DATE: 07/11/20

P-1

Pure Maths-1

Trigonometry-Graphs and Transformations Notes.

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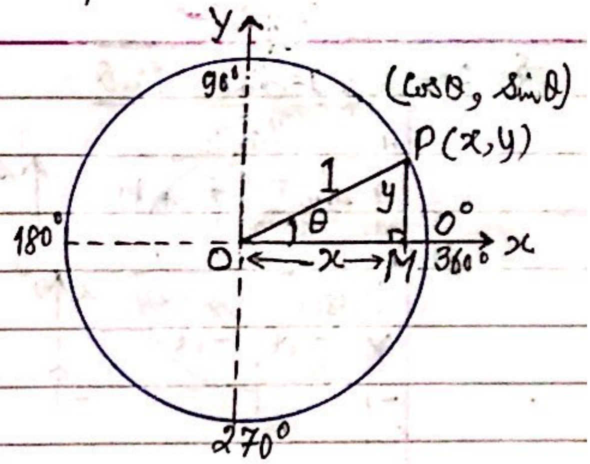
Trigonometry: Graphs and Transformations

§ Previous knowledge



Circular functions:

Draw a Unit circle, $OP = r = 1$
Let $P(x, y)$ is any point on the circle, angle $POX = \theta$, $PM \perp OX$.

we define: $\sin\theta = \frac{y}{1} = y$

$\cos\theta = \frac{x}{1} = x$ $\quad \therefore \underline{P(x,y) \equiv (\cos\theta, \sin\theta)}$

$\tan\theta = \frac{y}{x}$

$\therefore$ P is any point on the unit circle then its x-coordinate gives → $\cos\theta$; and y-coordinate gives → $\sin\theta$.

Values of some particular angles.

θ	$0°/0$ rad.	$30°/\frac{\pi}{6}$	$45°/\frac{\pi}{4}$	$60°/\frac{\pi}{3}$	$90°/\frac{\pi}{2}$ rad	$180°/\pi$	$270°/\frac{3\pi}{2}$	$360°/2\pi$ rad
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$90^-/90^+$ $\infty/-\infty$	0	$270^-/270^+$ $\infty/-\infty$	0

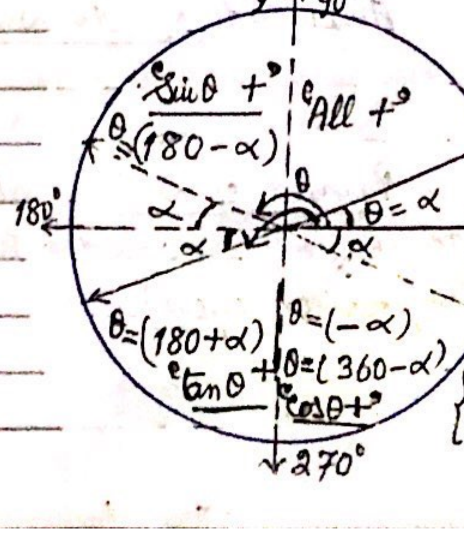
π radian $= 180°$
$1° = \frac{\pi}{180}$ radians

$\begin{cases} \sin(180-\alpha) = \sin\alpha \\ \cos(180-\alpha) = -\cos\alpha \end{cases}$

$\begin{cases} \sin(180+\alpha) = -\sin\alpha \\ \cos(180+\alpha) = -\cos\alpha \end{cases}$

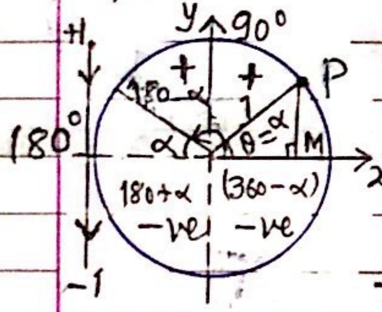
$\begin{cases} \sin(360-\alpha) = -\sin\alpha \\ \cos(360-\alpha) = \cos\alpha \end{cases}$

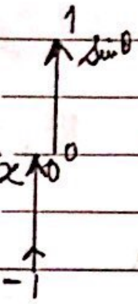
$\begin{cases} \sin(360+\alpha) = \sin\alpha \\ \cos(360+\alpha) = \cos\alpha \end{cases}$



$\begin{cases} \sin(-\alpha) = -\sin\alpha \\ \cos(-\alpha) = \cos\alpha \end{cases}$

§ Graph of $y = \sin\theta$





$0° \leq \theta \leq 360°$

$y = \sin\theta = PM$ "y" Increases

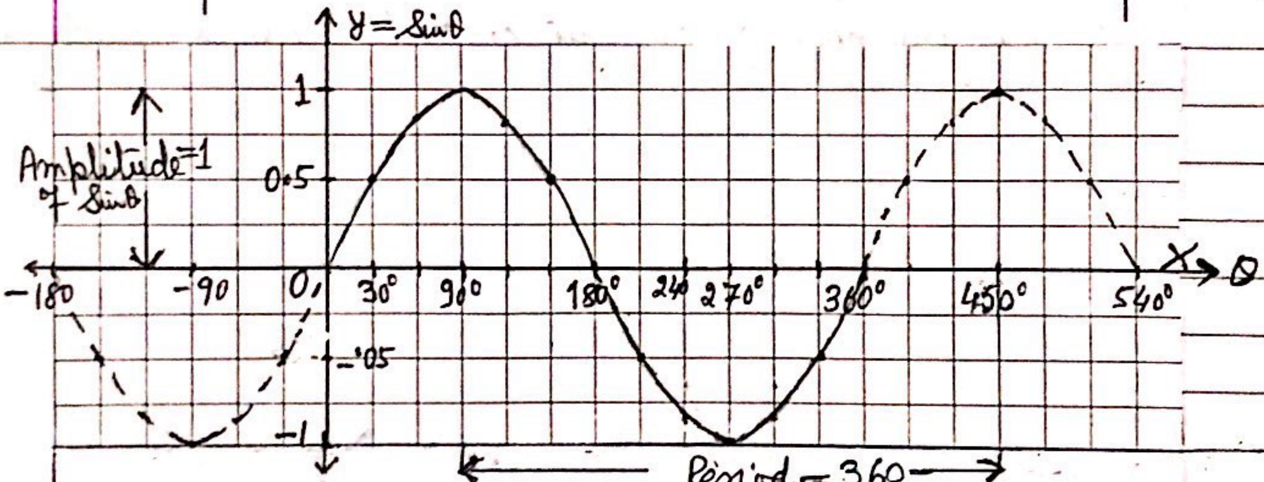
$\theta : 0° \to 90° \Rightarrow \sin\theta : 0 \to 1$

$\theta : 90° \to 180° \Rightarrow \sin\theta : 1 \xrightarrow{\text{dec.}} 0$

$\theta : 180° \to 270° \Rightarrow \sin\theta : 0 \xrightarrow{\text{dec.}} -1$ } decrease $1 \to -1$

$\theta : 270 \to 360 \Rightarrow \sin\theta : -1 \xrightarrow{\text{Inc.}} 0$

	−ve			+				+			−ve			−ve			+		
	IVth			$\theta = \alpha$ Ist quadrant				IInd $\theta = 180 - \alpha$			IIIrd $\theta = 180 + \alpha$			IVth			Ist $\theta = 360 + \alpha$		
θ	−90°	−60°	−30°	0°	30°	60°	90°	120°	150	180	210	240	270°	300	330	360	390	420	450
$\sin\theta$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
	−1	−.87	−.5	0	0.5	0.87	1	0.87	0.5	0	−0.5	−0.87	−1	−.87	−.5	0	0.5	0.87	1



{ Domain of $\sin\theta$ is $\theta \in R$ "Set of real numbers"
{ Range of $\sin\theta$ $-1 \leq \sin\theta \leq 1$ or $|\sin\theta| \leq 1$

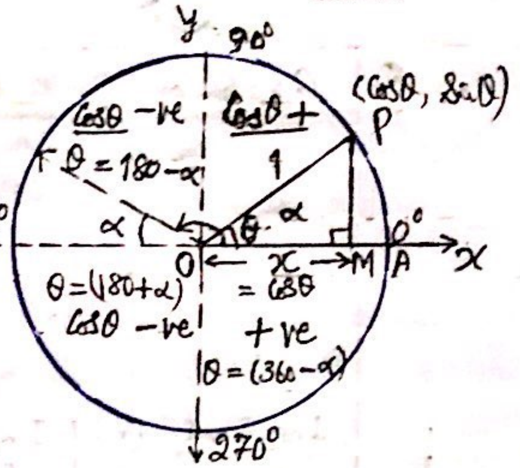
<u>Amplitude of $y = \sin\theta$ is '1'</u> (Amplitude is the distance between the maximum (or Min) and the principal axis) OX

<u>Period of $\sin\theta = 360°$ (or 2π radians)</u>
(The period of periodic function is the length of one repetition cycle).

Note:
for $y = a\sin bx$
Period $= 360°/b$
Amplitude $= a$.

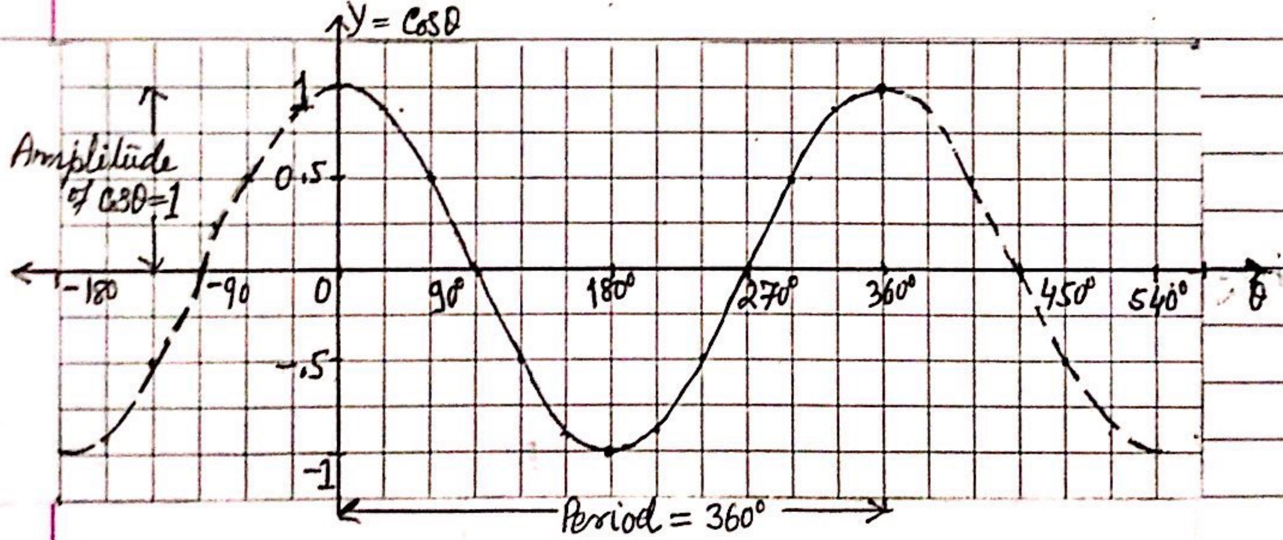
§ Graph of $y = \cos\theta$; $0^\circ \le \theta \le 360^\circ$

$\cos\theta = \frac{x}{1} = OM$ ✓



θ : $0^\circ \to 90^\circ \Rightarrow \cos\theta : 1 \to 0$
$90^\circ \to 180^\circ \Rightarrow \cos\theta : 0 \to -1$ } Decreases $1 \to -1$
$180^\circ \to 270^\circ \Rightarrow \cos\theta : -1 \to 0$
$270^\circ \to 360^\circ \Rightarrow \cos\theta : 0 \to 1$ } Increases -1 to 1.

	+			+				−ve			−ve			+			+		
	IVth ($-\alpha$)			$\theta=\alpha$ Ist quad.				$\theta=180-\alpha$ IInd			$\theta=180+\alpha$ IIIrd			$\theta=360-\alpha$ IVth			$\theta=360+\alpha$ Ist		
θ	-90°	-60°	-30°	0	30°	60°	90°	120°	150°	180°	210°	240°	270°	300	330°	360°	390°	420°	450°
$\cos\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0
	0	0.5	.87	1	0.87	0.5	0	−.5	−.87	−1	−.87	−.5	0	0.5	.87	1	.87	0.5	0



{ Domain of $\cos\theta$ is $\theta \in R$ "set of real numbers"
Range of $\cos\theta$ is $-1 \le \cos\theta \le 1$ or $|\cos\theta| \le 1$

<u>Amplitude of $y = \cos\theta$ is 1</u>

Period of $\cos\theta = 360^\circ$ (or 2π radians)

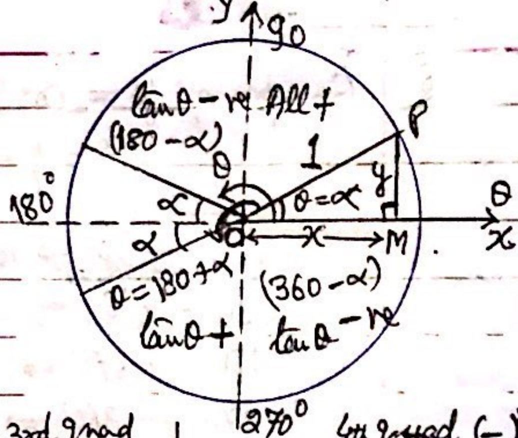
<u>Note</u>: $y = a\cos bx$

(i) Period $= \frac{360^\circ}{b}$

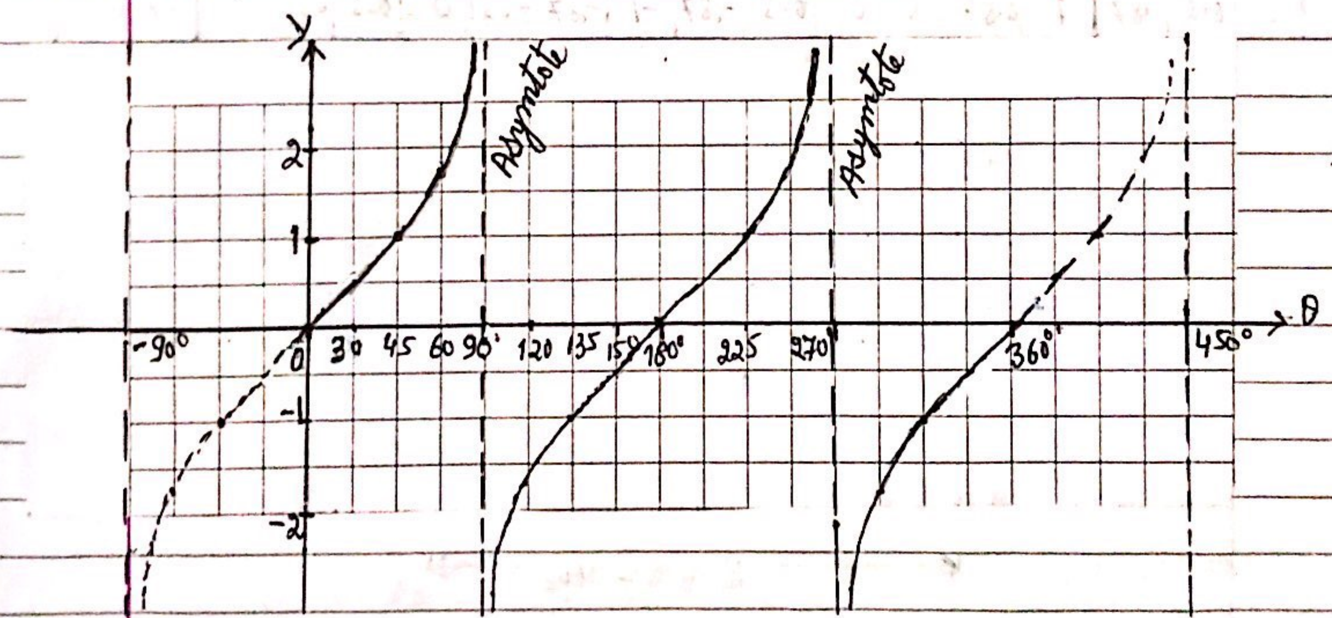
(ii) Amplitude $= a$

§ Graph of $y = \tan\theta$.

$$\tan\theta = \frac{PM}{OM} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$



	Ist quad. (+)					IInd quad. −ve					+ 3rd quad				4th quad (−)				
	$\theta = \alpha$ +					$\tan\theta = \tan(180-\alpha) = -\tan\alpha$					$\tan\theta = \tan(180+\alpha) = \tan\alpha$				$\tan\theta = \tan(360-\alpha) = -\tan\alpha$				
θ	0°	30°	45°	60°	90^-	90^+	120^+	135°	150°	180°	210°	225°	240°	270^-	270^+	300°	315°	330°	360°
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$+\infty$	$-\infty$	$-\sqrt{3}$	−1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$+\infty$	$-\infty$	$-\sqrt{3}$	−1	$-\frac{1}{\sqrt{3}}$	0
	0	0.58	1	1.7	$+\infty$	$-\infty$	−1.7	−1	−.58	0	.58	1	1.7	$+\infty$	$-\infty$	−1.7	−1	−.58	0



For the graph of $y = \tan\theta$
(i) Period $= 180°$ (π radians)
(ii) No Max or Min (amplitude ✗)
(iii) Asymtotes at $\pm 90°$ and $\pm 270°$

Note:
For $y = a\tan bx$
Period $= \dfrac{180°}{b}$

§ Inverse Trigonometric Function:

Given $x = \sin\theta$, $\theta \in R$

$\Rightarrow \theta = \sin^{-1} x \qquad -1 \le x \le 1$

but to each value of x, there should be a unique value of θ.

$\therefore \theta = \sin^{-1} x$; $-1 \le x \le 1$

$-90^\circ \le \theta \le 90^\circ$

Example:

Given: $\sin\theta = \frac{1}{2}$

$\Rightarrow \sin^{-1}\frac{1}{2} = \theta$

Now $\sin 30^\circ = \frac{1}{2}$

also $\sin 150^\circ = \frac{1}{2}$

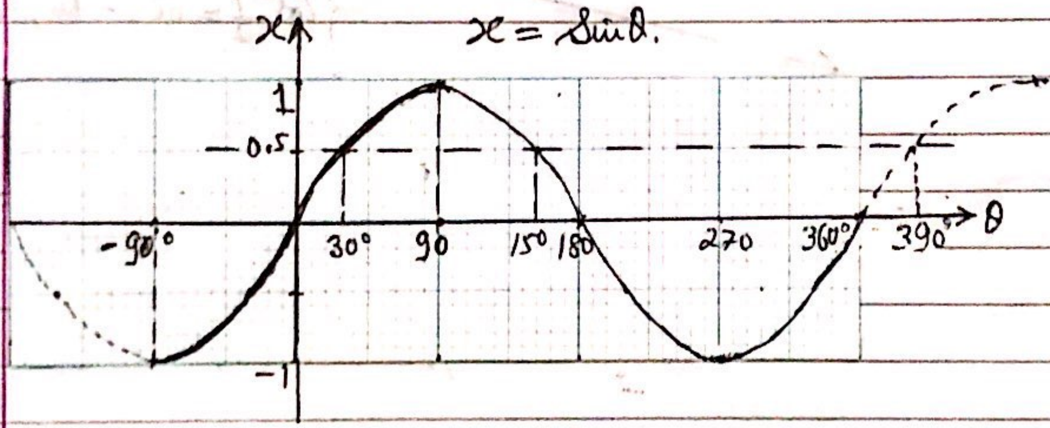
$\sin 390^\circ = \frac{1}{2}$

$\therefore \sin^{-1}\frac{1}{2} = 30^\circ, \cancel{150^\circ}, \cancel{390^\circ} \cdots$

defined as principal value branch.

$\therefore \sin^{-1}\frac{1}{2} = 30^\circ$ (Only) (or $\frac{\pi}{6}$)

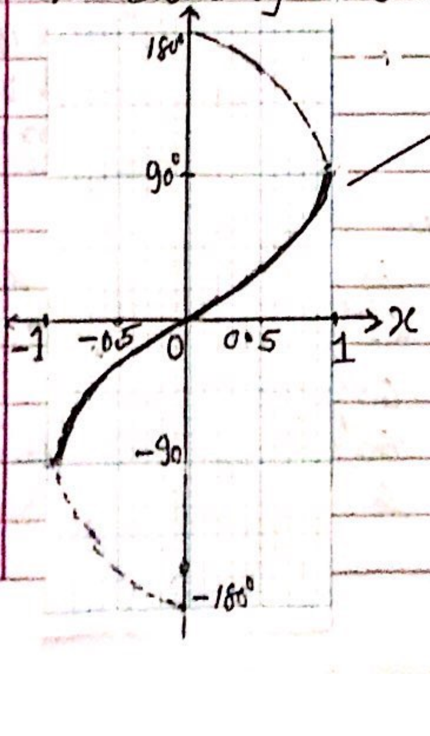
Let us see the graphs of $\sin\theta$, and $\sin^{-1}x$;



$x = \sin\theta$, $-1 \le x \le 1$

here when we restrict $-90^\circ \le \theta \le 90^\circ$ there is exactly one value of x,

$\therefore \theta = \sin^{-1} x$ has domain $-1 \le x \le 1$ and range $-90^\circ \le \theta \le 90^\circ$ (Princip. value Branch)



Now

$\sin^{-1} 0 = 0$

$\sin^{-1}\frac{1}{2} = 30^\circ$ (or $\frac{\pi}{6}$)

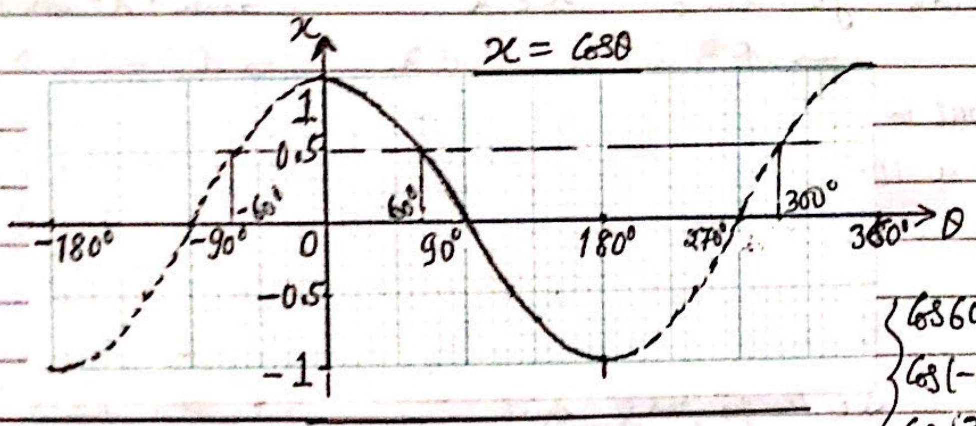
$\sin^{-1} 1 = 90^\circ$ ($\frac{\pi}{2}$)

$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ (or -30°)

$\sin^{-1}(-1) = -\frac{\pi}{2}$ (or -90°)

§ **Inverse trigonometric function:** $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

$-1 \le \cos\theta \le 1$ or $-1 \le x \le 1$



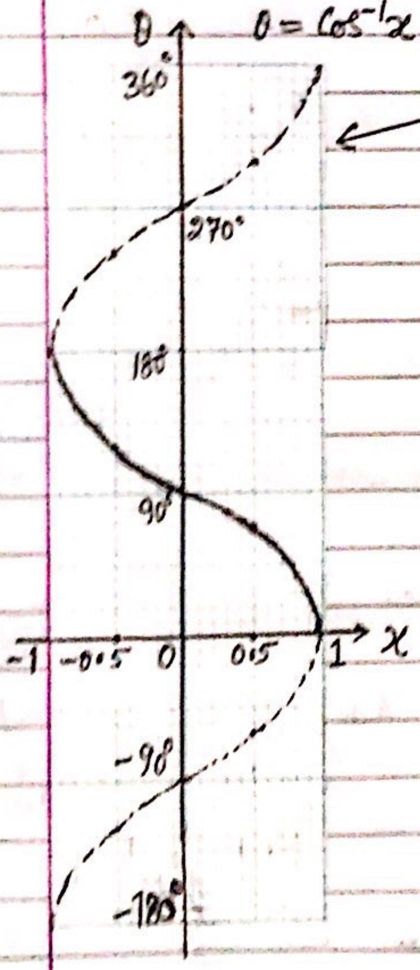
$\cos 60 = \frac{1}{2}$

$\cos(-60) = \frac{1}{2}$

$\cos 300 = \frac{1}{2}$

$\cos^{-1}\frac{1}{2} = 60°, -60°, 300°$

$\theta = \cos^{-1} x$ Graph of $\cos^{-1} x$



when $0 \le \theta \le 180°$

for each value of x : $-1 \le x \le 1$,

there is a unique value of $\cos\theta$.

$\therefore \theta = \cos^{-1} x$

(i) Domain

$-1 \le x \le 1$

(ii) Range.

$0 \le \theta \le 180°$

is the principal value **branch.**

Example.

$\cos^{-1} 0 = 90°$ (or $\frac{\pi}{2}$ radian)

$\cos^{-1}\frac{1}{2} = 60°$ (or $\frac{\pi}{3}$)

$\cos^{-1} 1 = 0°$

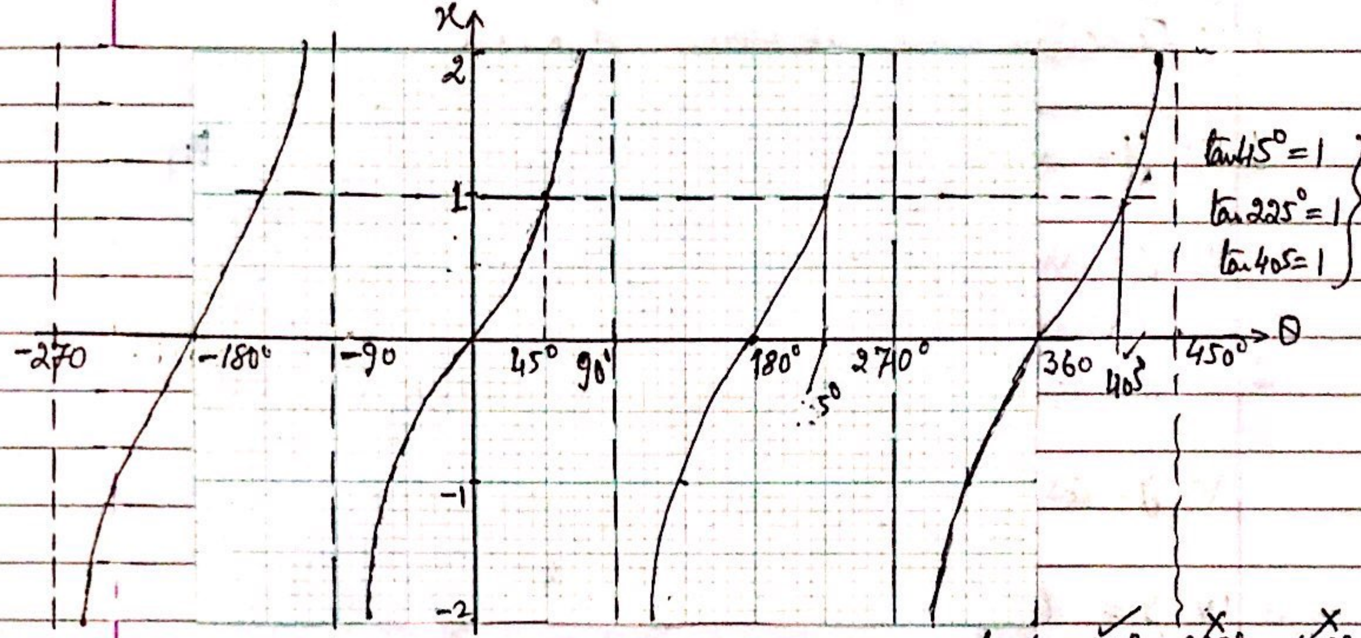
$\cos^{-1}(-1) = 180°$ (π rad).

$\cos^{-1}(-\frac{1}{2}) = 120°$ ($\frac{2\pi}{3}$ rad.)

§ **Inverse Trigonometric Function:** $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$.

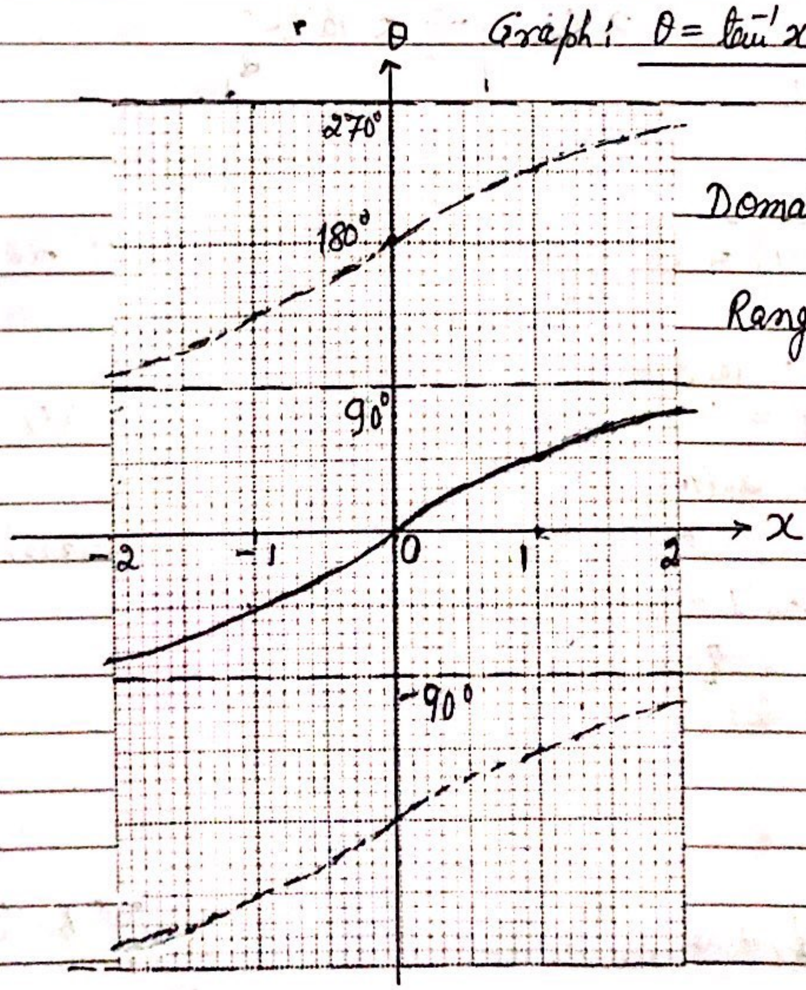
$-\infty < \tan\theta < \infty$ or $-\infty < x < 1$

Graph – $x = \tan\theta$



$\tan^{-1} 1 = 45°$, $225°$, $405°$

Graph: $\theta = \tan^{-1}x$.



$\theta = \tan^{-1}x$

Domain: $-\infty < x < \infty$

Range. $-90° < \tan^{-1}x < 90°$ is the principal value branch.

{To each value of x; $-\infty < x < \infty$, there is a unique θ : $-90° < \theta < 90°$

Now:

$\tan^{-1} 1 = 45°$ (or $\frac{\pi}{4}$)

$\tan^{-1} -1 = -45°$ (or $-\frac{\pi}{4}$)

$\tan^{-1}\sqrt{3} = 60°$ ($\frac{\pi}{3}$)

$\tan^{-1} -\sqrt{3} = -60°$ ($-\frac{\pi}{3}$)

$\tan^{-1} 0 = 0$.

§ Domain and Range of trigonometric and inverse trig. functions

	Function	Domain	Range
1	(i) $y = \sin x$	$x \in R$	$-1 \le \sin x \le 1$
	(ii) $y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$
2.	(i) $y = \cos x$	$x \in R$	$-1 \le \cos x \le 1$
	(ii) $y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le \cos^{-1} x \le \pi$
3.	(i) $y = \tan x$	$-\infty < x < \infty$	$\tan x \in R$
	(ii) $y = \tan^{-1} x$	$x \in R$	$-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$

Example 1: Find the value of x satisfying the equation:
$\sin^{-1}(x-1) = \tan^{-1} 3$. [W-14 | 11 | Q2]

Solution: $\sin^{-1}(x-1) = \tan^{-1} 3$

$\Rightarrow \sin^{-1}(x-1) = 1.25 \quad \left(-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}\right) \quad \left(\frac{\pi}{2} = 1.57\right)$

$\Rightarrow x - 1 = \sin(1.125) \quad \left(-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}\right)$

$x = 1 + 0.95 = 1.95$ ✓ (angles in radians)

Example 2: Evaluate: $\tan^{-1} 1 + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

Solution: $\tan^{-1} 1 = \frac{\pi}{4} \quad \left(-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}\right)$ — ①

$\cos^{-1}\left(-\frac{1}{2}\right) = \theta \Rightarrow \cos\theta = -\frac{1}{2} \quad (0 \le \cos^{-1} x \le \pi)$

$= -\cos\frac{\pi}{3} \Rightarrow \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ — ②

$\sin^{-1}\left(-\frac{1}{2}\right) = \theta \Rightarrow \sin\theta = -\frac{1}{2} \quad \left(-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}\right)$

$= -\sin\frac{\pi}{6} \Rightarrow \theta = -\frac{\pi}{6}$ — ③

$\therefore \tan^{-1} 1 + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{4} + \frac{2\pi}{3} + \left(-\frac{\pi}{6}\right) = \frac{9\pi}{12} = \frac{3\pi}{4}$ ✓ (from ①, ②, ③)

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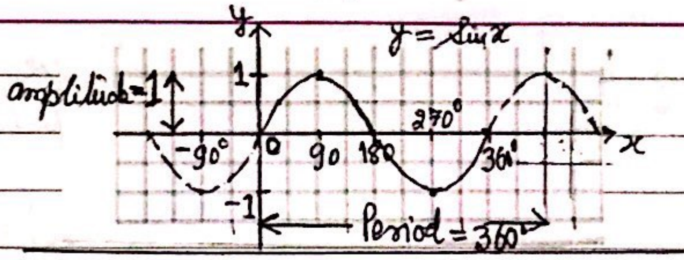
Transformations of trigonometric functions. P-9

§ 1 $y = \sin x$, $0 \le x \le 360$

Period $= 360°$

Amplitude $= 1$.



§ 2. (i) Stretch along y-axis:

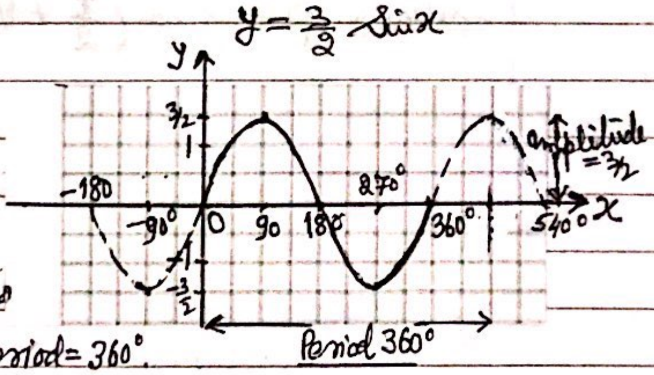
$y = a \sin x$

Amplitude $= a$

Period $= 360°$

Example: $y = \frac{3}{2} \sin x$,
is obtained with a stretch factor of $\frac{3}{2}$, applied to $y = \sin x$ along y-axis

Amplitude $= \frac{3}{2}$ ✓; Period $= 360°$.



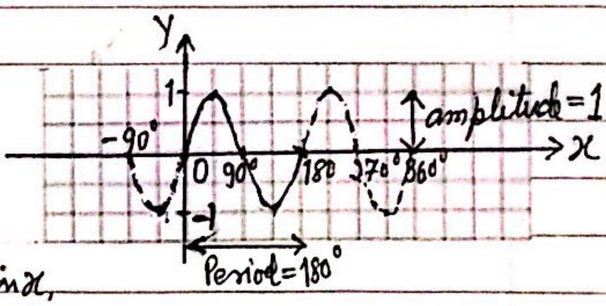
2 (ii) Stretch along x-axis:

$y = \sin bx$; Amplitude $= 1$

Amplitude $= 1$ and Period $= \frac{360°}{b}$.

Example: $y = \sin 2x$.
is obtained by a stretch of scale factor of $\frac{1}{2}$ along x-axis to $y = \sin x$.

Period $= \frac{360°}{2} = 180°$



2. (iii)

Stretch along x-axis and y-axis both:

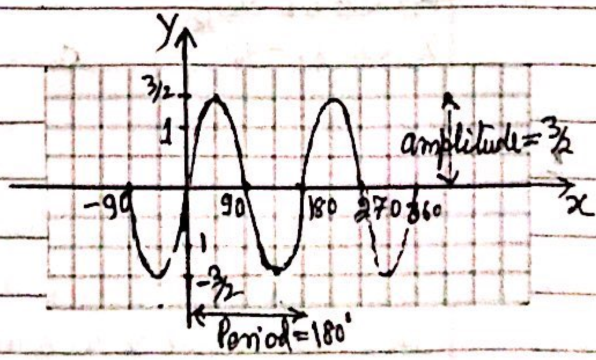
$y = a \sin bx$; amplitude $= a$

Period $= \frac{360}{b}$

Example: $y = \frac{3}{2} \sin 2x$,

amplitude $= \frac{3}{2}$

Period $= \frac{360°}{2} = 180°$.



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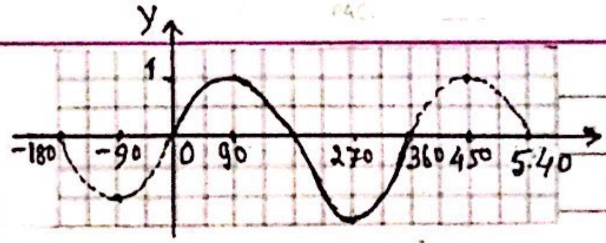
§ $y = \sin x$; $0 \le x \le 360$

Period $= 360°$

Amplitude $= 1$



§ 3. (i) **Translation along X-axis:**

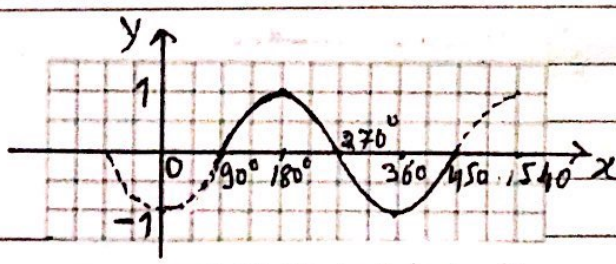
$y = \sin(x-h)$

The graph of $y = \sin(x-h)$ is a translation of $y = \sin x$ in the positive direction of X-axis $= \begin{pmatrix} h \\ 0 \end{pmatrix}$

Example: $y = \sin(x - 90°)$ is the translation of $y = \sin x$ by $\begin{pmatrix} 90 \\ 0 \end{pmatrix}$

Period $= 360°$ } same

Amplitude $= 1.$



→ Shift by 90° in the positive direction of X-axis

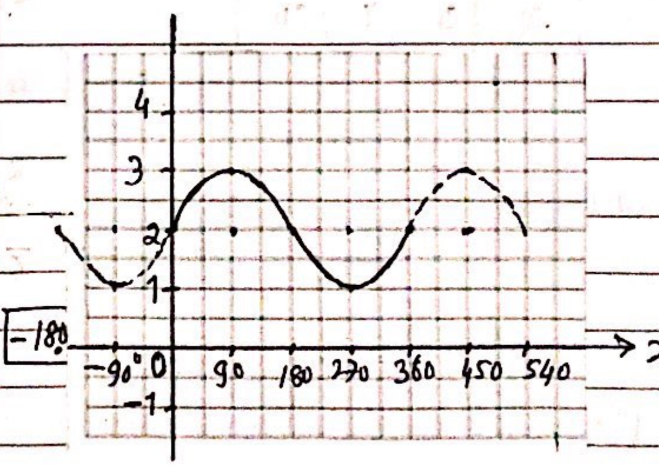
3(ii) **Translation along Y-axis:**

$y = \sin x + k.$

The graph of $y = \sin x + k$ is translation of $y = \sin x$ along Y axis by k unit $= \begin{pmatrix} 0 \\ k \end{pmatrix}$

upwards if $k > 0$

or downwords if $k < 0$.



Example: $y = \sin x + 2$

is a translation $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ upwards by 2 units. Period $= 360$, } Same

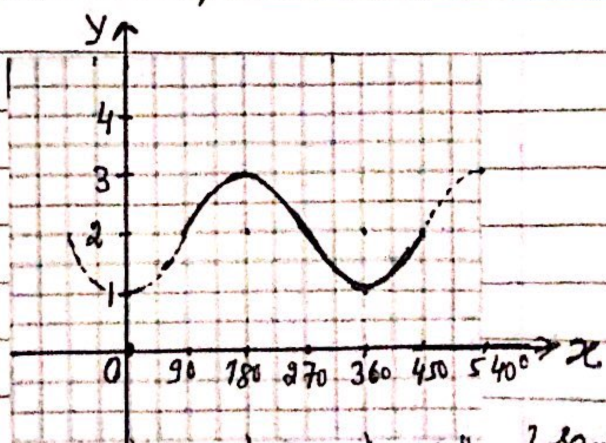
Amplitude $= 1$

3(iii) **Translation along X axis and Y-axis both:** $y = \sin(x-h) + k$

Example: $y = \sin(x - 90°) + 2$,

$y = \sin x$ is shifted in the +ve direction of X-axis and upward by 2 units along Y-axis.

Translation $= \begin{pmatrix} 90° \\ 2 \end{pmatrix}$



Period $= 360°$ } Same.

Amplitude $= 1$

§ 4 Transformations of trigonometric function:

Stretch and translation both:

$$y = a\sin(bx - h) + k \longrightarrow \text{amplitude} = a, \quad \text{Period} = \frac{360}{b}$$

Example: $y = 2\sin\left(\frac{x}{2} + 90^\circ\right) + 3$

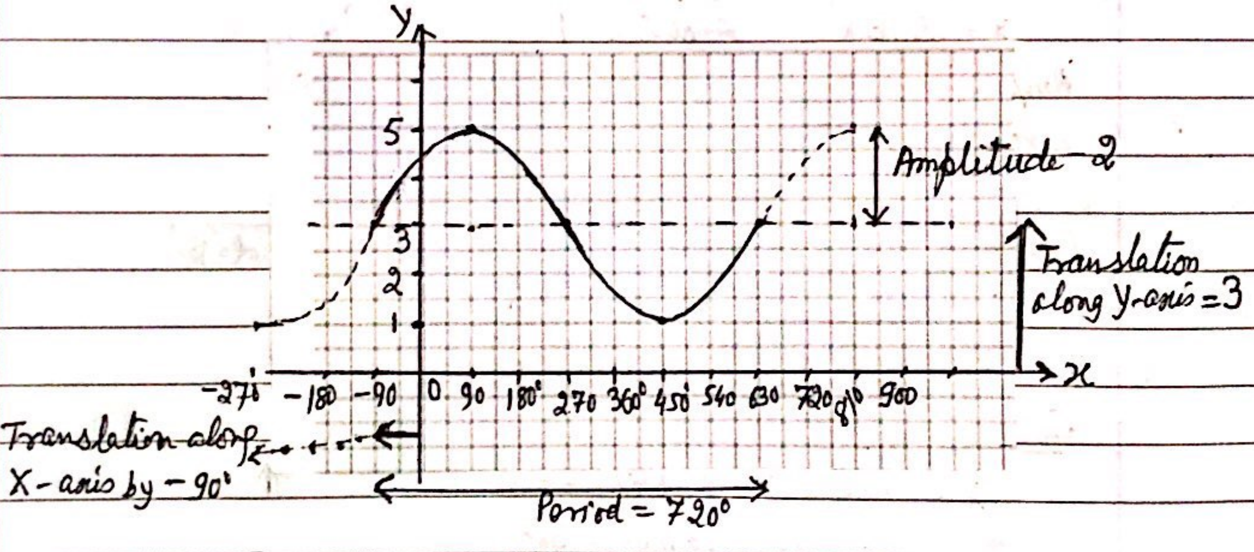
is a transformation of $y = \sin x$

(i) Stretch along y-axis, stretch factor 2. → Amplitude = 2

(ii) Stretch along X-axis of stretch factor 2, Period $= \frac{360^\circ}{\frac{1}{2}} = 720^\circ$

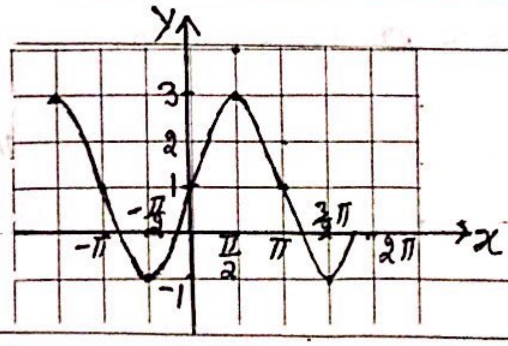
(iii) translation along X-axis by -90, $\begin{pmatrix}-90\\0\end{pmatrix}$

(iv) translation along y-axis by 3 → $\begin{pmatrix}0\\3\end{pmatrix}$ $\Bigg\}$ $\begin{pmatrix}-90\\3\end{pmatrix}$



Example 3:

The diagram shows part of the graph $y = a + b\sin x$. State the value of the constants a and b.



Solution: Translation along y-axis. $a = 1$ ✓

stretch along y-axis $b = 2$

[S-14/ 11/ Q1] ---[2]

Example 4: The function $f: x \to 5 + 3\cos\left(\frac{1}{2}x\right)$ is defined for $0 \leq x \leq 2\pi$

(i) Solve the equation $f(x) = 7$, giving your answer correct to 2 d.p. --[3]

(ii) Sketch the graph of $y = f(x)$ --[2]

(iii) Explain why f has an inverse. --[1]

(iv) Obtain an expression for $f^{-1}(x)$ --[3]

S-15 / 1.1 / Q8

Solution (i) $f(x) = 7 \Rightarrow 5 + 3\cos\left(\frac{x}{2}\right) = 7$ $\quad 0 \leq x \leq 2\pi \Rightarrow 0 \leq \frac{x}{2} \leq \pi$

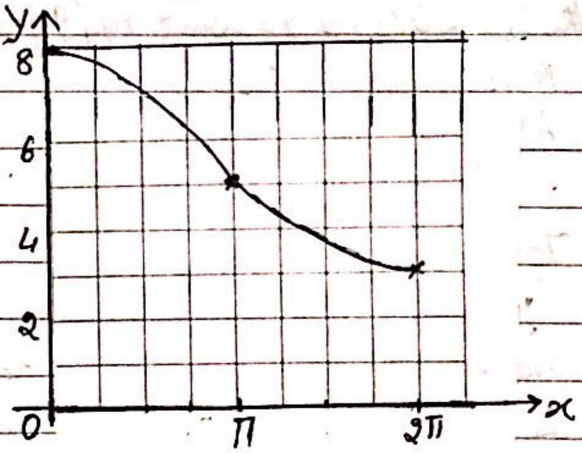
$\Rightarrow 3\cos\frac{x}{2} = 2$

$\Rightarrow \cos x/2 = 2/3 = \cos 0.841$

$\therefore \frac{x}{2} = 0.841 \Rightarrow x = \underline{1.68}$ ✓ rad. (only)

(ii) $f(x) = 5 + 3\cos\left(\frac{x}{7}\right)$

x	0	π	2π
$f(x)$	8	5	2



(iii) $f(x)$ is a decreasing function $0 \leq x \leq 2\pi$ (or no turning point)

$\therefore f(x)$ has an inverse.

(iv) $y = 5 + 3\cos x/2 \Rightarrow \cos x/2 = \frac{y-5}{2}$

Interchange x & y

$\Rightarrow \cos\frac{y}{2} = \frac{x-5}{2} \Rightarrow \frac{y}{2} = \cos^{-1}\left(\frac{x-5}{2}\right)$

$y = 2\cos^{-1}\left(\frac{x-5}{2}\right)$

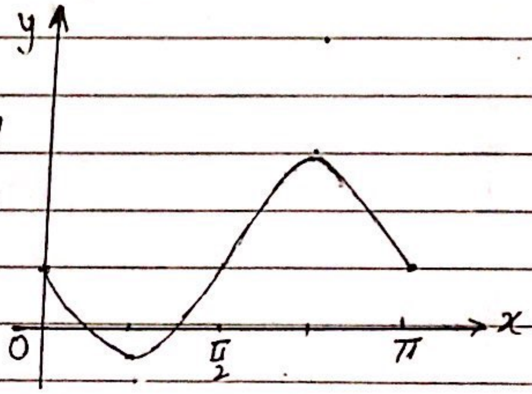
or $f^{-1}(x) = 2\cos^{-1}\left(\frac{x-5}{2}\right)$ ✓

Example 5. Functions f and g are such that,

$$f(x) = 2 - 3\sin 2x \quad \text{for } 0 \leq x \leq 2\pi$$
$$g(x) = -2f(x) \quad \text{for } 0 \leq x \leq \pi$$

(a) State the range of f and g. ---[3]

The diagram shows the graph of $y = f(x)$



(b) Sketch on this diagram, the graph of $y = g(x)$ ---[2]

The function $h(x)$ is such that.

$h(x) = g(x+\pi)$ for $-\pi \leq x \leq 0$

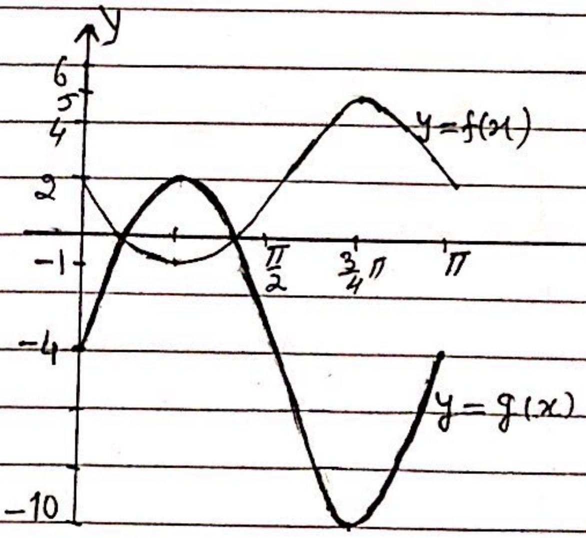
(c) Describe fully the sequence of transformation that maps the curve $y = f(x)$ on to $y = h(x)$ ---[3]

[S-20/12/Q9]

Solution: (a) $-1 \leq f(x) \leq 5$

$-10 \leq g(x) \leq 2$

(b)



(c) $h(x) = g(x+\pi)$ for $-\pi \leq x \leq 0$

from $f(x)$ to $h(x)$

(i) Reflection in x-axis.

(ii) Stretch factor 2 in the y-direction.

(iii) Translation by $-\pi$ in the x-direction (or Translation by $\begin{pmatrix} -\pi \\ 0 \end{pmatrix}$)