Def.: Quadratic function:
A polynomial of second degree,
$$ax^2+bx+c \qquad a\neq 0,\ x\in\mathbb{R}.$$
is called a quadratic function.

Example: (i) $5x^2-8x+13$ (ii) $3x^2+7x$ (iii) $6x^2$
are all quadratic functions.

§ Factorization of a quadratic function:
$$ax^2+bx+c$$

Particular case: when $a=1$

To factorize: x^2+bx+c

Case I when c is +
Take two factors of c whose sum is b. (both + or both −ve)

(i) x^2+5x+6
$= x^2+3x+2x+6$
$= x(x+3)+2(x+3)$
$= (x+3)(x+2)$ ✓

(ii) $x^2-7x+10$
$x^2-5x-2x+10$
$x(x-5)-2(x-5)$
$(x-5)(x-2)$ ✓

Case II when c is −ve take two factors of c whose difference is b.

(iii) x^2-x-6
$= x^2-3x+2x-6$
$= x(x-3)+2(x-3)$
$= (x-3)(x+2)$ ✓

(iv) $x^2+9x-52$
$= x^2+13x-4x-52$
$= x(x+13)-4(x+13)$
$= (x+13)(x-4)$ ✓

for ax^2+bx+c
Consider ac.

Case III If ac is +

(i') $3x^2+11x+6$ $\quad \left\{\begin{array}{l}6\times 3=18\\ 9\times 2=18\\ 9+2=11\end{array}\right.$
$=3x^2+9x+2x+6$
$=3x(x+3)+2(x+3)$
$=(x+3)(3x+2)$ ✓

(ii') $4x^2-16x+7$ $\quad \left\{\begin{array}{l}7\times 4=28\\ 14\times 2=28\end{array}\right.$
$=4x^2-14x-2x+7$
$=2x(2x-7)-1(2x-7)$
$=(2x-7)(2x-1)$ ✓

Case III when ac is −ve take two factors whose difference is b.

(iii') $2x^2-x-10$ $\quad \left\{\begin{array}{l}2\times 10=20\\ 5\times 4=20\end{array}\right.$
$=2x^2-5x+4x-10$
$=x(2x-5)+2(2x-5)$
$=(2x-5)(x+2)$ ✓

(iv') $8x^2-2x-15$ $\quad \left\{\begin{array}{l}8\times 15=120\\ 12\times 10=120\end{array}\right.$
$=8x^2-12x+10x-15$
$=4x(2x-3)+5(2x-3)$
$=(2x-3)(4x+5)$ ✓

§ Quadratic Equation:

$$ax^2+bx+c=0 \qquad a\neq 0,\ a,b,c\in R$$

§ To Solve a quadratic equation:

(1) (Splitting the middle term).

(i) Solve: $3x^2+11x+6=0$

$$3x^2+9x+2x+6=0$$
$$3x(x+3)+2(x+3)=0$$
$$(x+3)(3x+2)=0$$
$$\Rightarrow (x+3)=0 \text{ or } 3x+2=0$$
$$x=-3 \text{ or } -2/3 \checkmark$$

(ii) $8x^2-2x-15=0$

$$8x^2-12x+10x-15=0$$
$$4x(2x-3)+5(2x-3)=0$$
$$(2x-3)(4x+5)=0$$
$$\Rightarrow 2x-3=0 \text{ or } 4x+5=0$$
$$x=3/2 \text{ or } x=-\frac{5}{4} \checkmark$$

(2) Using Quadratic formula:

$$ax^2+bx+c=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Discriminant: $D=b^2-4ac$

Nature of Roots (Solution) of Quad. Eqn:

(i) $D>0$, Two real and distinct roots, $(D=b^2-4ac)$

(ii) $D=0$ Two real and equal roots

(iii) $D<0$ No real roots.

(3) Determine the nature of roots of the following quad. equations and find the real roots if possible:

(i) $2x^2 - x - 10 = 0$

$a = 2,\ b = -1,\ c = -10$

$D = b^2 - 4ac$
$= (-1)^2 - 4 \times 2 \times (-10)$
$= 1 + 80$
$= 81 > 0$

$\therefore$ It has two real roots.

$x = \dfrac{-b \pm \sqrt{D}}{2a} = \dfrac{-(-1) \pm \sqrt{81}}{2 \times 2}$

$= \dfrac{1 \pm 9}{4} = \dfrac{5}{2}$ or -2 ✓

(ii) $4x^2 + 20x + 25 = 0$

$a = 4,\ b = 20,\ c = 25$

$D = b^2 - 4ac$
$= (20)^2 - 4 \times 4 \times 25$
$= 400 - 400 = 0$

$D = 0$

$\therefore$ Two real and equal roots

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-20 \pm \sqrt{0}}{2 \times 4}$

Two repeated roots $= -\frac{5}{2}, -\frac{5}{2}$

or Exactly one root $-\frac{5}{2}$ ✓

(iii) $x^2 + x + 1 = 0$

$a = 1, b = 1, c = 1$

$D < 0$

$D = b^2 - 4ac$
$= (1)^2 - 4 \times 1 \times 1$
$= -3 < 0$

No Real Roots ✓

§ To Express a Quadratic Polynomial in the form.
$a(x+b)^2+c$

1 Examples when coff of $x^2 = 1$

$\left[(x+p)^2 = x^2+2px+p^2 \right.$
here the constant term
$p^2 = \left(\frac{\text{Coeff of } x}{2}\right)^2$

(i) x^2+6x+7
$= x^2+6x+\left(\frac{6}{2}\right)^2-9+7$
$= (x^2+6x+3^2)-2$
$= (x+3)^2-2$ ✓

Constant $= \left(\frac{6}{2}\right)^2 = 3^2$

(ii) $x^2-7x+10$
$x^2-7x+\left(\frac{7}{2}\right)^2-\frac{49}{4}+10$
$= \left(x-\frac{7}{2}\right)^2-\frac{9}{4}$ ✓

Constant $= \left(\frac{7}{2}\right)^2 = \frac{49}{4}$
add & Subtract

2. Coeff of $x^2 \neq 1$

(i) $-x^2+6x-5$
$= -[x^2-6x+5]$
$= -[x^2-6x+3^2-9+5]$
$= -[(x-3)^2-4]$
$= 4-(x-3)^2$ ✓

add & Sub.
Const $= \left(\frac{\text{Coeff of } x}{2}\right)^2$
$= \left(\frac{6}{2}\right)^2 = 3^2$

(ii) $2x^2+5x-3$
$= 2\left[x^2+\frac{5}{2}x-\frac{3}{2}\right]$
$= 2\left[x^2+\frac{5}{2}x+\left(\frac{5}{4}\right)^2-\frac{25}{16}-\frac{3}{2}\right]$
$= 2\left[\left(x+\frac{5}{4}\right)^2-\frac{49}{16}\right]$
$= 2\left(x+\frac{5}{4}\right)^2-\frac{49}{8}$ ✓

add & Sub
$\left(\frac{5}{2}\times\frac{1}{2}\right)^2 = \left(\frac{5}{4}\right)^2$ or $\frac{25}{16}$

§ To Express a quad. poly. in the form,

$$(ax+b)^2+c$$

$(ax+b)^2 = a^2x^2+2abx+b^2 = (ax+b)^2$

Constant term $b^2 = \left(\frac{\text{coeff of } x}{2a}\right)^2$

Example (i) $4x^2+6x+1$

$= (2x)^2 + 2\times 2x\times\frac{3}{2}x + \left(\frac{3}{2}\right)^2 - \frac{9}{4} + 1$

$= \left(2x+\frac{3}{2}\right)^2 - \frac{5}{4}$ ✓

Constant $= \left(\frac{6}{2\times 2}\right)^2 = \left(\frac{3}{2}\right)^2$ or $\frac{9}{4}$

(ii) $9x^2-6x-2$

$= (3x)^2 - 2\times 3x\times 1 - 2$

$= (3x)^2 - 2\times 3x\times 1 + 1 - 1 - 2$

$= (3x-1)^2 - 3$ ✓

Const $= \left(\frac{6}{3\times 2}\right)^2 = 1^2$

---

§ To Solve Equations reducible to quad equ^n:

Example 1. Solve: $x^4 - 2x^2 - 3 = 0 \quad : x\in R$

Let $x^2 = y \Rightarrow$ $y^2 - 2y - 3 = 0$

$y^2 - 3y + y - 3 = 0$

$y(y-3) + 1(y-3) = 0$

$(y-3)(y+1) = 0$

$\Rightarrow y = 3$ or $y = -1$

$\Rightarrow x^2 = 3$ or $x^2 = -1$ ✗ as $x^2 \geq 0$

$\Rightarrow x = \pm\sqrt{3} = \sqrt{3}$ or $-\sqrt{3}$ ✓

2. Solve: $x - \sqrt{x} - 6 = 0$ Let $\sqrt{x} = y \Rightarrow x = y^2$

$\Rightarrow y^2 - y - 6 = 0$

$y^2 - 3y + 2y - 6 = 0$

$y(y-3) + 2(y-3) = 0$

$(y-3)(y+2) = 0$

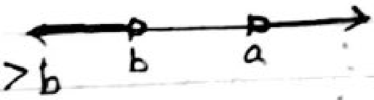
$y = 3$ or $y = -2$

$\sqrt{x} = 3$ or $\sqrt{x} = -2$ ✗

$x = 9$ ✓

§ Solution of Quadratic Inequalities:

Some Properties of Inequalities:



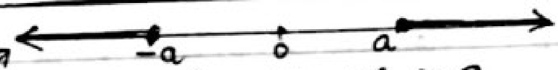
1. $(x-a)(x-b) > 0$: $a > b$

$\Rightarrow x < b$ or $x > a$

2. $(x-a)(x-b) \leqslant 0$

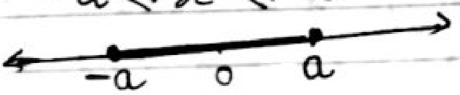
$\Rightarrow b \leqslant x \leqslant a$

3. $a > 0$



(i) $x^2 \geqslant a^2 \Rightarrow x < -a$ or $x > a$

(ii) $x^2 \leqslant a^2 \Rightarrow -a \leqslant x \leqslant a$

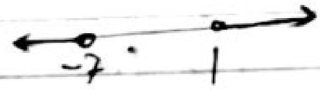


Example (i) Solve: $x^2+6x+2 > 9$ $\boxed{W-16/11/Q1(ii)}$

$\Rightarrow x^2+6x+2-9 > 0$

$\Rightarrow x^2+6x-7 > 0$

$\Rightarrow (x+7)(x-1) > 0$



$x = -7$ or $+1$

$\Rightarrow x < -7$ or $x > 1$

(ii) Solve: $4x^2-12x > 7$ $\boxed{S-14/11/Q2}$

or $4x^2-12x-7 > 0$

$(2x-7)(2x+1) > 0$

$x = \frac{7}{2}, -\frac{1}{2}$

$\Rightarrow x < -\frac{1}{2}$ or $x > \frac{7}{2}$

(iii) Solve $x^2-x-2 > 0$ $\boxed{W-13/13/Q1}$

$(x-2)(x+1) > 0$

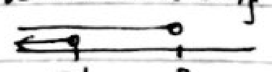
$2, -1$

$x < -1$ or $x > 2$

$\because ab > 0 \Rightarrow \{a > 0$ and $b > 0\}$ or $\{a < 0$ and $b < 0\}$

$\Rightarrow \{x-2 > 0$ and $x+1 > 0\}$ or $\{x-2 < 0$ and $x+1 < 0\}$

$\Rightarrow \{x > 2$ and $x > -1\}$ or $\{x < 2$ and $x < -1\}$



$x > 2$ or $x < -1$

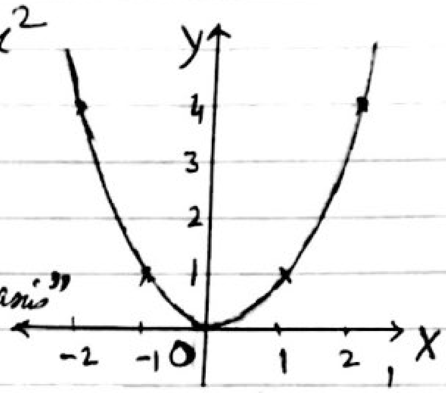
§ To Draw the graph of a Quadratic function:

$$y = ax^2 + bx + c$$

Case I Coeff. of x^2, $a > 1$, Graph is parabola opening upwards.

Example 1. Draw the graph of $y = x^2$

x	-2	-1	0	1	2
y	4	1	0	1	4



Vertex at $O(0,0)$

"axis of Symmetry of graph is Y-axis"

Example 2 draw the graph of,

$y = x^2 - 5x + 6$

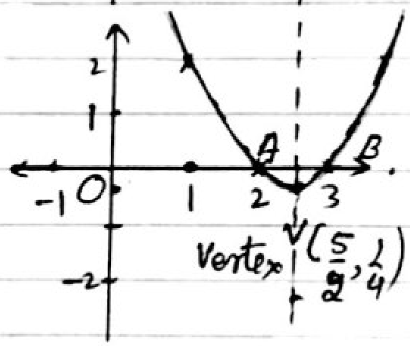
$= (x-2)(x-3)$

$y = 0$ at $x = 2$ and 3

Passes through $(2,0)$ & $(3,0)$ ✓

$x = 1, y = 2$, $(1,2)$

$x = 4, y = 2$ $(4,2)$



x		1	2	$5/2$	3	4
y		2	0	$\frac{1}{4}$	0	2

Vertex $V(\frac{5}{2}, \frac{1}{4})$

line of Symmetry is $x = \frac{5}{2}$

we observe that x-coordinate of Vertex is the mid point of the points A, B where the curve cuts X-axis.

Alternate Method

$y = x^2 - 5x + 6$. By Expressing quad in completing Square.

$= x^2 - 5x + \left(\frac{5}{2}\right)^2 - \frac{25}{4} + 6$

$= \left(x - \frac{5}{2}\right)^2 - \frac{1}{4}$

Vertex, $V = \left(\frac{5}{2}, -\frac{1}{4}\right)$

line of Symmetry is $x = \frac{5}{2}$

The function has a minimum at its Vertex

Note 1.

$y = (x-b)^2 + c$

V is at (b, c)

Note: $y = x^2$ by translation $\binom{b}{c}$ gives $y = (x-b)^2 + c$

Case II Graph of $y=ax^2+bx+c$ when $a<0$

The graph opens downwards, and has a Maximum at the vertex at $x=-\frac{b}{2a}$

Example 3.

(i) Draw the graph of: $y=-x^2+6x-5$, by expressing it in the form $a(x+b)^2+c$. [3]

The function $f: x \to -x^2+6x-5$ is defined for $x \geq m$, where m is a constant.

(ii) State the smallest value of m for which f is one-one. [1]

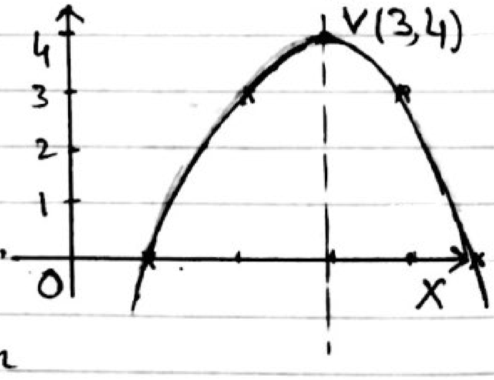
(iii) for the case when $m=5$, find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. $\boxed{SP-17/01/Q9}$ [4]

Solution: (i) $y=-x^2+6x-5$

$=-[x^2-6x+5]=-[x^2-6x+3^2-9+5]$

$\therefore\ y=-[(x-3)^2-4]=(-1)(x-3)^2+4$ ✓

$y=4-(x-3)^2$ has Vertex at $(3,4)$



(ii) $f(x)=-x^2+6x-5 \quad x \geq m$

$=4-(x-3)^2$

smallest value of $m=3$ ✓

as for $x \geq 3$ the curve is decreasing hence One-One

(iii) $f(x)=4-(x-3)^2$ for $x \geq m$

the curve is one-one ($\because$ decreasing) (f^{-1} exists)

$y=4-(x-3)^2$

$\Rightarrow (x-3)^2=4-y$

$(x-3)=\pm\sqrt{4-y}$

$x=3+\sqrt{4-y}$

$\therefore\ f^{-1}(x)=3+\sqrt{4-x}$ ✓

domain of f^{-1} is $x \leq 0$

"P_1" Quadratics Notes

§ To find the points of Intersection of a line and a quadratic function:

Q1 Find the set of Values of k for which the equation $2x^2 + 3kx + k = 0$ has distinct real roots. [4]

$\boxed{M-17/12/Q1}$

Solution $2x^2 + 3kx + k = 0$

has distinct real roots if the discriminant $b^2 - 4ac > 0$

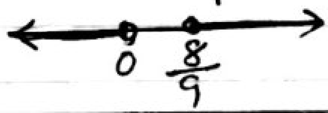
$a = 2$, $b = 3k$, $c = k$

$\Rightarrow (3k)^2 - 4 \times 2 \times k > 0$

$\Rightarrow 9k^2 - 8k > 0$

$\Rightarrow k(9k - 8) > 0$ root of corresponding equation are $0, \frac{8}{9}$

$\Rightarrow \quad k < 0$ or $k > \frac{8}{9}$



Q2 (a) Find the values of the constant m for which $y = mx$ is a tangent to the curve $y = 2x^2 - 4x + 8$ --- [3]

(b) The function f is defined for $x \in R$, by $f(x) = x^2 + ax + b$, where a and b are constants. The solution of the equation $f(x) = 0$ are $x = 1$ and $x = 9$. Find

(i) the Values of a and b [2]

(ii) the coordinates of the Vertex of the curve $y = f(x)$ [2]

$\boxed{S-16/11/Q6}$

Solution: $y = 2x^2 - 4x + 8$ and $y = mx$

for line is tangent to a curve this will intersect at one point

for Intersection; $2x^2 - 4x + 8 = mx$

$\Rightarrow 2x^2 - (4+m)x + 8 = 0$

for tangent $b^2 - 4ac = 0$

$a = 2$, $b = -(4+m)$, $c = 8$

$\Rightarrow \{-(4+m)\}^2 - 4 \times 2 \times 8 = 0$

$\Rightarrow m^2 + 8m - 48 = 0 \Rightarrow m = 4; -12$ ✓

(continued)

Q2(b) on next page

Q2(b)(i) $f(x) = x^2 + ax + b$

given $f(1) = 0 \Rightarrow 1 + a + b = 0 \Rightarrow a + b = -1$

and $f(9) = 0 \Rightarrow 81 + 9a + b = 0 \Rightarrow 9a + b = -81$

Solving $a = -10,\ b = 9$ ✓

(ii) for $a = -10,\ b = 9$

$y = f(x) = x^2 - 10x + 9$. —— (1)

Curve intersects x-axis at $x = 1$ and $x = 9$

$\therefore$ x-coord. of Vertex $= \frac{1+9}{2} = 5$

fr (1) $y = (5)^2 - 10 \times 5 + 9$

$y = -16$

Vertex $(5, -16)$ ✓

Alternate method

$f(x) = x^2 - 10x + 9$ completing the square

$= x^2 - 10x + 5^2 - 25 + 9$

$= (x-5)^2 - 16$

$\therefore$ Vertex $V(5, -16)$

$\because f(x) = (x-a)^2 + b$, $V(a, b)$

Q3 (i) Express $9x^2 - 12x + 5$ in the form, $(ax+b)^2 + C$ [3]

(ii) determine whether $3x^3 - 6x^2 + 5x - 12$ is an increasing function, a decreasing function or neither. [3]

W-14/13/Q3

Solu. (i) $9x^2 - 12x + 5 = (3x)^2 - 2 \times 3x \times 2 + 2^2 - 4 + 5$

$= (3x-2)^2 + 1$ ✓ —— (1)

(ii) Let $f(x) = 3x^3 - 6x^2 + 5x - 12$

$\Rightarrow f'(x) = 9x^2 - 12x + 5$

$= \{(3x-2)^2 + 1\} > 0$

$\Rightarrow f'(x) > 0$ for $x \in R$

$\because (3x-2)^2 \geq 0$, $(3x-2)^2 + 1 > 0$

$\therefore f(x)$ is an increasing function.

x —— x —— x