P3 CH-18: Integration Notes

Formulae

1 (i) $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x$ [let $x = \tan\theta$]

(ii) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x$ [let $x = \sin\theta$]

(iii) $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x$ [let $x = \sec\theta$]

2. (i) $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$ [let $x = a\tan\theta$]

(ii) $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$ [use partial fractions]

(iii) $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|$ ["]

3. (i) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}$ [let $x = a\sin\theta$]

(ii) $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x+\sqrt{x^2-a^2}\right|$ [let $x = a\sec\theta$]

(iii) $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \ln\left|x+\sqrt{a^2+x^2}\right|$ [let $x = a\tan\theta$]

4. (i) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

(ii) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|$

⊛ (iii) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|$

5. $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|$

6. Integration By Parts: $\int u\cdot v\,dx = u\cdot\int v\,dx - \int\left(\frac{du}{dx}\cdot\int v\,dx\right)dx$

1 (i) $\int \frac{1}{1+x^2}\,dx$

let $x = \tan\theta$ or $\theta = \tan^{-1}x$

$\left\{ \frac{dx}{d\theta} = \sec^2\theta \right.$

$= \int \frac{1}{1+\tan^2\theta} \cdot \frac{dx}{d\theta} \cdot d\theta$

$= \int \frac{1}{\sec^2\theta} \cdot \sec^2\theta \cdot d\theta$

$= \int 1\, d\theta$

$= \theta$

$= \tan^{-1}x + k$

---

2 (i) $\int \frac{1}{a^2+x^2}\,dx$

Let $x = a\tan\theta \Rightarrow \theta = \tan^{-1}\frac{x}{a}$

$\frac{dx}{d\theta} = a\sec^2\theta$

$= \int \frac{1}{a^2+a^2\tan^2\theta} \cdot \frac{dx}{d\theta} \cdot d\theta$

$= \int \frac{1}{a^2\sec^2\theta} \cdot a\sec^2\theta\, d\theta$

$= \frac{1}{a}\int 1\, d\theta$

$= \frac{1}{a}\theta$

$= \frac{1}{a}\tan^{-1}\frac{x}{a} + k$

---

1 (ii) $\int \frac{1}{\sqrt{1-x^2}}\,dx$

Let $x = \sin\theta$ or $\theta = \sin^{-1}x$

$\frac{dx}{d\theta} = \cos\theta$

$= \int \frac{1}{\sqrt{1-\sin^2\theta}} \cdot \frac{dx}{d\theta} \cdot d\theta$

$= \int \frac{1}{\cos\theta} \cdot \cos\theta\, d\theta$

$= \int 1\, d\theta$

$= \theta$

$= \sin^{-1}x + k$

1 (iii) $\int \frac{1}{x\sqrt{x^2-1}}\,dx$

Put $x = \sec\theta$ or $\theta = \sec^{-1}x$

$\left\{ \frac{dx}{d\theta} = \sec\theta\tan\theta \right.$

$= \int \frac{1}{\sec\theta\sqrt{\sec^2\theta-1}} \cdot \frac{dx}{d\theta} \cdot d\theta$

$= \int \frac{1}{\sec\theta\tan\theta} \times \sec\theta\tan\theta\, d\theta$

$= \int 1\, d\theta$

$= \theta$

$= \sec^{-1}x + K$

---

3 (ii) $\int \frac{1}{\sqrt{x^2-a^2}}\,dx$

Put $x = a\sec\theta$; ~~$\sec\theta =$~~ $\sec\theta = \frac{x}{a}$

$\left\{ \frac{dx}{d\theta} = a\sec\theta\tan\theta \right.$

$= \int \frac{1}{\sqrt{a^2\sec^2\theta - a^2}} \frac{dx}{d\theta} \cdot d\theta$

$= \int \frac{1}{\cancel{a}\tan\theta} \times \cancel{a}\sec\theta \cdot \tan\theta\, d\theta$

$= \int$ ~~tan~~ $\sec\theta\, d\theta$

$= \ln(\sec\theta + \tan\theta)$

$= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right)$

$= \ln\left(x + \sqrt{x^2-a^2}\right) - \ln a + c$

$= \ln\left(x + \sqrt{x^2-a^2}\right) + K$

$\tan\theta = \sqrt{\sec^2\theta - 1}$

$= \sqrt{\frac{x^2}{a^2} - 1}$

$= \frac{\sqrt{x^2-a^2}}{a}$

---

3 (iii) $\int \frac{1}{\sqrt{a^2+x^2}}\,dx$

Put $x = a\tan\theta \Rightarrow \tan\theta = \frac{x}{a}$

$\frac{dx}{d\theta} = a\sec^2\theta \Big\}$ $\sec\theta = \frac{\sqrt{x^2+a^2}}{a}$

$= \int \frac{1}{\sqrt{a^2+a^2\tan^2\theta}} \cdot \frac{dx}{d\theta} \cdot d\theta$

$= \int \frac{1}{a\sec\theta} \times a\sec^2\theta\, d\theta$

$= \int \sec\theta\, d\theta = \ln(\sec\theta + \tan\theta) = \ln\left[\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right]$

$= \ln\left[\sqrt{x^2+a^2} + x\right] - \ln a + k$

$= \ln\left(x + \sqrt{x^2+a^2}\right) + K$

3(i) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$

Put $x = a\sin\theta \Rightarrow \theta = \sin^{-1}\frac{x}{a}$

$\frac{dx}{d\theta} = a\cos\theta\, d\theta$

$= \int \frac{1}{\sqrt{a^2-a^2\sin^2\theta}}\cdot\frac{dx}{d\theta}\, d\theta$

$= \int \frac{1}{a\cos\theta}\cdot a\cos\theta\, d\theta$

$= \int 1\, d\theta$

$= \theta$

$= \sin^{-1}\frac{x}{a} + k$

2(ii) $\int \frac{1}{x^2-a^2}\,dx$

$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{(x-a)} + \frac{B}{(x+a)}$

or multi by $(x-a)$

$\frac{1}{x+a} = A + \frac{(x-a)}{(x+a)}.B$

Put $x = a$

$\frac{1}{2a} = A.$

Similarly. $B = \frac{-1}{2a}$

$= \int\left(\frac{1}{2a(x-a)} - \frac{1}{2a(x+a)}\right)dx$

$= \frac{1}{2a}\left[\ln(x-a) - \ln(x+a)\right]$

$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + k$

2(iii) $\int \frac{1}{a^2-x^2}\,dx$

$= \int\left(\frac{1}{2a(a+x)} + \frac{1}{2a(a-x)}\right)dx$ using Partial fractions

$= \frac{1}{2a}\left[\ln(a+x) + \frac{\ln(a-x)}{-1}\right]$

$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + k$

4(i). $\int \sqrt{a^2-x^2}\,dx$

Put $x = a\sin\theta$; ~~θ = sin⁻¹ x/a~~

$\frac{dx}{d\theta} = a\cos\theta\, d\theta$

$\sin\theta = \frac{x}{a}$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$= \sqrt{1-\frac{x^2}{a^2}}$

$\therefore = \frac{\sqrt{a^2-x^2}}{a}$

$= \int \sqrt{a^2-a^2\sin^2\theta}\cdot\frac{dx}{d\theta}\cdot d\theta$

$= \int a\cdot\cos\theta\cdot a\cos\theta\cdot d\theta$

$= a^2\int \cos^2\theta\, d\theta$

$= a^2\int \left(\frac{1+\cos 2\theta}{2}\right) d\theta$

$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]$

$= \frac{a^2}{2}\left[\theta + \sin\theta\cdot\cos\theta\right]$

$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}\right]$

$= \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + K$

Similarlity we can work out 4(ii) and 4(iii)

P3 CH-18 : Integration $\boxed{\text{Notes}}$ Page-6

4′(1) $\boxed{\text{Definite Integral}}$ Adjust the limits

$$\int_0^3 \sqrt{9-x^2}\,dx$$

let $x = 3\sin\theta$; $\theta = \sin^{-1}\frac{x}{3}$

$\frac{dx}{d\theta} = 3\cos\theta$

$\begin{cases} x=0 \Rightarrow \theta=0 \\ x=3 \Rightarrow \theta=\frac{\pi}{2} \end{cases}$

$$= \int_0^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta}\,\frac{dx}{d\theta}.d\theta$$

$$= \int_0^{\frac{\pi}{2}} 3\cos\theta \times 3\cos\theta\,d\theta$$

$$= 9.\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= \frac{9}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{9}{2}\left[\theta + \sin\theta\cdot\cos\theta\right]_0^{\pi/2}$$

$$= \frac{9}{2}.\left[\left(\frac{\pi}{2}+0\right)-(0)\right]$$

$$= \frac{9\pi}{4}$$ ✓

Definite Integral (Substitution)

Example 1. $\displaystyle\int_1^6 \frac{1}{4+x^2}\,dx$

Let $\begin{cases} x = 2\tan u \Rightarrow u = \tan^{-1}\frac{x}{2} \\ \frac{dx}{du} = 2\sec^2 u \end{cases}$

$x = 1 \Rightarrow u = \tan^{-1}\frac{1}{2}$

$x = 6 \Rightarrow u = \tan^{-1} 3$

$$= \int_{\tan^{-1}\frac{1}{2}}^{\tan^{-1}3} \frac{1}{(4+4\tan^2 u)} \cdot \frac{dx}{du} \cdot du$$

$$= \int_{\tan^{-1}\frac{1}{2}}^{\tan^{-1}3} \frac{1}{4\sec^2 u} \times 2\sec^2 u\,du$$

$$= 2\int_{\tan^{-1}\frac{1}{2}}^{\tan^{-1}3} 1\,du = 2\left[u\right]_{\tan^{-1}\frac{1}{2}}^{\tan^{-1}3}$$

$$= 2\left[\tan^{-1}3 - \tan^{-1}\tfrac{1}{2}\right]$$

$$= 2\times\tan^{-1}\left(\frac{3-\frac{1}{2}}{1+3\times\frac{1}{2}}\right)$$

$$= 2(\tan^{-1}1)$$

$$= 2\times\frac{\pi}{4} = \frac{\pi}{8}$$

$\because$ $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Example 2 $\displaystyle\int_e^{e^2} \frac{1}{x(\ln x)^2}\,dx$

$\begin{cases} x = e^u \Rightarrow u = \ln x \\ \frac{dx}{du} = e^u \end{cases}$

$\begin{cases} x = e \Rightarrow u = 1 \\ x = e^2 \Rightarrow u = 2 \end{cases}$

$$= \int_1^2 \frac{1}{e^u \cdot u^2} \cdot \frac{dx}{du} \cdot du$$

$$= \int_1^2 \frac{1}{\cancel{e^u} \cdot u^2} \cdot \cancel{e^u}\,du$$

$$= \int_1^2 u^{-2}\,du = \left[-\frac{1}{u}\right]_1^2 = -\left[\frac{1}{2} - 1\right] = \frac{1}{2}$$

(P3) CH-18: Integration

$\boxed{\text{Reverse Substitution}}$

Example 3 $\displaystyle\int x^2\sqrt{1+x^3}\,dx$

Put $u = 1+x^3$

$\dfrac{du}{dx} = 3x^2$

$$= \frac{1}{3}\int \sqrt{1+x^3}\cdot 3x^2\,dx$$

$$= \frac{1}{3}\int \sqrt{u}\;\frac{du}{dx}\,dx$$

$$= \frac{1}{3}\int u^{1/2}\,du = \frac{1}{3}\cdot\frac{u^{3/2}}{3/2}$$

$$= \frac{2}{9}(1+x^3)^{3/2} + k$$

Example 4 $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\sin 2x}{(1+\cos 2x)}\,dx$

$\left\{\begin{array}{l} \text{let } u = 1+\cos 2x \\ \dfrac{du}{dx} = -2\sin 2x \end{array}\right.$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{-2\sin 2x}{(1+\cos 2x)}\cdot dx$$

$\left\{\begin{array}{l} x = 0 \Rightarrow u = 2 \\ x = \frac{\pi}{6} \Rightarrow u = 3/2 \end{array}\right.$

$$= -\frac{1}{2}\int_2^{3/2} \frac{1}{u}\cdot\frac{du}{dx}\cdot dx$$

$$= -\frac{1}{2}\int_2^{3/2} \frac{1}{u}\,du$$

$$= -\frac{1}{2}\left[\ln u\right]_2^{3/2}$$

$$= -\frac{1}{2}\left[\ln 3/2 - \ln 2\right] = -\frac{1}{2}\ln\frac{3}{4}$$

$$= \frac{1}{2}\ln\frac{4}{3}$$

Integration by Parts:

$$\int u \cdot v \, dx = u \cdot \int v \, dx - \int \left(\frac{du}{dx} \cdot \int v \, dx \right) dx$$

Note

$\ln x$, $\sin^{-1} x$

$1, x, x^2$

$\sin x$, $\cos x$, e^x

I-method →	$e^x \cdot \sin x$
	$e^x \cdot \cos x$

$$\boxed{\int e^x \cdot \{ f(x) + f'(x) \} \, dx = e^x \cdot f(x)}$$

Example 5. $\int x \sin x \, dx$

$$= x \cdot \int \sin x \, dx - \int \left(\frac{d}{dx} x \cdot \int \sin x \, dx \right) dx$$

$$= x(-\cos x) - \int 1(-\cos x) \, dx$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + K \quad \text{—— (i)}$$

Example 6. $\int x^2 \cos x \, dx$

$$= x^2 \cdot \int \cos x \, dx - \int \left(\frac{d}{dx} x^2 \cdot \int \cos x \, dx \right) dx$$

$$= x^2 \cdot \sin x - \int 2x \cdot \sin x \, dx$$

$$= x^2 \sin x - 2 \int x \sin x \, dx$$

$$= x^2 \sin x - 2 \left[-x \cos x + \sin x \right]$$

(Sin (i) Repeat from Example 5)

$$= x^2 \sin x + 2x \cos x - 2 \sin x + K \checkmark$$

P3 CH-18: Integration [Note]

Integration by Parts

Example 7. $\int \ln x\, dx$

$$= \int \ln x \cdot 1\, dx$$

$$= \ln x \cdot \int 1\, dx - \int \left(\frac{d}{dx} \ln x \cdot \int 1\, dx \right) dx$$

$$= \ln x \cdot x - \int \frac{1}{\cancel{x}} \cdot \cancel{x}\, dx$$

$$= \ln x \cdot x - x$$

$$= x \ln x - x + k$$

Example 8 $\int_1^e x \ln x\, dx$

$$= \ln x \cdot \int x\,dx \Big]_1^e - \int_1^e \left(\frac{d}{dx} \ln x \cdot \int x\,dx \right) dx$$

$$= \left[\frac{x^2}{2} \ln x \right]_1^e - \int_1^e \frac{1}{x} \cdot \frac{x^2}{2}\, dx$$

$$= \left[\frac{e^2}{2} \ln e - 0 \right] - \frac{1}{2} \int_1^e x\, dx$$

$$= \frac{e^2}{2} \times 1 - \frac{1}{2} \left[\frac{x^2}{2} \right]_1^e$$

$$= \frac{e^2}{2} - \frac{1}{4} (e^2 - 1)$$

$$= \frac{e^2}{4} + \frac{1}{4}$$

$$= \frac{1}{4} (e^2 + 1)$$

Integration by Parts

I-method

Example 9 $I = \int e^{2x} \cdot \sin x\,dx$

$$I = e^{2x} \cdot \int \sin x\,dx - \int\left(\frac{d}{dx}e^{2x} \cdot \int \sin x\,dx\right)dx$$

$$= e^{2x}(-\cos x) - \int 2e^{2x}(-\cos x)\,dx$$

$$= -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx$$

$$= -e^{2x}\cos x + 2\left[e^{2x} \cdot \int \cos x\,dx - \int\left(\frac{d}{dx}e^{2x} \cdot \int \cos x\right)dx\right.$$

$$= -e^{2x}\cos x + 2\left[e^{2x}\sin x - \int 2e^{2x}\sin x\,dx\right.$$

$$I = \text{— — — — ——} - 2\int e^{2x}\sin x\,dx$$

$$I = -e^{2x}\cos x + 2\left[e^{2x}\sin x - 2I\right]$$

$$I = -e^{2x}\cos x + 2e^{2x}\sin x - 4I$$

$$5I = e^{2x}\left[2\sin x - \cos x\right]$$

$$I = \frac{e^{2x}}{5}\left[2\sin x - \cos x\right] + K$$

Example 10 $\int \sin^{-1}x\,dx$

$$\left\{\begin{array}{l} \text{Let } u = \sin^{-1}x \\ \text{or } x = \sin u \\ \frac{dx}{du} = \cos u \end{array}\right.$$

$$= \int u \cdot \frac{dx}{du} \cdot du$$

$$= \int u \cdot \cos u\,du$$

$$= u \cdot \int \cos u\,du - \int\left(\frac{d}{du}u \cdot \int \cos u\,du\right)$$

$$= u \cdot \sin u - \int 1 \times \sin u\,du$$

$$= u\sin u + \cos u$$

$$= x\sin^{-1}x + \sqrt{1-\sin^2 u}$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + K$$