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P-1

# Pure Maths-1

# Differentiation Notes.

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**P1**

**Differentiation**

Everyday I leave for school at around 7:30 am, to reach the school at 8:00 am. School is approx 20 km away from my house.

One day, as I crossed a red light signal, all of a sudden a traffic inspector appeared and asked me to stop the car and issued me a speed - challan. I was surprised, as the speed limit allowed is 60 km/h and I was driving at a speed of 40 km/h.

$$\text{Distance} = 20 \text{ km}$$

$$\text{Time} = 30 \text{ min} = \tfrac{1}{2} \text{ h}$$

$$\text{Speed} = \frac{\text{Distance}}{\text{time}}$$

$$= \frac{20}{1/2}$$

$$= 40 \text{ km/h}$$

I am still wondering why at all the speed challan was issued? Traffic cop was not listening to my argument that my speed, 40 km/h was within limit.

If you have an answer, please help me ?

# Differentiation

Sir Isaac Newton and Gottfried Leibniz both independently developed the concept of calculus in the middle of 17th Century.

Differential calculus is about finding the rate of change of one quantity with respect to another quantity. Graphically speaking differential calculus is about finding the "gradient" (or slope) of tangent to the graph of a function $y = f(x)$

**Example 1.** Given, the distance travelled by a car is given a function of time, then how to find the instantaneous rate of change of distance with time i.e. velocity of car.

**Example 2.** Given area of a square 'A' and length of a side 'x', both can vary, then what is the rate of change of area 'A' with change in length of side 'x'. $x \to x+\delta x$

we know $A = x^2$ ; $A+\delta A = (x+\delta x)^2$

Let $\delta x$ denotes a very small change in $x$ and $\delta A$ denotes the corresponding change in area, $\frac{\delta A}{\delta x} = ?$

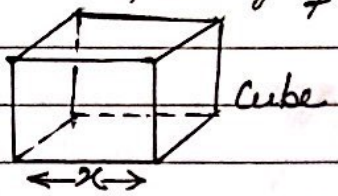
| $x$ | $A$ | $x+\delta x$ | $A+\delta A$ | $\delta x$ | $\delta A$ | $\delta A/\delta x$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1.001 | 1.002001 | 0.001 | 0.002001 | $\frac{0.002001}{0.001} = 2.001 = 2\times 1$ approx. |
| 2 | 4 | 2.001 | 4.004001 | 0.001 | 0.004001 | $\frac{0.004001}{0.001} = 4.001 = 2\times 2$ approx |
| 3 | 9 | 3.001 | 9.006001 | 0.001 | 0.006001 | $\frac{0.006001}{0.001} = 6.001 = 2\times 3$ approx. |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ = ⋮ |
| $x$ | $x^2$ | $(x+\delta x)$ | $(x+\delta x)^2 = x^2+2x\cdot\delta x + \delta x^2$ | $\delta x$ | $2x\delta x+\delta x^2 = \delta x(2x+\delta x)$ | $\frac{\delta x(2x+\delta x)}{\delta x} = (2x+\delta x) = 2\times x$ approx. as $\delta x$ is very small. |

∴ Rate of change of $A = x^2$ with respect to $x$ is $2x$, denoted by $\frac{dA}{dx} = 2x$ ✓ or $\frac{d}{dx}x^2 = 2x$ ✓

$\boxed{\text{Rate of change of area of square at } x=8;\ \left(\frac{dA}{dx}\right)_{x=8} = 2\times 8 = 16}$

# Differenentiation

**Example 3.** Given the volume of a cube 'V' and the length of its one side '$x$', both can vary, then what is the rate of change of Volume 'V' with change in $x$.




$$V = x^3$$

$$x \rightarrow x + \delta x$$

$$V \rightarrow V + \delta V$$

what is the value of $\frac{\delta V}{\delta x}$ when $\delta x$ is very small.

| $x$ | $V$ | $x+\delta x$ | $V+\Delta V$ | $\Delta x$ | $\Delta V$ | $\delta V/\delta x$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1.001 | 1.003003 | 0.001 | 0.003003 | $\frac{0.003003}{0.001} = 3.003 = 3\times 1^2$ approx |
| 2 | 8 | 2.001 | 8.012006 | 0.001 | 0.012006 | $\frac{0.012006}{0.001} = 12.006 = 3\times 2^2$ approx |
| 3 | 27 | 3.001 | 27.027009 | 0.001 | 0.027009 | $\frac{0.027009}{0.001} = 27.009 = 3\times 3^2$ approx. |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x$ | $x^3$ | $x+\delta x$ | $(x+\delta x)^3$ | $\Delta x$ | $3x^2\cdot\delta x + 3x\cdot\delta x^2 + \delta x^3$ $= \delta x(3x^2 + 3x\cdot\delta x + \delta x^2)$ | $= \frac{\delta x(3x^2+3x\cdot\delta x+\delta x^2)}{\delta x} = 3\cdot x^2$ as $\delta x$ is very small. |

$\therefore$ Rate of change of V with respect to $x = 3x^2$
or differential of Volume with respect to $x$, denoted by,

$V(x) = x^3 \Rightarrow V'(x) = 3x^2$ or $\frac{dV}{dx} = 3x^2$ or $\frac{d}{dx}x^3 = 3x^2$

Rate of change of Volume at $x=4$; $\left(\frac{dV}{dx}\right)_{x=4} = 3\times 4^2 = 48$ ✓

§ In general.

| | | |
|---|---|---|
| $y = x^n$ <br> $\frac{dy}{dx} = n\cdot x^{n-1}$ | $\frac{d}{dx}x^n = n\cdot x^{n-1}$ | $f(x) = x^n$ <br> $f'(x) = nx^{n-1}$ |

$\frac{dy}{dx}$ denotes the derivative of $y$ w.r.t. $x$.

# Differentiation

## § Differentiation Formulae:

(i) $\frac{d}{dx} x^n = n x^{n-1}$ ✓ or $\begin{cases} y = x^n \\ \frac{dy}{dx} = n x^{n-1} \end{cases}$ or $\begin{cases} f(x) = x^n \\ f'(x) = n \cdot x^{n-1} \end{cases}$

(Power function)

(ii) $y = a\,f(x)$

$\frac{dy}{dx} = a \cdot f'(x)$

(iii) $\frac{dc}{dx} = 0$ (c is a constant)

(iv) $\frac{d}{dx} ax = a$ $\left\{ \because \frac{d}{dx} x = 1 \right.$

(v) $\frac{d}{dx} a x^n = a n x^{n-1}$

(vi) $\frac{d}{dx} \sqrt{x} = \frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ ✓

(vii) $\frac{d}{dx} \frac{1}{x} = \frac{d}{dx} x^{-1} = -1 x^{-2} = \frac{-1}{x^2}$ ✓

(viii) $y = a \cdot f(x) + b \cdot g(x) - c \cdot h(x)$

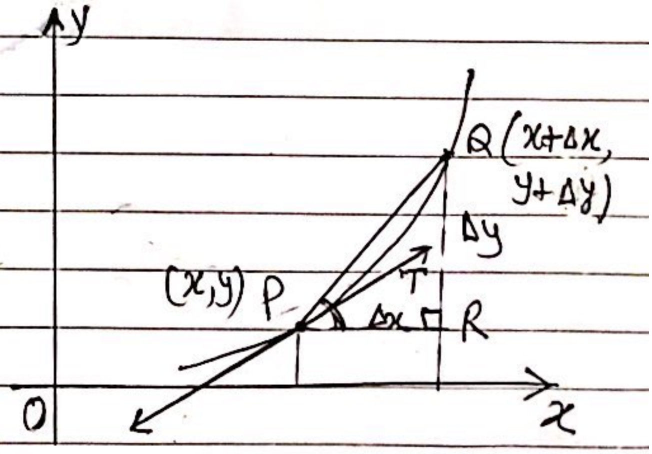
$\frac{dy}{dx} = a f'(x) + b g'(x) - c\,h'(x)$

**Example:** $y = x^5 - 3x^4 + 7x^2 + 8$

$\frac{dy}{dx} = 5x^4 - 3 \times 4x^3 + 7 \times 2x + 0$

$= 5x^4 - 12x^3 + 14x$ ✓

## § Geometric interpretation of $\frac{dy}{dx}$.




Given a function $y = f(x)$

Then $\frac{dy}{dx}$ or $f'(x)$ denotes the gradient (or slope) of the tangent to the curve $y = f(x)$ at any point $P(x, y)$ on the curve.

$f'(x) = \frac{dy}{dx} = \lim_{x \to 0} \frac{\Delta y}{\Delta x}$ = Gradient of the tangent to the curve at any point $(x, y)$.

as $\Delta x \to 0$, $\Delta y \to 0$ and the chord $PQ \to$ tangent $PT$.

"PT is the tangent at P."

# Differentiation

Example 4. Differentiate with respect to $x$ (or w.r.t $x$)

(i) $\frac{1}{x^3}$ (ii) $x^2(1+x)$ (iii) $\frac{1+x}{x^2}$ (iv) $\frac{x^2+5x}{3\sqrt{x}}$

(i) $\frac{d}{dx}\frac{1}{x^3} = \frac{d}{dx}x^{-3} = -3x^{-3-1} = -3x^{-4} = \frac{-3}{x^4}$ ✓

Since $\frac{d}{dx}x^n = n \cdot x^{n-1}$

(ii) $\frac{d}{dx}x^2(1+x) = \frac{d}{dx}(x^2+x^3) = 2x+3x^2$ ✓

(iii) $\frac{d}{dx}\frac{1+x}{x^2} = \frac{d}{dx}\left(\frac{1}{x^2}+\frac{x}{x^2}\right) = \frac{d}{dx}\left(x^{-2}+x^{-1}\right) = -2x^{-3}-1x^{-2}$

$= \frac{-2}{x^3} - \frac{1}{x^2}$ ✓

(iv) $\frac{d}{dx}\left(\frac{x^2+5x}{3\sqrt{x}}\right) = \frac{1}{3}\frac{d}{dx}\left(\frac{x^2}{x^{1/2}}+\frac{5x}{x^{1/2}}\right) = \frac{1}{3}\frac{d}{dx}\left(x^{3/2}+5x^{1/2}\right)$

$= \frac{1}{3}\left[\frac{3}{2}x^{1/2} + 5\times\frac{1}{2}x^{-\frac{1}{2}}\right]$

$= \frac{1}{3}\left[\frac{3}{2}\sqrt{x} + \frac{5}{2\sqrt{x}}\right]$ ✓

Example 5. Find the value of $\frac{dy}{dx}$ at the given points.

(a) $y = x^4$ at $x=2$

$\frac{dy}{dx} = 4x^3$

$\left(\frac{dy}{dx}\right)_{x=2} = 4\times 2^3 = 32$ ✓

(b) $y = 5x^3 - 3x^2 + 7x + 6$ at $x=5$

$\frac{dy}{dx} = 5\times 3x^2 - 3\times 2x + 7\times 1 + 0$

$= 15x^2 - 6x + 7$

$\left(\frac{dy}{dx}\right)_{x=5} = 15\times 5^2 - 6\times 5 + 7 = 375 - 30 + 7 = 352$ ✓

Example 6. Given $f(x) = 4\sqrt{x}$ find $f'(9)$

$f'(x) = 4\times\frac{1}{2\sqrt{x}} = \frac{2}{\sqrt{x}}$

$\therefore f'(9) = \frac{2}{\sqrt{9}} = \frac{2}{3}$ ✓

$f(x) = 4\sqrt{x} = 4x^{\frac{1}{2}}$

$f'(x) = 4\times\frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{2}{\sqrt{x}}$

## § Derivative of composite functions (Using Chain Rule)

Given $y = f(u)$ and $u = g(x)$

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ ✓

§ $y = (ax+b)^n$ to find $\frac{dy}{dx}$.

$y = u^n$ Let $u = ax+b$

$\frac{dy}{du} = nu^{n-1}$ and $\frac{du}{dx} = a$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (Chain rule)

$= nu^{n-1} \cdot a$

$\therefore \frac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$ ✓ —— ①

**Example 7:** $f(x) = (3-5x)^8$, find the value of $f'(0)$.

**Solution:** $f(x) = (3-5x)^8 \Rightarrow f'(x) = -5 \times 8(3-5x)^7$

$\therefore f'(0) = -40 \times 3^7 = -87480$ ✓

**Example 8:** $y = \sqrt{2x^3+4x^2+1}$ ; find $\frac{dy}{dx}$.

**Solution:** $y = \sqrt{u}$ and $u = 2x^3+4x^2+1$

$\frac{dy}{du} = \frac{1}{2\sqrt{u}}$ $\frac{du}{dx} = 6x^2+8x$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{1}{2\sqrt{u}} \times (6x^2+8x) = \frac{\cancel{2}(3x^2+4x)}{\cancel{2}\sqrt{2x^3+4x^2+1}}$

$\therefore \frac{dy}{dx} = \frac{3x^2+4x}{\sqrt{(2x^3+4x^2+1)}}$

Example 9: Find $\frac{dy}{dx}$; for $y = \frac{1}{\sqrt{2x^2+3x-10}}$

$y = \frac{1}{\sqrt{u}}$ , Let $u = 2x^2+3x-10$

$\frac{du}{dx} = 4x+3$

or $y = u^{-\frac{1}{2}}$ $= (4x+3)$

$\frac{dy}{du} = -\frac{1}{2} u^{-\frac{1}{2}-1}$

$= -\frac{1}{2u^{3/2}}$ (Using chain rule)

$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{2u^{3/2}} \times (4x+3)$

$= -\frac{1}{2} \cdot \frac{(4x+3)}{(2x^2+3x-10)^{3/2}}$ ✓

Example 10. Differentiate $(x^2+x+1)^4$.

$y = (x^2+x+1)^4$

Now $y = u^4$ , Let $u = x^2+x+1$

$\frac{dy}{du} = 4u^3$ ; $\frac{du}{dx} = 2x+1$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (Using chain rule)

$= 4u^3 \times (2x+1)$

$= 4(2x+1)(x^2+x+1)^3$ ✓

Example 11: It is given that a curve has equation $y = f(x)$, where $f(x) = x^3 - 2x^2 + x$ ; find the set of values of $x$, for which the gradient of the curve is less than 5. ---[4]

Solution: $f(x) = x^3 - 2x^2 + x$; gradient of the curve $= f'(x)$

$f'(x) = 3x^2 - 4x + 1 < 5 \Rightarrow 3x^2 - 4x - 4 < 0$

$(3x+2)(x-2) < 0$ $[x = 2, -\frac{2}{3})$

$\Rightarrow -\frac{2}{3} < x < 2$ ✓

is the required set of values of $x$.

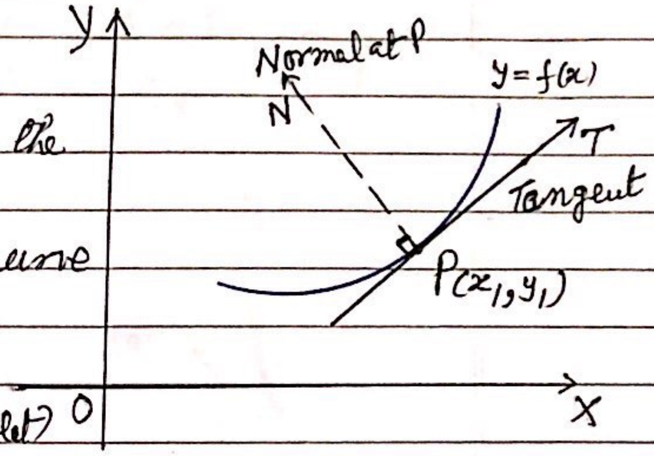
## § Tangent and Normal to a curve:

Given a curve $y = f(x)$

Let $P(x_1, y_1)$ is a point on the the curve.

$PT$ is tangent line to the curve at $P(x_1, y_1)$.

Gradient of the tangent to the curve at $P = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = m$ (let)




$\therefore$ Equation of tangent $PT$; $\underline{y - y_1 = m(x - x_1)}$ —— (1)

Now $PN$ is perpendicular to $PT$ at $P$.

$\therefore$ $PN$ is normal to the curve at $P$.

Gradient of normal $= -\frac{1}{m}$ $\left[m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}\right.$

$\therefore$ Equation of normal, $\underline{y - y_1 = -\frac{1}{m}(x - x_1)}$. —— (2)

**Example 12:** Given a point $P(1,2)$ on a curve $y = x^2 + 1$

(i) Find the equation of tangent to the curve at $(1,2)$

(ii) Find the equation of normal at $P(1,2)$.

**Solution:** $y = x^2 + 1$

$\frac{dy}{dx} = 2x$

$m = \left(\frac{dy}{dx}\right)_{(1,2)} = 2 \times 1 = 2$

(i) $\therefore$ equation of tangent at $(1,2)$

$y - y_1 = m(x - x_1)$

$y - 2 = 2(x - 1)$

or $\underline{y - 2x = 0}$ ✓

(ii) gradient of normal $= -\frac{1}{m} = -\frac{1}{2}$

$\therefore$ Equation of normal at $(1,2)$

$y - y_1 = -\frac{1}{m}(x - x_1)$

$y - 2 = -\frac{1}{2}(x - 1)$

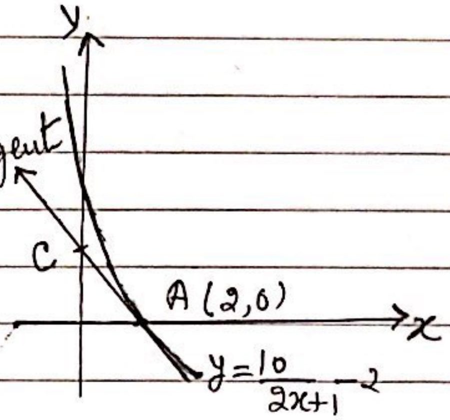
$2y - 4 = -x + 1$

or $\underline{x + 2y = 5}$ ✓

**Example 13.** The curve $y = \frac{10}{2x+1} - 2$ intersects the x-axis at A. The tangent to the curve at A intersects y-axis at C.

(i) Show that the equation of AC is $5y + 4x = 8$ --[5]

(ii) Find the distance AC. --[2]




**Solution:** Given equation of curve.

$$y = \frac{10}{2x+1} - 2 \quad \text{--- (1)}$$

Intersects X-axis at A.

$y = 0$ in (1), $\frac{10}{2x+1} - 2 = 0$

$\Rightarrow x = 2$ at A

$\therefore A(2,0)$.

differentiating (1) $\frac{dy}{dx} = -10(2x+1)^{-2} \times 2$ [ from (1) $y = 10(2x+1)^{-1} - 2$ ]

$\frac{dy}{dx} = -\frac{20}{(2x+1)^2}$ [ $\frac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$ ]

at A, $\left(\frac{dy}{dx}\right)_{(2,0)} = \frac{-20}{5^2} = -\frac{4}{5} = m$ (let)

$\therefore$ Equation of tangent at $A(2,0)$

AC: $y - y_1 = m(x - x_1)$

$\Rightarrow y - 0 = -\frac{4}{5}(x-2) \Rightarrow \underline{5y + 4x = 8}$ ✓ --- (2)

Now tangent (2) intersects y-axis at C, put $x = 0$

$y = 8/5$

$\therefore C(0, 8/5)$ and $A(2,0)$

$\therefore AC = \sqrt{2^2 + \left(\frac{8}{5}\right)^2} = \sqrt{4 + \frac{64}{25}} = \sqrt{\frac{164}{25}}$

$AC = \underline{2.56}$ ✓

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**Example 14:** The equation of a curve is $y = 2 + \frac{3}{2x-1}$

(i) Obtain an expression for $\frac{dy}{dx}$ ---[2]

(ii) Show that the normal to curve at P passes through origin, given at the point P on the curve, $x = 2$. $\boxed{W-15/12/Q7}$ ---[4]

**Solution:** Curve; $y = 2 + \frac{3}{(2x-1)}$ —— ①

(i) or $y = 2 + 3(2x-1)^{-1}$

differentiating w.r.t $x$, $\frac{dy}{dx} = -3(2x-1)^{-2} \times 2$

$= \frac{-6}{(2x-1)^2}$ —— ②

(ii) at $x = 2$, from ① $y = 2 + \frac{3}{(2\times2-1)} = 3 \Rightarrow P(2,3)$

from ② $\left(\frac{dy}{dx}\right)_{x=2} = \frac{-6}{(2\times2-1)^2} = \frac{-6}{9} = -\frac{2}{3} = m$ (let)

$\therefore$ Gradient of the Normal $= -\frac{1}{m} = \frac{3}{2}$ at $P(2,3)$

Equation of the Normal at P, $y - y_1 = -\frac{1}{m}(x - x_1)$

$y - 3 = \frac{3}{2}(x-2)$

$\Rightarrow y = \frac{3}{2}x$ —— ③

which passes through origin $(0,0)$ as for ③ $0 = 0$ True.

**Example 15:** The tangent to the curve $y = x^3 - 9x^2 + 24x - 12$ at a point A is parallel to the line $y = 2 - 3x$, Find the equation of the tangent at A. $\boxed{S-18/13/Q8}$ ---[6]

**Solution:** Curve; $y = x^3 - 9x^2 + 24x - 12$ —— ①

line ; $y = 2 - 3x$ —— ②

Gradient of line ② $m_1 = -3$ —— ③

diff. ① $\frac{dy}{dx} = 3x^2 - 18x + 24 = m_2$ —— ④

Given tangent to the curve is Parallel to the line ② $\therefore m_1 = m_2$

$\Rightarrow 3x^2 - 18x + 24 = -3$

$\Rightarrow 3x^2 - 18x + 27 = 0$

$\Rightarrow x^2 - 6x + 9 = 0$

$\Rightarrow (x-3)^2 = 0$

$x = 3$

from ① $y = 3^3 - 9\times3^2 + 24\times3 - 12 = 6$

$\therefore A(3,6)$, from ③ $m = -3$

$\therefore$ Equation of tangent to the curve at $A(3,6)$

$y - y_1 = m(x - x_1)$

$y - 6 = -3(x-3)$

$\Rightarrow y + 3x - 15 = 0$

Example 16: The equation of a curve is $y = 3 + 4x - x^2$

(i) Show that the equation of the normal to the curve at the point $(3,6)$ is $2y = x + 9$ --[4]

(ii) Given that the normal meets the coordinate axes at points A and B, find the coordinates of the mid point of AB. --[2]

(iii) Find the coordinates of the point at which the normal meets the curve again. --[4]

Solution: Curve: $y = 3 + 4x - x^2$ —— ①

(i) diff. $\frac{dy}{dx} = 4 - 2x$

at $x = 3$, $\left(\frac{dy}{dx}\right)_{x=3} = 4 - 2\times3 = -2 \Rightarrow m = -2$

$\therefore$ gradient of the normal at $(3,6)$ $= -\frac{1}{m} = \frac{1}{2}$ ✓

$\therefore$ Equation of the normal at $(3,6)$ is $y - 6 = \frac{1}{2}(x-3)$

$\Rightarrow \underline{2y = x + 9}$ —— ②

(ii) Normal ② meets x-axis for $y = 0 \Rightarrow x = -9$, $A(-9,0)$ ✓

and meets y-axis for $x = 0 \Rightarrow y = \frac{9}{2}$, $B\left(0, \frac{9}{2}\right)$

Mid point of AB $\left(\frac{-9+0}{2}, \frac{0+9/2}{2}\right) = \underline{\left(-\frac{9}{2}, \frac{9}{4}\right)}$ ✓

(iii) To find the point of intersection of curve and the normal.

$y = 3 + 4x - x^2$ —— ①

fr② $2y = x + 9$ —— ②

from ① & ② $\quad 2(3 + 4x - x^2) = x + 9$

$\Rightarrow 2x^2 - 7x + 3 = 0$

$\Rightarrow (x-3)(2x-1) = 0$

$x = \frac{1}{2}$ or $x = 3^{x}$ is already given

$\therefore$ normal meets the $\underline{\text{curve again}}$

at $x = \frac{1}{2}$, $y = \frac{1}{2}\left(\frac{1}{2} + 9\right)$ fr②

$= \frac{19}{4}$

$\therefore$ Point $\left(\frac{1}{2}, \frac{19}{4}\right)$ or $\underline{\left(\frac{1}{2}, 4\frac{3}{4}\right)}$ ✓

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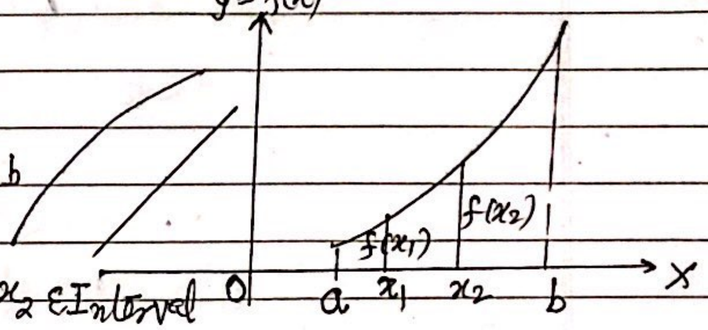
# § Increasing and Decreasing Functions:

## § Increasing function:



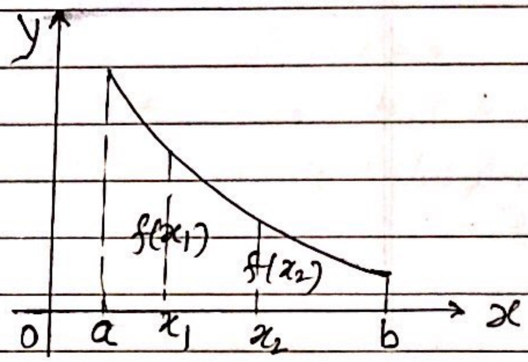

Given $y=f(x)$, $a<x<b$

such that for all $x_1, x_2 \in$ Interval

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \quad (\text{or } y_1 < y_2)$$

Note: In this interval. $f'(x)$ or $\frac{dy}{dx} > 0$ ✓

## § Decreasing function:




$x_1, x_2 \in$ Interval $a<x<b$
such that

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$
$$\text{or } (y_1 > y_2)$$

Note: for decreasing function
$f'(x)$ or $\frac{dy}{dx} < 0$ in the given interval.

§ Note: If a function $f(x)$ is increasing (or decreasing) in in an interval then it is one-one function in that interval and $f^{-1}(x)$ exists in that interval.

**Example 17:** It is given that $f(x) = (2x-5)^3 + x$, for $x \in R$, show that $f$ is an increasing function. --[3]

**Solution:** $f(x) = (2x-5)^3 + x$

diff. $f'(x) = 3(2x-5)^2 \times 2 + 1$

or $f'(x) = 6(2x-5)^2 + 1 >$ for $x \in R$. $[\because (2x-5)^2 \geq 0 \ \forall x \in R$

$[6(2x-5)^2 + 1 > 0 \ \forall x \in R]$

$\therefore$ $f(x)$ is increasing for all reals.

Example 18: It is given that $f(x)=\frac{1}{x^3}-x^3$, for $x>0$, show that $f$ is a decreasing function. ---[3]

Solution: $f(x)=\frac{1}{x^3}-x^3$ , $x>0$

or $f(x)=x^{-3}-x^3$

diff $f'(x)=-3x^{-4}-3x^2$

$=-3\left[\frac{1}{x^4}+x^2\right]<0$

$\therefore$ f(x) is a decreasing function.

$\because$ $x^2>0$ for $\forall x\in R$
$x^4>0$
$\Rightarrow\left(\frac{1}{x^4}+x^2\right)>0$
$\Rightarrow -3\left(\frac{1}{x^4}+x^2\right)<0$ $\forall x\in R$.

---

Example 19: The function $f$ is such that $f(x)=x^3-3x^2-9x+2$, for $x>n$, where $n$ is an integer. It is given that $f$ is an increasing function. Find the least possible value of $n$. --[4]

Solution: $f(x)=x^3-3x^2-9x+2$ ; $x>n$

$f'(x)=3x^2-6x-9>0$ for f(x) to be increasing

$3(x^2-2x-3)>0$

$(x-3)(x+1)>0$

$\Rightarrow$ $x>3$ ✓ or $x<-1$ ✗

$\therefore$ Least Value of $n=3$ ✓

---

Example 20: $f(x)=x^3-3x^2+7x-8$ ; for $x\in R$

Find $f'(x)$ and state, with a reason, whether $f$ is an increasing, a decreasing function or neither. W-15/13/Q3/ ---[3]

Solution: $f(x)=x^3-3x^2+7x-8$

$f'(x)=3x^2-6x+7$

$=3[x^2-2x+1-1]+7$

$=3(x-1)^2+4>0$

$\therefore$ f(x) is an increasing function.

$\because$ $(x-1)^2\geq 0$ $\forall x\in R$
$(x-1)^2+4>0$, $\forall\in R$.

Example 21. The function f is defined by:

$$f(x) = \frac{1}{x+1} + \frac{1}{(x+1)^2} \quad \text{for } x > -1$$

(i) Find $f'(x)$. ---[3]

(ii) State, with a reason, whether f is an increasing function, a decreasing function or neither. ---[1]

[S-15/13/Q8]

Solution: $f(x) = \frac{1}{(x+1)} + \frac{1}{(x+1)^2}$ for $x > -1$

or $f(x) = (x+1)^{-1} + (x+1)^{-2}$ for $x > -1$ or $(x+1) > 0$

$\Rightarrow f'(x) = -(x+1)^{-2} - 2(x+1)^{-3}$

$= -\left[\frac{1}{(x+1)^2} + \frac{2}{(x+1)^3}\right] < 0 \quad \left[\because \begin{array}{l}(x+1)^2 > 0\\ (x+1)^3 > 0\end{array}\right.$

$\therefore$ f(x) is a decreasing function.

Example 22:

$$f(x) = x^3 + 2x^2 - 4x + 7 \quad \text{for } x \geq -2$$

determine, showing all necessary working, whether f is an increasing or decreasin function or neither. ---[4]

[W-18/13/Q2]

Solution: $f(x) = x^3 + 2x^2 - 4x + 7$ for $x \geq -2$

$f'(x) = 3x^2 + 4x - 4$

$(x+2)(3x-2)$ — (1) [critical point are $-2$, $2/3$]

f(x) is decreasing for $f'(x) < 0 \Rightarrow (x+2)(3x-2) < 0$

$\therefore$ f(x) is decreasing $\Rightarrow$ $-2 < x < 2/3$ — (2) ✓

And increasing for $f'(x) > 0$

fm (1) $\quad \cancel{x < -2}$ or $x > 2/3$ ✓ as $x \geq -2$ — (3)

(not defined)

Now f(x) is decreasing $-2 < x < \frac{2}{3}$ and

is increasing for $x > 2/3$

as f(x) is Increasing and partly decreasing for $x \geq -2$

$\therefore$ f(x) is neither

## § Rate of Change:

(i) Rate of change of $y$ w.r.t $x = \frac{dy}{dx}$

(ii) Connected rate of change:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

(iii) $$\frac{dy}{dx} \times \frac{dx}{dy} = 1 \Rightarrow \frac{dx}{dy} = \frac{1}{dy/dx}.$$

---

**Example 23:** A stone is droped in a quiet lake and waves move in circles at a speed of 4 cm per second. At the instant when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

**Solution:** Let radius $= r$, given $\frac{dr}{dt} = 4 \text{ cms}^{-1}$ —— ①

To find $\frac{dA}{dt}$ when $r = 10$.

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} \quad \text{—— ①}$$

Now area of circular wave $A = \pi r^2$

Diff. $\frac{dA}{dr} = 2\pi r$

$$\left(\frac{dA}{dr}\right)_{r=10} = 2\pi \times 10 = 20\pi$$

$\therefore$ from ① & ② $\frac{dA}{dt} = 20\pi \times 4 = \underline{80\pi \text{ cm}^2\text{s}^{-1}}$ ✓

---

**Example 24:** A balloon, which always remains spherical, has a variable radius. Find the rate at which its volume is increasing with the radius when radius is 10 cm.

**Solution:** Volume of sphere $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dr} = \frac{4}{3}\pi \times 3r^2 = 4\pi r^2$$

$\therefore$ Rate of change of $V = \left(\frac{dV}{dr}\right)_{r=10} = 4\pi \times 10^2 = \underline{400\pi \text{ cm}^3\text{s}^{-1}}$ ✓

**Example 25:** The point $P(x,y)$ is moving along the curve, $y = x^2 - \frac{10}{3}x^{3/2} + 5x$, in such a way that the rate of change of $y$ is constant. Find the value of $x$ at the point at which the rate of change of $x$ is equal to half the rate of change $y$. ---[7]

[S-16/13/Q7]

**Solution:** $y = x^2 - \frac{10}{3}x^{3/2} + 5x$ —— ①

Also given $\frac{dx}{dt} = \frac{1}{2}\frac{dy}{dt} \Rightarrow \frac{dy}{dt} \Big/ \frac{dx}{dx} = 2 \Rightarrow \frac{dy}{dx} = 2$ —— ②

diff. ① $\frac{dy}{dx} = 2x - \frac{10}{3} \times \frac{3}{2} x^{1/2} + 5 = 2$ from ②

$\Rightarrow 2x - 5x^{1/2} + 3 = 0$

$2y^2 - 5y + 3 = 0$ [ Let $x^{1/2} = y$

$(2y-3)(y-1) = 0$

$y = 3/2,\ y = 1$

or $x^{1/2} = 3/2,\ x^{1/2} = 1$

i.e. $x = 9/4$ ✓ or $x = 1$ ✓

---

**Example 26:** The equation of a curve is $y = 2 + \frac{3}{2x-1}$

(i) Obtain an expression for $\frac{dy}{dx}$. ---[2]

At the point P on the curve $x=2$

(ii) A point moves along the curve in such a way that its x-coordinate is decreasing at a constant rate of 0.06 units per second. Find the rate of change of the y-coordinate as the point passes through P. ---[2]

[W-16/12/Q7]

**Solution(i)** $y = 2 + \frac{3}{2x-1}$

or $y = 2 + 3(2x-1)^{-1}$

$\frac{dy}{dx} = -3(2x-1)^{-2}$

$= \frac{-3\times 2}{(2x-1)^2}$ ✓

at P,

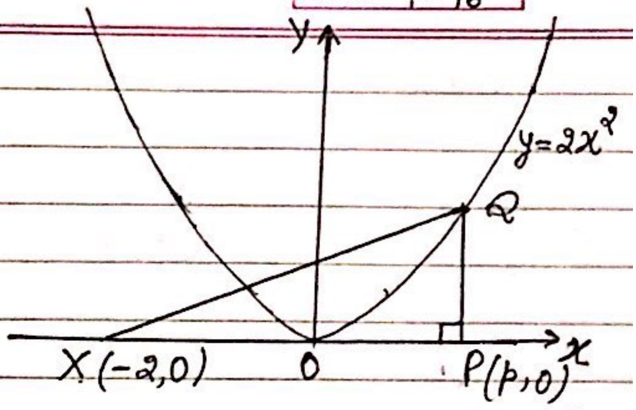
$\left(\frac{dy}{dx}\right)_{x=2} = \frac{-3\times 2}{3^2} = -\frac{2}{3}$ ✓ —— ①

Given $\frac{dx}{dn} = -0.06$ —— ②

Now $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$ (from ① & ②)

$= -\frac{2}{3} \times -0.06$

$\therefore \frac{dy}{dt} = 0.04$ ✓




Example 27: The diagram shows the curve $y=2x^2$ and the points $X(-2,0)$ and $P(p,0)$. The point Q lies on the curve and PQ is parallel to the y-axis.

(i) Express the area, A, of the triangle XPQ in term of p. ---[2]

The point P moves along the x-axis at a constant rate of 0.02 units per seconds and Q moves along the curve so that PQ remains parallel to the y-axis.

(ii) Find the rate at which A is increasing when $p=2$. ---[3]

[S-15/11/Q2]

Solution: $y=2x^2$ — ①, $X(-2,0)$ and $P(p,0)$, $Q(p,2p^2)$

(i) Area of triangle XPQ, $A=\frac{1}{2}\cdot(2+p)\times 2p^2$ [$\because QP=2p^2$]

$A=(2p^2+p^3)$ — ②

(ii) To find $\frac{dA}{dt}=\frac{dA}{dp}\cdot\frac{dp}{dt}$ — ③

Given $\frac{dp}{dt}=0.02$ — ④

Diff ② $\frac{dA}{dp}=4p+3p^2$

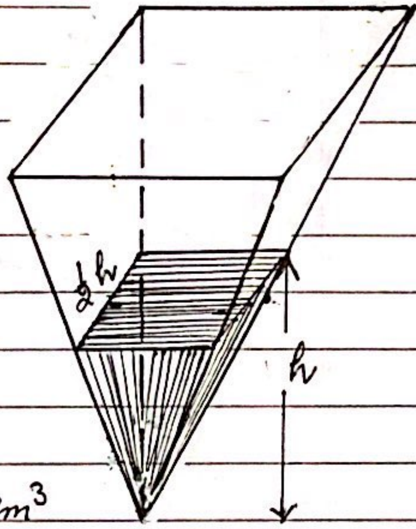
at $p=2$, $\left(\frac{dA}{dp}\right)_{p=2}=4\times2+3\times2^2=20$ — ⑤

$\therefore$ from ④ & ⑤ in ③

$\frac{dA}{dt}=20\times0.02$

$=0.4$ unit$^2$ per seconds.

Example 28: The diagram shows a water container in the form an inverted pyramid, which is such that, when the height of the water level is $h$ cm, the surface of the water is a square of side $\frac{1}{2}h$ cm.




(i) Express the volume of water in the container in terms of $h$. ---[1]

(ii) Water is steadily dripping into the container at a constant rate of 20 $cm^3$ per minute. Find the rate, in cm per minute, at which the water level is rising when the height of the water level is 10 cm. [M-17/12/Q3] ---[4]

Solution (i) Volume of water $= \frac{1}{3} \times$ area base $\times$ height

$$V = \frac{1}{3} \times \left(\frac{1}{2}h\right)^2 \times h = \frac{1}{12}h^3$$

(ii) $V = \frac{1}{12}h^3$

Diff. $\frac{dV}{dh} = \frac{1}{12} \times 3h^2$

when $h = 10$, $\left(\frac{dV}{dh}\right)_{h=10} = \frac{1}{4} \times 10^2 = 25$ —— ①

and $\frac{dV}{dt} = 20 \text{ cm}^3/\text{m}$ —— ② (Given)

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$\Rightarrow 20 = 25 \times \frac{dh}{dt} \quad (\text{fm } ① \,\&\, ②)$$

$$\Rightarrow \frac{dh}{dt} = \frac{20}{25} = 0.8 \text{ cm/minute}$$

## § Second Derivative:

$$y = f(x)$$

First derivative, $\frac{dy}{dx} = f'(x)$

second derivative $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$

**Example 29:** Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

(i) $y = x^3 - 5x^2 + 13x + 11$ : or $y = f(x)$

$f'(x)$ or $\frac{dy}{dx} = 3x^2 - 10x + 13$

$f''(x)$ or $\frac{d^2y}{dx^2} = 6x - 10.$

(ii) $y = (3x+7)^4$

$\frac{dy}{dx} = 4(3x+7)^3 \times 3$

$= 12(3x+7)^3$

$\frac{d^2y}{dx^2} = 12 \times 3(3x+7)^2 \times 3$

$= 108(3x+7)^2$

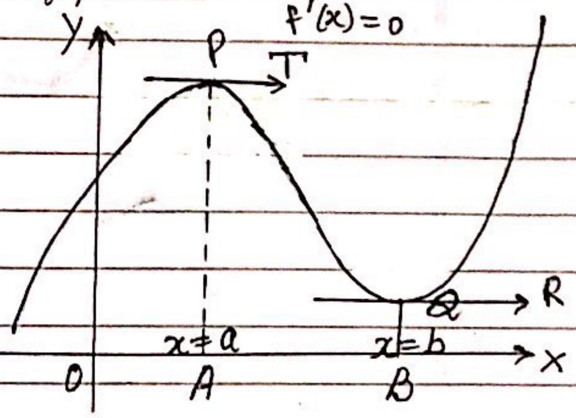
(iii) $y = \frac{1}{(ax+b)} = (ax+b)^{-1}$

$\frac{dy}{dx} = -(ax+b)^{-2} \times a = \frac{-a}{(ax+b)^2}$ ✓

$\frac{d^2y}{dx^2} = -(-2)a\,(ax+b)^{-3} \times a$

$= \frac{2a^2}{(ax+b)^3}$ ✓

## § Stationary point (or Turning point):




Given a function:

$$y = f(x)$$

The points P, Q, --- on the graph of $y=f(x)$ are such that the gradient of the curve at P, Q, --- is zero. (or $f'(x)=0$)

The points P, Q, --- are called stationary points.

## § To find stationary points:

$$y = f(x)$$

diff. $\quad \frac{dy}{dx} = f'(x)$

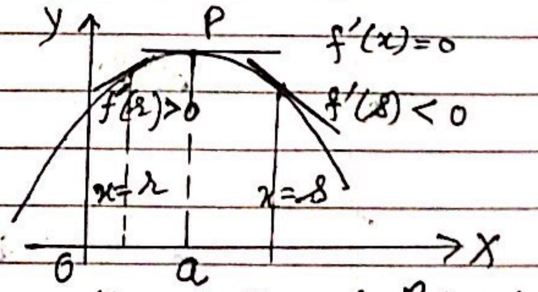
for stationary points,

Solve $f'(x)=0$

we get $x = a, b, ---$

## § To check the nature of stationary points (Max or Min).

/First derivative test/



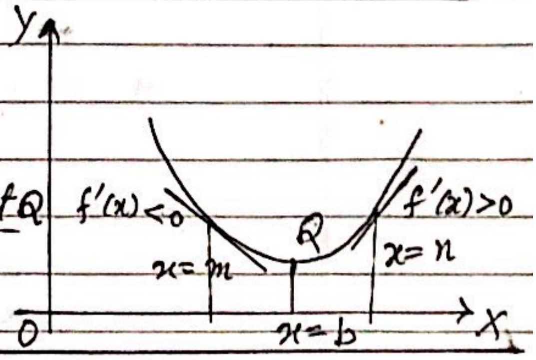

(i) At 'P' $x=a$ let $r<a<s$

on the left of $x=a$

$f'(r)>0$ and

on the right of $x=a$, $f'(s)<0$

$\therefore$ $f'(x)$ changes sign from + to -ve $\Rightarrow$ there is Max at P $(x=a)$




(ii) At Q let $m<b<n$

such that $f'(m)<0$ and $f'(n)>0$

or $f'(x)$ changes sign from -ve to + at Q $x=b$.

$\therefore$ There is a Min at Q.

**Note:** At a stationary point if $f'(x)$ does not change the sign. [NO-Max or Min. But point of Inflexion]

+ 0 +

- 0 -ve

**Example 30:** Find the coordinates of the stationary points on the curve $y = x^3 - 6x^2 + 9x - 8$ and determine the nature of these points. (First derivative test)

**Solution:**

$$y = x^3 - 6x^2 + 9x - 8 \quad \text{——} ①$$

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$
$$= 3(x^2 - 4x + 3)$$
$$= 3(x-1)(x-3)$$

for stationary points $\frac{dy}{dx} = 0 \Rightarrow 3(x-1)(x-3) = 0$
$\Rightarrow x = 1, 3$

from ① $x = 1 \rightarrow y = -4$
$x = 3 \rightarrow y = -8$

$\therefore$ Stationary point are $(1,-4)$ and $(3,-8)$

To check the nature of stationary points (Using first derivative test)

(i) at $(1,-4)$, $\frac{dy}{dx} = 3(x-1)(x-3)$

$x < 1$ $\left(\frac{dy}{dx}\right)_{x=0.9} = 3(-)(-) = +$

$x > 1$ $\left(\frac{dy}{dx}\right)_{x=1.1} = 3(+)(-) = -ve$

$\therefore$ at $x = 1$, $\frac{dy}{dx}$ changes sign from + to −ve

$\therefore$ Max at $(1,-4)$ ✓

(ii) at $(3,-8)$, $\frac{dy}{dx} = 3(x-1)(x-3)$

$x < 3$ $\left(\frac{dy}{dx}\right)_{x=2.9} = 3(+)(-) = -ve$

$x > 3$ $\left(\frac{dy}{dx}\right)_{x=3.1} = 3(+)(+) = +$

at $x = 3$, $\frac{dy}{dx}$ changes sign from −ve to +

$\therefore$ Min at $(3,-8)$ ✓

**Example 31:** Find the coordinates of the stationary points on the curve $y = x^3 - 6x^2 + 12x - 8$ and determine the nature of these points.

**Solution:**

$$y = x^3 - 6x^2 + 12x - 8 \quad \text{—— (1)}$$

$$\frac{dy}{dx} = 3x^2 - 12x + 12$$

$$= 3(x^2 - 4x + 4)$$

$$= 3(x-2)^2$$

for stationary points $\frac{dy}{dx} = 0 \Rightarrow 3(x-2)^2 = 0 \Rightarrow x = 2$ ✓

from (1), $x = 2 \Rightarrow y = 0$

$\therefore$ $(2, 0)$ is the stationary point.

To check the nature of the stationary point:

at $(2,0)$ $\quad \frac{dy}{dx} = 3(x-2)^2$

Now for $x < 2$ $\quad \left(\frac{dy}{dx}\right)_{x=1.9} = 3(-0.1)^2 > 0$

for $x > 2$ $\quad \left(\frac{dy}{dx}\right)_{x=2.1} = 3(0.1)^2 > 0$

$\therefore$ $\frac{dy}{dx}$ does not change sign at $(2,0)$

so there is no. Max or Min at $(2,0)$

it is a point of inflexion at $(2,0)$

§ <u>To check the nature of stationary point using second derivative test:</u>

Given $y = f(x)$ —— ①

diff. $\frac{dy}{dx} = f'(x)$ —— ②

diff again $\frac{d^2y}{dx^2} = f''(x)$ —— ③

To find the stationary points put $f'(x) = 0$

solve, $x = a, b, c, ---$

are the stationary points.

<u>Now to check the nature of stationary points:</u>

(using second derivative test)

(i) at $x = a$, find $f''(a) = +$ $>0$ confirms a Minimum

(ii) at $x = b$, find $f''(b) = -ve$ $<0$ confirms a Maximum.

(iii) at $x = c$, find $f''(c) = 0 \Rightarrow$ Second derivative test fails

[may be yet a Max/Min/or None.]

*Example 32:* A curve has equation: $y = \frac{8}{x} + 2x$

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ --[3]

(ii) Find the coordinates of the stationary points and state, with a reason, the nature of each stationary point. --[5]

[SP-17/01/Q5]

<u>Solution:</u> $y = \frac{8}{x} + 2x$ --(i)

(i) $\frac{dy}{dx} = -\frac{8}{x^2} + 2$ --(ii)

$\frac{d^2y}{dx^2} = -8(-2x^{-3}) = \frac{16}{x^3}$ --(iii)

(ii) for stationary points $\frac{dy}{dx} = 0$

$\Rightarrow -\frac{8}{x^2} + 2 = 0 \Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

$\therefore$ Stationary points $(2, 8), (-2, -8)$

To check the nature of stationary points:

(i) at $x = 2$, from (iii)

$\left(\frac{d^2y}{dx^2}\right)_{x=2} = \frac{16}{2^3} = 2 > 0 \therefore$ Min at $(2, 8)$ ✓

(ii) at $x = -2$, $\left(\frac{d^2y}{dx^2}\right)_{x=-2} = \frac{16}{(-2)^3} = -2 < 0$

$\therefore$ Max at $(-2, -8)$ ✓

**Example 33:** A curve has equation $y=\frac{1}{2}x^2-4x^{3/2}+8x$

(i) Find the $x$-coordinates of the stationary points. ---[5]

(ii) Find $\frac{d^2y}{dx^2}$. ---[1]

(iii) Find, showing all necessary working, the nature of each stationary point. [M-18/12/Q8] ---[2]

**Solution:** $y=\frac{1}{2}x^2-4x^{3/2}+8x$ ---- ①

(i) diff. $\frac{dy}{dx}=\frac{1}{2}\times 2x-4\times\frac{3}{2}x^{1/2}+8$

or $\frac{dy}{dx}=x-6x^{1/2}+8$ ---- ②

$=(x^{1/2}-4)(x^{1/2}-2)=0$ for stationary points $\left[\frac{dy}{dx}=0\right.$

$\Rightarrow x^{1/2}=4$ or $x^{1/2}=2$

$x=16$ or $x=4$ ✓

or Let $x^{1/2}=y$, $x=y^2$

from ②

$y^2-6y+8=0$

$(y-4)(y-2)=0$

$y=4$, $y=2$

$x^{1/2}=4$, $x^{1/2}=4$ ✓

(ii) diff. ② $\frac{d^2y}{dx^2}=1-6\times\frac{1}{2}x^{-1/2}=1-\frac{3}{\sqrt{x}}$

or $\frac{d^2y}{dx^2}=1-\frac{3}{\sqrt{x}}$ ---- ③

(iii) To check the nature of the stationary points

at $x=16$, from ③ $\left(\frac{d^2y}{dx^2}\right)_{x=16}=1-\frac{3}{\sqrt{16}}=\frac{1}{4}>0\Rightarrow$ Min. at $x=16$ ✓

at $x=4$, from ③ $\left(\frac{d^2y}{dx^2}\right)_{x=4}=1-\frac{3}{\sqrt{4}}=-\frac{1}{2}<0\Rightarrow$ Max. at $x=4$ ✓

**Example 34:** The equation of a curve is $y=8\sqrt{x}-2x$

(i) Find the coordinates of the stationary points on the curve. ---[3]

(ii) Determine the nature of the stationary point. [S-17/12/Q9] ---[2]

**Solution:** $y=8\sqrt{x}-2x$ ---- ①

(i) diff $\frac{dy}{dx}=8\times\frac{1}{2\sqrt{x}}-2$

or $\frac{dy}{dx}=\frac{4}{\sqrt{x}}-2$ ---- ②

for stationary point $\frac{4}{\sqrt{x}}-2=0$

$\Rightarrow\sqrt{x}=2\Rightarrow x=4$

from ① $y=8$

∴ stationary point $(4,8)$ ✓

(ii) diff ② $\frac{d^2y}{dx^2}=-2x^{-3/2}$

or $\frac{d^2y}{dx^2}=\frac{-2}{x\sqrt{x}}$

$\left(\frac{d^2y}{dx^2}\right)_{x=4}=\frac{-2}{4\times 2}=-\frac{1}{4}<0\rightarrow$ Max

Max at $(4,8)$ ✓

**Example 35:** Variables $u$, $x$ and $y$ are such that $u = 2x(y-x)$ and $x+3y = 12$. Express $u$ in terms of $x$ and hence find the stationary value of $u$. [S-15/12/Q4] --- [5]

**Solution:** $u = 2x(y-x)$ —— ①

and $x+3y = 12 \Rightarrow 3y = 12-x$

$\Rightarrow y = \left(4 - \frac{x}{3}\right)$ —— ③

Put the value of $y$ from ③ in ①

$u = 2x\left[\left(4 - x/3\right) - x\right]$

or $u = 2x\left(4 - \frac{4}{3}x\right)$

or $u = 8x - \frac{8}{3}x^2$ —— ④

diff. $\frac{du}{dx} = 8 - \frac{16}{3}x = 0$ for stationary point $\frac{du}{dx} = 0$

$\Rightarrow x = 3/2$

from ④ stationary value of $u = 8 \times \frac{3}{2} - \frac{8}{3}\left(\frac{3}{2}\right)^2$

$u = 12 - 6 = 6$ ✓

$\therefore$ stationary value of $u = 6$ ✓

---

**Example 36:** The equation of a curve is $y = x^3 + ax^2 + bx$, where $a$ and $b$ are constants.
In case where the curve has no stationary points, show that $a^2 < 3b$.

**Solution:** curve: $y = x^3 + ax^2 + bx$

diff. $\frac{dy}{dx} = 3x^2 + 2ax + b$

for stationary points $\frac{dy}{dx} = 0 \Rightarrow 3x^2 + 2ax + b = 0$ —— ①

as the curve has no stationary point, equation ① has no solu.

$\Rightarrow (2a)^2 - 4 \times 3 \times b < 0$ [ $B^2 - 4AC < 0$

$4a^2 - 12b < 0$

$\Rightarrow a^2 - 3b < 0$

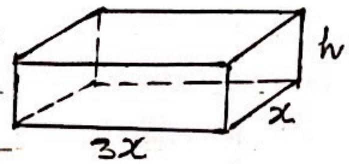
$\Rightarrow \underline{a^2 < 3b}$ ✓

Example 37: The base of a cuboid has sides of $x$ cm and $3x$ cm. The volume of the cuboid is 288 $cm^3$.

(i) Show that the total surface area of the cuboid, $A$ $cm^2$, is given by, $A = 6x^2 + \frac{768}{x}$ --- [3]

(ii) Given that $x$ can vary, find the stationary value of $A$ and determine its nature. [S-14 | 13 | Q9] --- [5]

Solution: T. Surface area of Cuboid $= 2[lb + bh + lh]$ — ①




(i) Given volume of cuboid $lbh = 288$

$$\Rightarrow x \times 3x \times h = 288$$

$$\text{or } h = \frac{288}{3x^2} = \frac{96}{x^2} \quad \text{— ②}$$

from ① and ②

The total surface area $= 2\left[3x \times x + x \times \frac{96}{x^2} + 3x \times \frac{96}{x^2}\right]$

$$\text{or } A = 6x^2 + \frac{768}{x} \quad \text{--- ③}$$

(ii) diff. ③ $\frac{dA}{dx} = 12x - \frac{768}{x^2}$ — ④ $\left[\because \frac{d}{dx}\frac{1}{x} = \frac{d}{dx}x^{-1} = -\frac{1}{x^2}\right]$

for stationary point, $\frac{dA}{dx} = 0 \Rightarrow 12x - \frac{768}{x^2} = 0$ from ④

$$\Rightarrow x^3 = 64 \Rightarrow \underline{x = 4} ✓$$

To check the nature of stationary point.

diff ④ $\frac{d^2A}{dx^2} = 12 - 768 \times \left(\frac{-2}{x^3}\right)$

$$\text{or } \frac{d^2A}{dx^2} = 12 + \frac{1536}{x^3}$$

At $x = 4$, $\left(\frac{d^2A}{dx^2}\right)_{x=4} = 12 + \frac{1536}{4^3} = 36 > 0$ $\therefore$ $\underline{A \text{ is Min at } x = 4}$ ✓

$\therefore$ Value of Min. S. area $= 6 \times 4^2 + \frac{768}{4}$ [Put $x = 4$ in ③

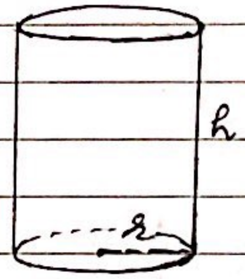
or Stationary value of $A$. $= 96 + 192 = \underline{288 \text{ cm}^2}$ ✓

Example 38: A vacuum flask (for keeping drinks hot) is modelled as a closed cylinder in which the internal radius is $r$ cm, and the internal height is $h$ cm. The volume of the flask is $1000\ cm^3$. A flask is most efficient when the total internal surface area, $A\ cm^2$, is a minimum.

(i) Show that $A = 2\pi r^2 + \dfrac{2000}{r}$ ---[3]

(ii) Given that $r$ can vary, find the value of $r$, correct to one decimal place, for which $A$ has a stationary value and verify that the flask is most efficient when $r$ takes this value

M-16/12/Q6 ---[5]




Solution: Volume $V = \pi r^2 h = 1000$

(i) $\Rightarrow h = \dfrac{1000}{\pi r^2}$ —— ①

Total surface area $= 2\pi r^2 + 2\pi r h$

$= 2\pi r^2 + 2\pi r \times \dfrac{1000}{\pi r^2}$ fm ①

or $A = 2\pi r^2 + \dfrac{2000}{r}$ —— ②

(ii) diff. ② $\dfrac{dA}{dr} = 4\pi r - \dfrac{2000}{r^2}$ —— ③

for stationary point $\dfrac{dA}{dr} = 0 \Rightarrow 4\pi r - \dfrac{2000}{r^2} = 0$ fm ③

$\Rightarrow r^3 = \dfrac{2000}{4\pi} = 159$

$r = 5.4$ cm.

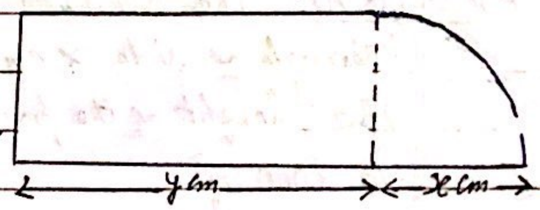
To check the nature of stationary point:

diff ③ $\dfrac{d^2A}{dr^2} = 4\pi + \dfrac{4000}{r^3}$

at $r = 5.4$ $\left(\dfrac{d^2A}{dr^2}\right)_{r=5.4} = \left(4\pi + \dfrac{4000}{159}\right) > 0$ hence A is Min

Surface area of the flask is Minimum, hence it is most efficient for $r = 5.4$ cm.

**Example 39:**




The diagram shows a metal plate consisting of a rectangle with sides $x$ cm and $y$ cm and a quarter-circle of radius $x$ cm. The perimeter of the plate is 60 cm.

(i) Express $y$ in terms of $x$. ---[2]

(ii) Show that the area of the plate, $A\,cm^2$, is given by, $A = 30x - x^2$ ---[2]

(iii) Given that $x$ can vary, find the value of $x$ at which $A$ is stationary. ---[2]

(iv) Find this stationary value of $A$, and determine whether it is a maximum or a minimum value. $\boxed{W-10/11/Q8}$ ---[2]

**Solution** (i) $2x + 2y + \frac{1}{2}\pi x = 60$

$2y = 60 - 2x - \frac{\pi x}{2}$

$y = 30 - x - \frac{\pi x}{4}$ ✓ —— (1)

(ii) $A = xy + \frac{1}{4}\pi x^2$ from (1)

$= x(30 - x - \frac{1}{4}\pi x) + \frac{1}{4}\pi x^2$

$= 30x - x^2 \cancel{- \frac{1}{4}\pi x^2} \cancel{+ \frac{1}{4}\pi x^2}$

$\therefore A = 30x - x^2$ —— (2)

(iii) diff (2) $\frac{dA}{dx} = 30 - 2x$ —— (3)

$A$ is stationary for $\frac{dA}{dx} = 0$

from (3) $30 - 2x = 0$

or $x = 15$ cm ✓

(iv) diff (3) $\frac{d^2A}{dx^2} = -2 < 0$ $\therefore$ Maximum at $x = 15$

from (2) Stationary value of $A = 30 \times 15 - 15^2 = 450 - 225 = 225\ cm^2$
at $x = 15$,

$\therefore$ Max value of $A = 225\,cm^2$ ✓

**Example 40:** The horizontal base of a solid prism is an equilateral triangle of side $x$ cm. The sides of the prism are vertical. The height of the prism is $h$ cm, and the volume of the prism is $2000\ cm^3$.

(i) Express $h$ in terms of $x$ and show that the total surface area of the prism, $A\ cm^2$, is given by, $A = \frac{\sqrt{3}}{2}x^2 + \frac{24000}{\sqrt{3}} x^{-1}$ ---[3]

(ii) Given that $x$ can vary, find the value of $x$ for which $A$ has a stationary value. ---[3]

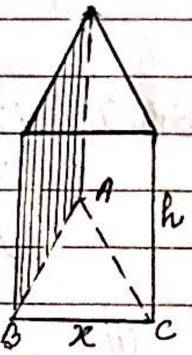
(iii) Determine, showing all necessary working, the nature of this stationary value. /S-17/11/Q6/ ---[2]

**Solution** (i) Volume of prism = area base × height

$= \text{area of } \triangle ABC \times h$

$= \frac{\sqrt{3}}{4}x^2 h = 2000$ (Given)

$\Rightarrow h = \frac{8000}{\sqrt{3}x^2}$ —— ①

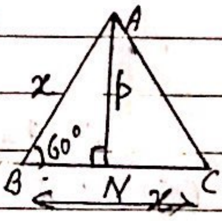



Total surface area = 2 × base $\triangle$ + 3 vertical sides (rectangles)

$= 2 \times \frac{\sqrt{3}}{4}x^2 + 3xh$

$= \frac{\sqrt{3}}{2}x^2 + 3x\left(\frac{8000}{\sqrt{3}x^2}\right)$

$A = \frac{\sqrt{3}}{2}x^2 + \frac{24000}{\sqrt{3}}x^{-1}$ —— ②




$AN \perp BC$, $AN = p$

$\frac{p}{x} = \sin 60$

$p = \frac{\sqrt{3}}{2}x$

Area of equilateral $\triangle$

$= \frac{1}{2} \times x \times \frac{\sqrt{3}}{2}x$

$= \frac{\sqrt{3}}{4}x^2$ ✓

(ii) Diff ② $\frac{dA}{dx} = \sqrt{3}x - \frac{24000}{\sqrt{3}}x^{-2}$ —— ③

for stationary point $\frac{dA}{dx} = 0$

for ③ $\Rightarrow \sqrt{3}x - \frac{24000}{\sqrt{3}x^2} = 0$

$\Rightarrow x^3 = 8000 \rightarrow x = 20$ cm ✓

For stationary value of $A$, $x = 20$ ✓

(iii) For nature, diff ③ $\frac{d^2A}{dx^2} = \sqrt{3} + \frac{48000}{\sqrt{3}x^3}$

$\left(\frac{d^2A}{dx^2}\right)_{x=20} = \sqrt{3} + \frac{48000}{\sqrt{3} \times 8000} > 0$

∴ Area of Prism is <u>Min</u> at <u>$x = 20$</u>.