·6 \times \frac{v^2}{0·4 \cos 60°}$

$\Rightarrow$ $v = \underline{1·07\,ms^{-1}}$.

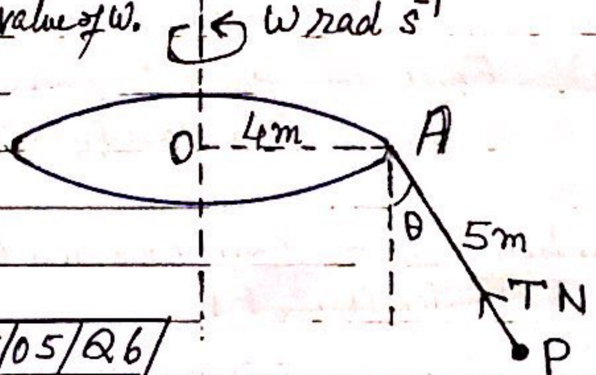
Example 13: A horizontal circular disc of radius 4m is free to rotate about a vertical axis through its centre O. One end of a light inextensible rope of length 5m is attached to a point A of the circumference of the disc and an object P of mass 24 kg is attached to the other end of the rope. When the disc rotates with constant angular speed ω rad s^{-1}, the rope makes an angle of θ radians with the vertical and the tension in the rope is T N. You may assume that the rope is always in the same vertical plane as the radius OA of the disc.

(i) Given that $\cos\theta = \frac{24}{25}$, find the value of ω.

(ii) Given instead that the speed of P is twice the speed of the point A, find

(a) the value of T.

(b) the speed of P.



W-05/05/Q6

Solution: $\cos\theta = \frac{24}{25}$, $\sin\theta = \frac{7}{25}$

(i) $r = PR = QR + PQ$
$= (4 + 5\sin\theta)$

at P $\quad r = (4 + 5 \times \frac{7}{25}) = 5.4m$

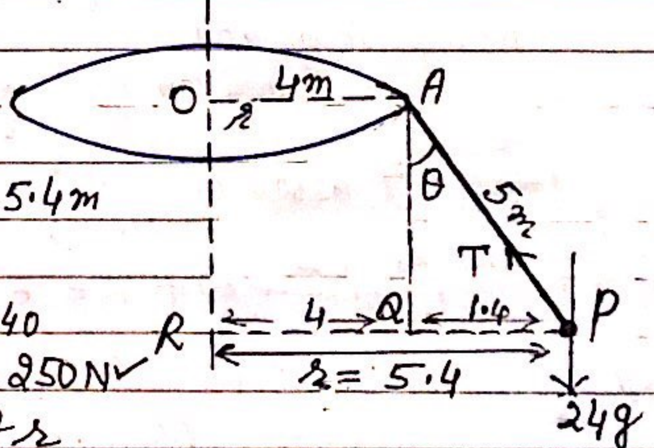
Vertical components,

$T\cos\theta = 24g \Rightarrow T \times \frac{24}{25} = 240$

$\Rightarrow T = 250N$

Horizontally: $T\sin\theta = m\omega^2 r$

$\Rightarrow 250 \times \frac{7}{25} = 24 \times \omega^2 \times 5.4 \Rightarrow \underline{\omega = 0.735}$



(ii) Velocity at A $V_1 = V$ $\quad \omega = V_1 r_1$
Velocity at P $V_2 = 2V$ $\quad \omega = V_2 r_2$

$V_2 r_2 = V_1 r_1 \Rightarrow r_2 = \frac{V_1 r_1}{V_2} = \frac{V}{2V} \times 4$

$\Rightarrow r_2 = PR = 8m \Rightarrow PQ = 4m$
$\Rightarrow AQ = 3m$

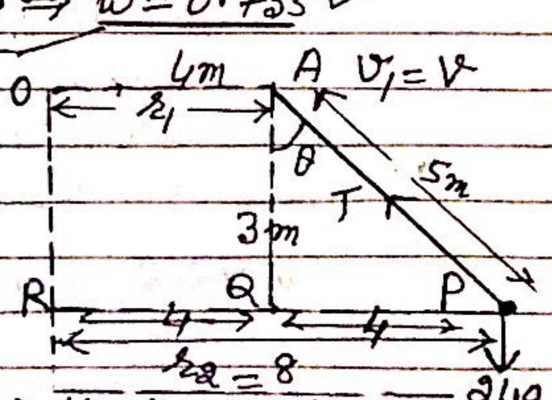
(a) $T\cos\theta = mg = 24 \times 10$

$T \times 3/5 = 240 \Rightarrow$

$\Rightarrow T = 400N.$

(b) $T\sin\theta = mV^2/r$

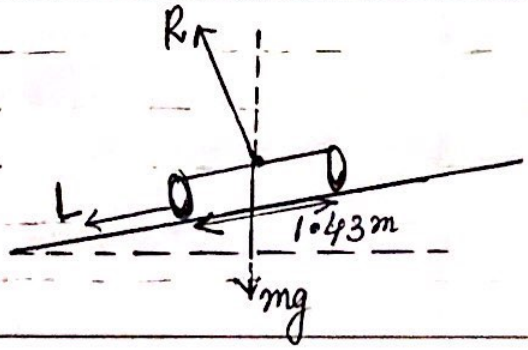
$400 \times \frac{4}{5} = 24 \times \frac{V^2}{8} \Rightarrow \underline{V = 10.5 ms^{-1}}$



Car (or train) moving round a bend. P-13

Example 14: A bend on a railway track has a radius of 500m and is to be banked so that a train can negotiate it at 96 kmh^{-1} without the need for a lateral force between its wheels and the rail. The distance between the rails is 1·43 m. How much higher should outside rail be than the inner one.

Solution: When there is no lateral thrust $L = 0$



Horizontally $R\sin\alpha = \frac{mv^2}{r}$ — ①

Vertically $R\cos\alpha = mg$ — ②

$① \div ② \Rightarrow \tan\alpha = \frac{v^2}{rg}$

$= \frac{(80/3)^2}{500 \times 10}$

$\tan\alpha = 0.1422$

$\Rightarrow \alpha = 8.1^\circ$

$v = 96\ kmh^{-1}$

$= \frac{96 \times 1000}{60 \times 60}$

$v = \frac{80}{3}\ ms^{-1}$

$r = 500\ m$

∴ The out side rail should be raised by $1.43\sin\alpha$ m.

$= 1.43 \times \sin 8.1$

$= 0.201\ m$

$= 20$ Cm. ✓

← × — × →