P3

Pure Maths 3

Integration

Notes

Suresh Goel
(Director)
Alliance World School,
Noida, Delhi NCR, India.

Integration (Revision)

§ Indefinite Integral:

§ (i) Given $\frac{dy}{dx} = f'(x) = x^n$; $n \neq -1$

$$\therefore y = f(x) = \int x^n \, dx = \frac{x^{n+1}}{(n+1)} + c$$

(ii) for $\frac{dy}{dx} = f'(x) = (ax+b)^n$; $n \neq -1$

$$y = f(x) = \frac{(ax+b)^{n+1}}{a(n+1)}$$

Example 1. $\int (x^3 - 5x^2 + 2x + 7)\, dx$

$$= \frac{1}{4}x^4 - \frac{5}{3}x^3 + \frac{\cancel{2}x^2}{\cancel{2}} + 7x + c \checkmark$$

Example 2.

$$\int (2x+3)^5 \, dx = \frac{(2x+3)^6}{2 \times 6} + c$$

$$= \frac{1}{12}(2x+3)^6 + c \checkmark$$

$\int a\, dx = ax + c$

$\int 5\, dx = 5x + c$

$\int \sqrt{x}\, dx = \frac{x^{3/2}}{3/2} + c$

$\int \frac{1}{x^2} dx = \int x^{-2} dx$

$= -\frac{1}{x} + c$

§ Definite Integral: $\int_a^b f(x)\, dx = \left[F(x)\right]_a^b = F(b) - F(a)$

Given.

$\left[\int f(x)\, dx = F(x)\right]$

Example 3. $\int_1^3 (x^3 + 2x\, dx) = \left[\frac{x^4}{4} + \frac{\cancel{2}x^2}{\cancel{2}}\right]_1^3$

$$= \left[\left(\frac{3^4}{4} + 3^2\right) - \left(\frac{1^4}{4} - 1^2\right)\right]$$

$$= 28 \checkmark$$

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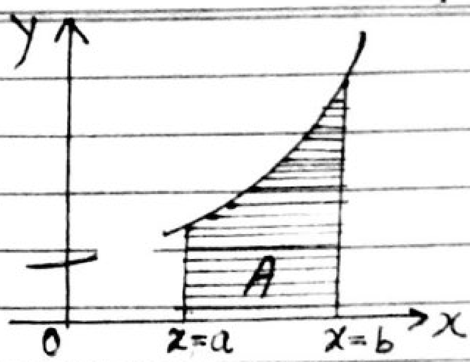
§ Area Under a Curve $y = f(x)$

Case I. Area under a curve $y = f(x)$, above x-axis and between the ordinates $x = a$ and $x = b$

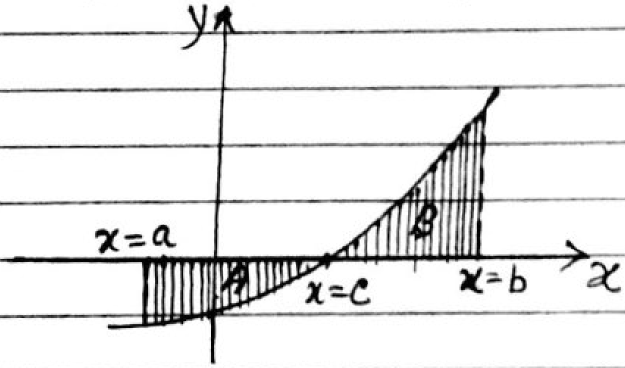
$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx$$ ✓



Note: If the area is below x-axis (Partially or wholly).
Let the curve intersects the x-axis at $x = c$.
Then the shaded area

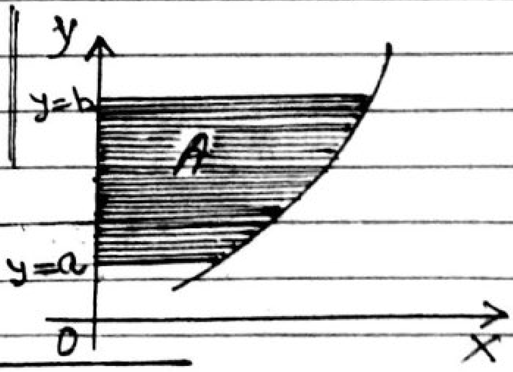
$$= \left| \int_a^c y\,dx \right| + \int_c^b y\,dx$$

(Area below x-axis $-ve$)
[so we take modulus]



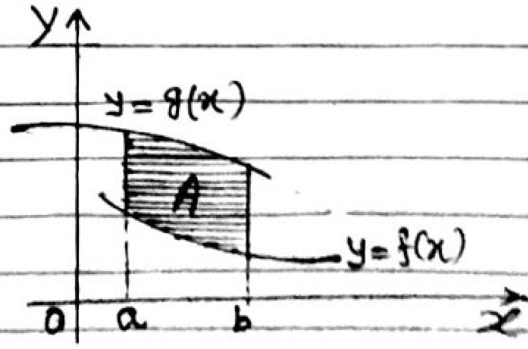
Case II. Given $y = f(x)$; Express $x = g(y)$
Area between the curve $x = g(y)$, y-axis and the lines $y = a$ & $y = b$

$$A = \int_a^b x\,dy = \int_a^b g(y)\,dy$$ ✓



Case III. Area between two curves:
Given curves $y = g(x)$ and $y = f(x)$

$$A = \int_a^b \{g(x) - f(x)\}\,dx$$



Integration

§. Exponential and Logarithmic functions:

$\lim_{x\to 0} (1+x)^{1/x} = e = 2.718$ (approx)

$\lim_{h\to 0} \frac{(e^h - 1)}{h} = 1$

1. (i) $\int e^x dx = e^x + K$

(ii) $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + K$

$\frac{d}{dx} e^x = e^x$

$\frac{d}{dx} e^{ax+b} = a.e^{ax+b}$

2. (i) $\int \frac{1}{x} dx = \ln x + c \qquad x > 0$

(ii) $\int \frac{1}{(ax+b)} dx = \frac{1}{a} \ln(ax+b) + c$

here $(ax+b) > 0$

$\ln x = \log_e x, \; x \in R, \; x > 0$

$\frac{d}{dx} \ln x = \frac{1}{x}$

3. (i) $\int \frac{1}{x} dx = \ln(-x) + c \qquad$ when $x < 0$

(ii) $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln(-ax-b) + c \qquad$ when $(ax+b) < 0$

Example 1. $\int_0^1 \frac{1}{(x-2)} dx = [\ln(2-x)]_0^1 = -\ln 2$

Example 2. Let $f(x) = \frac{3x^2 - 4}{x^2(3x+2)}$ $\qquad \boxed{S\text{-}17/33/Q9}$

(i) Express $f(x)$ in partial fractions. ---[5]

(ii) Hence show that $\int_1^2 f(x)\,dx = \ln\left(\frac{25}{8}\right) - 1$. ---[5]

Solution(i) $\frac{3x^2-4}{x^2(3x+2)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{3x+2}$ —— ①

multiply ① by x^2

$\frac{(3x^2-4)}{(3x+2)} = ax + b + \frac{cx^2}{3x+2}$

Put $x = 0$, $\frac{-4}{2} = b \Rightarrow b = -2$ ✓

multiply ① by $x^2(3x+2)$

$3x^2 - 4 = ax(3x+2) - 2(3x+2) - 6x^2$

Put $x = 1 \Rightarrow a = 3$ ✓

multiply by $(3x+2)$ to ① $\quad \frac{3x^2-4}{x^2} = a(3x+2) - \frac{b(3x+2)}{x^2} + c$

Put $x = -2/3$, $\quad -6 = 0 - 0 + c \Rightarrow c = -6$ ✓

(Continued →)

Integration

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Example 2(i) ∴ Required partial fractions:

$$= \frac{3}{x} + \frac{-2}{x^2} + \frac{-6}{3x+2}$$

(ii) $\int_1^2 f(x)\,dx = \int_1^2 \left(\frac{3}{x} - 2x^{-2} - 6 \cdot \frac{1}{(3x+2)}\right) dx$

$$= \left[3\ln x - 2\frac{x^{-2+1}}{(-2+1)} - 6 \cdot \frac{\ln(3x+2)}{3}\right]_1^2$$

$$= \left[\ln x^3 + \frac{2}{x} - 2\ln(3x+2)\right]_1^2$$

$$= \left\{\ln 2^3 + \frac{2}{2} - 2\ln 8\right\} - \left\{\ln 1 + \frac{2}{1} - 2\ln 5\right\}$$

$$= \ln 8 + 1 - \ln 64 - 0 - 2 + \ln 25$$

$$= \ln\left(\frac{8 \times 25}{64}\right) - 1 = \ln\left(\frac{25}{8}\right) - 1$$

Example 3. Let $f(x) = \frac{4x^2 + 9x - 8}{(x+2)(2x-1)}$ W-17/31/Q8

(i) Express $f(x)$ in the form $= A + \frac{B}{(x+2)} + \frac{C}{(2x-1)}$ --- [4]

(ii) Hence show that $\int_1^4 f(x)\,dx = 6 + \frac{1}{2}\ln\left(\frac{16}{7}\right)$ -- [5]

Solution (i) $\frac{4x^2+9x-8}{(x+2)(2x-1)} = \frac{4x^2+9x-8}{2x^2+3x-2} = 2 + \frac{3x-4}{(x+2)(2x-1)}$ — ① [∴ A=2]

Consider $\frac{3x-4}{(x+2)(2x-1)} = \frac{B}{x+2} + \frac{C}{(2x-1)}$ — ②

multiply ② by $(x+2) \Rightarrow \frac{3x-4}{2x-1} = B + \frac{C(x+2)}{(2x-1)}$

Put $x = -2 \Rightarrow \frac{-10}{-5} = B \Rightarrow B = 2$

Again multiply ② by $(2x-1) \Rightarrow \frac{3x-4}{x+2} = \frac{B(2x-1)}{(x+2)} + C$

Put $x = \frac{1}{2} \Rightarrow \frac{-5/2}{5/2} = 0 + C \Rightarrow C = -1$

Hence for ① Required Partial fractions $= 2 + \frac{2}{x+2} + \frac{-1}{2x-1}$

(ii) $\int_1^4 f(x)\,dx = \int_1^4 \left(2 + \frac{2}{x+2} - \frac{1}{2x-1}\right) dx = \left[2x + 2\ln(x+2) - \frac{1}{2}\ln(2x-1)\right]_1^4$

$$= 6 + \frac{1}{2}\ln\left(\frac{16}{7}\right)$$

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Integration

§ Trigonometric functions

1 (i) $\int \sin x\, dx = -\cos x + c$ $\quad \because \frac{d}{dx}\cos x = -\sin x$

(ii) $\int \cos x\, dx = \sin x + C$ as $\frac{d}{dx}\sin x = \cos x$

(iii) $\int \sec^2 x\, dx = \tan x + c$ as $\frac{d}{dx}\tan x = \sec^2 x$

(iv) $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$ as $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$

(v) $\int \sec x \tan x\, dx = \sec x + c$ as $\frac{d}{dx}\sec x = \sec x \tan x$

(vi) $\int \operatorname{cosec} x \cdot \cot x\, dx = -\operatorname{cosec} x + C$ as $\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cdot \cot x$

2. (i) $\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + c$; $\frac{d}{dx}\sin(ax+b) = a\cos(ax+b)$

(ii) $\int \tan x\, dx = -\ln \cos x$ (or $= \ln \sec x$) $+ C$; $\frac{d}{dx}\ln \sec x = \tan x$

(iii) $\int \cot x\, dx = \ln \sin x + c$; $\frac{d}{dx}\ln \sin x = \cot x$

(iv) $\int \operatorname{cosec} x\, dx = \ln(\tan x/2) + c$ $\quad$ (v) $\int \sec x\, dx = \ln(\sec x + \tan) + c$

3. **Trigonometry for Integral:**

(i) $\begin{cases} \sin^2 x = \frac{1}{2}(1-\cos 2x) & \text{as } \cos 2x = 1 - 2\sin^2 x \\ \sin^2 x/2 = \frac{1}{2}(1-\cos x) & \text{as } \cos x = 1 - 2\sin^2 x/2 \end{cases}$

(ii) $\begin{cases} \cos^2 x = \frac{1}{2}(1+\cos 2x) & \text{as } \cos 2x = 2\cos^2 x - 1 \\ \cos^2 x/2 = \frac{1}{2}(1+\cos x) & \text{as } \cos x = 2\cos^2 x/2 - 1 \end{cases}$

(iii) $\tan^2 x = \sec^2 x - 1$

(iv) $\cot^2 x = \operatorname{cosec}^2 x - 1$

(v) $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

(vi) $\sin^3 x = \frac{1}{4}[3\sin x - \sin 3x]$ as $\sin 3x = 3\sin x - 4\sin^3 x$

(vii) $\cos^3 x = \frac{1}{4}[3\cos x + \cos 3x]$ as $\cos 3x = 4\cos^3 x - 3\cos x$.

(viii) $\sin^4 x = (\sin^2 x)^2 = [\frac{1}{2}(1-\cos 2x)]^2 = \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$

(ix) $\cos^4 x = (\cos^2 x)^2 = [\frac{1}{2}(1+\cos 2x)]^2 = \ldots$

Integration

Example 4 (i) Using the expansion of $\cos(3x+x)$ and $\cos(3x-x)$, show that $\frac{1}{2}(\cos 4x + \cos 2x) = \cos 3x \cos x$ --- [3]

[M-18/32/Q3]

(ii) Hence show that
$$\int_{-\frac{1}{6}\pi}^{\frac{1}{6}\pi} \cos 3x \cdot \cos x \, dx = \frac{3}{8}\sqrt{3} \quad \text{--- [3]}$$

Solution (i)
$$\begin{cases} \cos(3x+x) = \cos 3x \cos x - \sin 3x \sin x & [\text{Using } \cos(A+B) \text{ formula}] \\ \cos(3x-x) = \cos 3x \cos x + \sin 3x \sin x \end{cases}$$

add $\cos(3x+x) + \cos(3x-x) = 2\cos 3x \cos x$

or $\frac{1}{2}[\cos 4x + \cos 2x] = \cos 3x \cos x$ —— (1)

(ii)
$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos 3x \cdot \cos x \, dx = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}(\cos 4x + \cos 2x)\, dx$$
$$= \frac{1}{2}\left[\frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$
$$= \frac{1}{8}\left[\sin 4x + 2\sin 2x\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$
$$= \frac{1}{8}\left[\left(\sin\frac{2}{3}\pi + 2\sin\frac{\pi}{3}\right) - \left(\sin -\left(\frac{2\pi}{3}\right) + 2\sin\left(-\frac{\pi}{3}\right)\right)\right]$$
$$= \frac{1}{8}\left[\left(\frac{\sqrt{3}}{2} + 2\times\frac{\sqrt{3}}{2}\right) - \left(-\frac{\sqrt{3}}{2} - \frac{2\sqrt{3}}{2}\right)\right] = \frac{1}{8}\left[3\sqrt{3}\right]$$
$$= \frac{3\sqrt{3}}{8} \checkmark$$

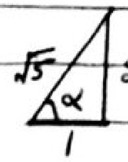
Example 5 (i) Express $\cos\theta + 2\sin\theta$ in the form $R\cos(\theta-\alpha)$, where $R>0$ and $0<\alpha<\frac{1}{2}\pi$. Give the exact value of R and $\tan\alpha$. --- [3]

(ii) Hence showing all necessary working, show that
$$\int_0^{\frac{1}{4}\pi} \frac{15}{(\cos\theta + 2\sin\theta)^2}\, d\theta = 5 \quad \text{--- [5]}$$

[S-18/33/Q7]

Solution (i) $\cos\theta + 2\sin\theta$
$$= \sqrt{5}\left[\frac{1}{\sqrt{5}}\cos\theta + \frac{2}{\sqrt{5}}\sin\theta\right]$$
$$= \sqrt{5}\left[\cos\theta\cos\alpha + \sin\theta\sin\alpha\right]$$
$$= \sqrt{5}\cos(\theta-\alpha)$$

$\therefore R = \sqrt{5}$, $\tan\alpha = 2$



$R\cos\alpha = 1$
$R\sin\alpha = 2$
$R = \sqrt{5}$
$\tan\alpha = 2$
$\cos\alpha = \frac{1}{\sqrt{5}}$, $\sin\alpha = \frac{2}{\sqrt{5}}$

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Example 5(ii): $\int_0^{\frac{1}{4}\pi} \frac{15}{(\cos\theta + 2\sin\theta)^2} d\theta$

$= \int_0^{\frac{1}{4}\pi} \frac{15}{\left(\sqrt{5}\cos(\theta-\alpha)\right)^2} d\theta$

[∵ from Part (i) $\cos\theta + 2\sin\theta = \sqrt{5}\cos(\theta-\alpha)$ and $\tan\alpha = 2$]

$= 15\int_0^{\frac{1}{4}\pi} \frac{1}{5}\sec^2(\theta-\alpha)\, d\theta$

$= 3\left[\tan(\theta-\alpha)\right]_0^{\frac{\pi}{4}}$

$= 3\left[\tan\left(\frac{\pi}{4}-\alpha\right) - \tan(-\alpha)\right]$

$= 3\left[\frac{\tan\frac{\pi}{4} - \tan\alpha}{1+\tan\frac{\pi}{4}\tan\alpha} + \tan\alpha\right]$ ($\tan\frac{\pi}{4} = 1$, $\tan\alpha = 2$)

$= 3\left[\frac{(1-2)}{1+1\times 2} + 2\right] = 3\left[-\frac{1}{3}+2\right] = 3\times\frac{5}{3}$

$= 5$ ✓

Example 6. (i) Prove that if $y = \frac{1}{\cos\theta}$, then $\frac{dy}{d\theta} = \sec\theta.\tan\theta$ --[2]

(ii) Prove the identity $\frac{1+\sin\theta}{1-\sin\theta} = 2\sec^2\theta + 2\sec\theta\tan\theta - 1$ --[3]

(iii) Hence find the exact value of $\int_0^{\frac{\pi}{4}} \frac{1+\sin\theta}{1-\sin\theta} d\theta$ ---[4] [S-17/32/Q7]

Solution (i) $y = \frac{1}{\cos\theta} \Rightarrow \frac{dy}{d\theta} = \frac{\cos\theta \times \frac{d}{d\theta}1 - 1\times\frac{d}{d\theta}\cos\theta}{\cos^2\theta}$

Using quotient rule

$= \frac{0 - (-\sin\theta)}{\cos^2\theta} = \frac{1}{\cos\theta}\times\frac{\sin\theta}{\cos\theta}$

$\therefore \frac{d}{d\theta}\sec\theta = \sec\theta.\tan\theta$ ✓

(ii) $\frac{1+\sin\theta}{1-\sin\theta} = \frac{1+\sin\theta}{1-\sin\theta}\times\frac{1+\sin\theta}{1+\sin\theta}$

$= \frac{(1+\sin\theta)^2}{1-\sin^2\theta} = \frac{1+2\sin\theta+\sin^2\theta}{\cos^2\theta}$

$= \frac{1}{\cos^2\theta} + 2\frac{\sin\theta}{\cos\theta}\times\frac{1}{\cos\theta} + \tan^2\theta$

$= \sec^2\theta + 2\tan\theta.\sec\theta + (\sec^2\theta - 1)$

$= 2\sec^2\theta + 2\sec\theta\tan\theta - 1$ ✓

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Example 6 (iii) consider $\int_0^{\frac{1}{4}\pi} \frac{1+\sin\theta}{1-\sin\theta}\,d\theta$

$$= \int_0^{\frac{\pi}{4}} (2\sec^2\theta + 2\sec\theta\tan\theta - 1)\,d\theta \quad \left\{ \text{Using Part (ii) and Part (i)} \right.$$

$$= \left[2\tan\theta + 2\sec\theta - \theta\right]_0^{\pi/4}$$

$$= \left[\left(2\tan\frac{\pi}{4} + 2\sec\frac{\pi}{4} - \frac{\pi}{4}\right) - \left(2\tan 0 + 2\sec 0 - 0\right)\right]$$

$$= \left(2 + 2\sqrt{2} - \frac{\pi}{4}\right) - \left(0 + 2 - 0\right)$$

$$= 2\sqrt{2} - \frac{1}{4}\pi \checkmark$$

Example 7. (i) Prove the identity $\tan 2\theta - \tan\theta = \tan\theta\sec 2\theta$ --- [4]

(ii) Hence show that $\int_0^{\frac{1}{6}\pi} \tan\theta\sec 2\theta\,d\theta = \frac{1}{2}\ln\frac{3}{2}$ --- [4]

W-15/31/Q5

Solution: (i) $\tan 2\theta - \tan\theta = \frac{\sin 2\theta}{\cos 2\theta} - \frac{\sin\theta}{\cos\theta}$

$$= \frac{\sin 2\theta\cos\theta - \cos 2\theta\cdot\sin\theta}{\cos 2\theta\cos\theta}$$

$$= \frac{\sin(2\theta-\theta)}{\cos 2\theta\cos\theta} = \frac{\sin\theta}{\cos\theta}\times\frac{1}{\cos 2\theta} = \tan\theta\sec 2\theta \checkmark$$

(ii) $\int_0^{\frac{1}{6}\pi} \tan\theta\sec 2\theta\,d\theta = \int_0^{\frac{1}{6}\pi} (\tan 2\theta - \tan\theta)\,d\theta$ [from Part (i)]

$$= \left[\frac{\ln\sec 2\theta}{2} - \ln\sec\theta\right]_0^{\frac{1}{6}\pi}$$

$$= \frac{1}{2}\left[\ln\sec 2\theta - 2\ln\sec\theta\right]_0^{\pi/6}$$

$$= \frac{1}{2}\left[\ln\sec\frac{\pi}{3} - 2\ln\sec\frac{\pi}{6}\right) - \left(\ln\sec 0 - 2\ln\sec 0\right)\right]$$

$$= \frac{1}{2}\left[\left(\ln 2 - 2\ln\frac{2}{\sqrt{3}}\right) - (0)\right]$$

$$= \frac{1}{2}\left[\ln 2 - \ln\frac{4}{3}\right] = \frac{1}{2}\ln 2\times\frac{3}{4} = \frac{1}{2}\ln\frac{3}{2} \checkmark$$

Integration

§ Integration by Substition:

(It is the reverse process of differentiation by Chain rule)

(i) To find $\int x^2 \cos x^3 dx$

[Consider $\frac{d}{dx} \frac{1}{3}\sin(x^3) = x^2 \cdot \cos x^3$

$= \frac{1}{3}\int \cos x^3 \cdot 3x^2 dx$

{ Put $u = x^3$
diff. $\frac{du}{dx} = 3x^2$
or $3x^2 dx = du$

$= \frac{1}{3}\int \cos u \, du$

$= \frac{1}{3} \sin u + C$

$= \frac{1}{3}\sin x^3 + C$ ✓

(ii) Find $\int \cos^3 x \cdot \sin x \, dx$

{ Put $u = \cos x$
$\frac{du}{dx} = -\sin x$
or $-du = \sin x \, dx$

$= \int u^3 \cdot (-du)$

$= -\left[\frac{u^4}{4}\right]$

$= -\frac{1}{4} \cdot \cos^4 x + C$ ✓

(iii) Find $\int (3x^5+7x+1)^6 (15x^4+7) dx$

{ Put $u = 3x^5+7x+1$
$\frac{du}{dx} = 15x^4+7$
[or $du = (15x^4+7)dx$

$= \int u^6 du$

$= \frac{u^7}{7} + C$

$= \frac{1}{7}(3x^5+7x+1)^7 + C$

(iv) To find $\int \sqrt{\tan x} \sec^2 x \, dx$

{ Put $\tan x = u$
diff $\sec^2 x \, dx = du$

$= \int \sqrt{u} \, du$

$= \int u^{1/2} du = \frac{2}{3}u^{3/2} = \frac{2}{3}(\tan x)^{3/2} + C$ ✓

Integration

§ Definite Integration by substitution:

(Adjust the limits)

(i) Evaluate. $\int_0^{\frac{1}{4}\pi} \tan x \sec^2 x\, dx$

$\begin{cases} \text{Put } \tan x = u \quad \text{—①} \\ \text{diff} \\ \sec^2 x\, dx = du \end{cases}$

from ① $\begin{cases} x = 0, \ u = 0 \\ x = \frac{\pi}{4}, \ u = 1 \end{cases}$

$= \int_0^1 u\, du$

$= \left[\frac{u^2}{2}\right]_0^1$

$= \frac{1}{2} - 0 = \frac{1}{2}$ ✓

(ii) $\int_2^3 \frac{2x+3}{(x^2+3x-7)^5} dx$

$\begin{cases} \text{Put } u = x^2+3x-7 \\ \text{diff } du = (2x+3)\, dx \end{cases}$

$\begin{cases} x = 1, \ u = 3 \\ x = 3, \ u = 11 \end{cases}$

$= \int_3^{11} \frac{1}{u^5} du$

$= \int_3^{11} u^{-5} du = \left[-\frac{1}{4} u^{-}\right]_3^{11} = -\frac{1}{4}\left[\frac{1}{u^4}\right]_3^{11}$

$= -\frac{1}{4}\left[\frac{1}{14641} - \frac{1}{81}\right]$

$= -\frac{1}{4}\left[\frac{81 - 14641}{14641 \times 81}\right]$

$= \frac{14560}{4 \times 1185921} = \frac{3640}{1185921}$ ✓

Example 8: Use the substitution $u = 4 - 3\cos x$, to find the exact value of $\int_0^{\frac{1}{2}\pi} \frac{9\sin 2x}{\sqrt{(4-3\cos x)}} dx$ $\boxed{W\text{-}15/33/Q5}$ -- [8]

Solution: $\int_0^{\frac{1}{2}\pi} \frac{9\sin 2x}{\sqrt{(4-3\cos x)}} dx$

$\begin{cases} \text{Put } u = 4 - 3\cos x \\ \text{diff. } du = 3\sin x \\ 3\cos x = (4-u) \end{cases}$ $\begin{cases} x = 0, u = 1 \\ x = \frac{1}{2}\pi, u = 4 \end{cases}$

$= \int_0^{\frac{\pi}{2}} \frac{9 \times 2\sin x \cos x}{\sqrt{(4-3\cos x)}} dx = \int_1^4 \frac{2(4-u)\, du}{\sqrt{u}} = \int_1^4 (8u^{-\frac{1}{2}} - 2u^{\frac{1}{2}})\, du = \left[16u^{\frac{1}{2}} - \frac{4}{3}u^{\frac{3}{2}}\right]_1^4$

$= 20/3$ ✓

Integration

Example 9. Let $I = \int_{\frac{1}{4}}^{3/4} \sqrt{\left(\frac{x}{1-x}\right)}\, dx$

(i) Using substitution $x = \cos^2\theta$ show that $I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} 2\cos^2\theta\, d\theta$ --- [4]

(ii) Hence find the exact value of I --- [4]

$\boxed{S\text{-}18 / 31 / Q5}$

Solution: (i) $\int_{\frac{1}{4}}^{3/4} \sqrt{\left(\frac{x}{1-x}\right)}\, dx$

Let $x = \cos^2\theta$

diff $\frac{dx}{d\theta} = -2\cos\theta\sin\theta$

or $dx = -2\cos\theta\sin\theta\, d\theta$

adjust the limits

$x = \frac{1}{4},\ \theta = \frac{\pi}{3}$

$x = \frac{3}{4}\quad \theta = \frac{\pi}{6}$

$$= \int_{\frac{\pi}{3}}^{\pi/6} \sqrt{\left(\frac{\cos^2\theta}{1-\cos^2\theta}\right)} \cdot (-2\cos\theta\sin\theta\, d\theta)$$

$$= \int_{\pi/3}^{\pi/6} -2 \times \frac{\cos\theta}{\cancel{\sin\theta}} \times \cos\theta\, \cancel{\sin\theta}\, d\theta$$

$$= -\int_{\frac{\pi}{3}}^{\pi/6} 2\cos^2\theta\, d\theta = \int_{\pi/6}^{\pi/3} 2\cos^2\theta\, d\theta \checkmark \quad \text{--- } \left[\text{as} \int_a^b f(x)dx = -\int_b^a f(x)dx\right]$$

(ii) $\int_{\pi/6}^{\pi/3} 2\cos^2\theta\, d\theta = \int_{\frac{\pi}{6}}^{\pi/3} (1+\cos 2\theta)\, d\theta \quad \left[\because \cos 2\theta = 2\cos^2\theta - 1\right]$

$$= \left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/6}^{\pi/3}$$

$$= \left(\frac{\pi}{3} + \frac{1}{2} \times \frac{\sqrt{3}}{2}\right) - \left(\frac{\pi}{6} + \frac{1}{2} \times \frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{6}\pi \checkmark$$

Example 10. (i) Show that $\frac{2\sin x - \sin 2x}{1-\cos 2x} = \frac{\sin x}{1+\cos x}$ --- (4)

(ii) Hence showing all necessary working, find $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{2\sin x - \sin 2x}{1-\cos 2x}\, dx$ --- [4]

$\boxed{S\text{-}18 / 32 / Q4}$

Solution: (i) $\frac{2\sin x - \sin 2x}{1-\cos 2x}$

$$= \frac{2\sin x - 2\sin x\cos x}{2\sin^2 x} = \frac{2\sin x(1-\cos x)}{2\sin^2 x}$$

$$= \frac{1-\cos x}{\sin x} \times \frac{(1+\cos x)}{(1+\cos x)} = \frac{1-\cos^2 x}{\sin x(1+\cos x)} = \frac{\sin^2 x}{\sin x(1+\cos x)} = \frac{\sin x}{(1+\cos x)} \checkmark$$

(ii) $\int_{\frac{\pi}{3}}^{\pi/2} \frac{2\sin x - \sin 2x}{1-\cos 2x}\, dx$

$$= \int_{\pi/3}^{\pi/2} \frac{\sin x}{1+\cos x}\, dx \quad \text{from Part (i)}$$

Put $1+\cos x = u$

diff $-\sin x\, dx = du$

$x = \frac{\pi}{3},\ u = 3/2$

$x = \frac{\pi}{2}\quad u = 1$

$$= \int_{3/2}^{1} \frac{-1\, du}{u} = \left[-\ln u\right]_{3/2}^{1}$$

$$= -\left[\ln 1 - \ln 3/2\right] = + \ln 3/2 \checkmark$$

Integration

§ Integration by Parts

$u = f(x) \cdot V = g(x)$

$$\int \underset{\text{Ist}}{u} \cdot \underset{\text{IInd}}{v}\, dx = u \cdot \int v\,dx - \int\left(\frac{d}{dx}u \cdot \int v\,dx\right)dx$$

$\downarrow \begin{cases} \ln x, \\ 1, x, x^2 \\ \sin x, \cos x, e^x \end{cases}$

$\left\{ \text{I-method} \begin{matrix} e^x \cdot \cos x \\ e^x \cdot \sin x \end{matrix} \right.$

Example 11 (i) Showing all your working, find the value of $\int x \cos 3x\, dx$ --[4]

(ii) hence find the value of $\int_0^{\frac{\pi}{6}} x \cos 3x\,dx$ in terms of π --[1]

[S-18/33/Q4]

Solution (i) $\int x \cdot \cos 3x\, dx$

Integrate by parts

$$= x \cdot \int \cos 3x\, dx - \int\left(\frac{d}{dx}x \cdot \int \cos 3x\, dx\right)dx$$

$$= x \cdot \frac{\sin 3x}{3} - \int 1 \times \frac{\sin 3x}{3}\,dx$$

$$= \frac{1}{3}x \sin 3x - \frac{1}{3} \cdot \frac{-(\cos 3x)}{3}$$

$$= \frac{1}{9}\left[3x \sin 3x + \cos 3x\right]$$

(ii) $\int_0^{\pi/6} x \cos 3x\,dx = \frac{1}{9}\left[3x \sin 3x + \cos 3x\right]_0^{\pi/6}$ from Part (i)

$$= \frac{1}{9}\left[\left(3 \cdot \frac{\pi}{6} \cdot 1 + 0\right) - (1)\right]$$

$$= \frac{1}{9}\left(\frac{\pi}{2} - 1\right) = \frac{1}{18}(\pi - 2) \checkmark$$

P3

Integration

Example 12 (i) Find $\int \frac{\ln x}{x^3}\,dx$ --- [3]

(ii) Hence show that $\int_1^2 \frac{\ln x}{x^3}\,dx = \frac{1}{16}(3-\ln 4)$ --- [2]

W-18/32/Q3

Solution: (i) $\int \frac{\ln x}{x^3}\,dx = \int \ln x \cdot x^{-3}\,dx$

$$= \ln x \cdot \int x^{-3}\,dx - \int\left(\frac{d}{dx}\ln x \cdot \int x^{-3}\,dx\right)dx$$

$$= \ln x \cdot \frac{x^{-2}}{-2} - \int \frac{1}{x}\cdot\frac{x^{-2}}{-2}\,dx$$

$$= -\frac{\ln x}{2x^2} + \frac{1}{2}\int x^{-3}\,dx$$

$$= -\frac{\ln x}{2x^2} + \frac{1}{2}\cdot\frac{x^{-2}}{-2}$$

$$= -\frac{\ln x}{2x^2} - \frac{1}{4x^2}$$

(ii) $\int_1^2 \frac{\ln x}{x^3}\,dx = -\left[\frac{\ln x}{2x^2} + \frac{1}{4x^2}\right]_1^2$

$$= -\left[\left(\frac{\ln 2}{8} + \frac{1}{16}\right) - \left(0 + \frac{1}{4}\right)\right] = -\frac{1}{8}\ln 2 - \frac{1}{16} + \frac{1}{4}$$

$$= -\frac{1}{8}\ln 2 + \frac{3}{16} = \frac{1}{16}[3 - 2\ln 2] = \frac{1}{16}(3 - \ln 4) \checkmark$$

Example 13. Show that $\int \ln x\,dx = x\ln x - x + c$

M-17/32/Q10(ii)

Solution: $\int \ln x\,dx = \int \ln x \times 1\,dx$ (using integration by parts)

$$= \ln x \cdot \int 1\,dx - \int\left(\frac{d}{dx}\ln x \cdot \int 1\,dx\right)dx$$

$$= \ln x \cdot x - \int \frac{1}{x}\cdot x\,dx$$

$$= x\ln x - \int 1\,dx$$

$$= x\ln x - x + c \checkmark$$

(P3)

Integration

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Example 14. Find the exact value of $\int_0^{\frac{1}{2}\pi} \theta \cdot \sin\frac{1}{2}\theta \, d\theta$ ---[4]

[S-17/33/Q4]

Solution: $\int_0^{\frac{1}{2}\pi} \theta \cdot \sin\frac{1}{2}\theta \, d\theta$ (Integration by parts)

Consider $\int \theta \cdot \sin\frac{1}{2}\theta \, d\theta = \theta \cdot \int \sin\frac{1}{2}\theta \, d\theta - \int\left(\frac{d}{d\theta}\theta \cdot \int \sin\frac{1}{2}\theta \, d\theta\right) d\theta$ {using integration by parts.

$$= \theta\left(\frac{-\cos\theta/2}{1/2}\right) - \int 1 \times \left(\frac{-\cos\theta/2}{1/2}\right) d\theta$$

$$= -2\theta\cos\theta/2 + 2\int \cos\theta/2 \, d\theta$$

$$= -2\theta\cos\theta/2 + 2 \times \frac{\sin\theta/2}{1/2} = -2\theta\cos\frac{\theta}{2} + 4\sin\theta/2$$

$$\therefore \int_0^{\frac{\pi}{2}} \theta\sin\theta/2 \, d\theta = \left[-2\theta\cos\theta/2 + 4\sin\theta/2\right]_0^{\pi/2}$$

$$\left(-2 \times \frac{\pi}{2}\cos\frac{\pi}{4} + 4 \times \frac{1}{\sqrt{2}}\right) - (-0 + 0) = \frac{-\pi}{\sqrt{2}} + \frac{4}{\sqrt{2}}$$

$$= (4-\pi)/\sqrt{2} \checkmark$$

Example 15. Find the exact value of $\int_0^{\frac{1}{2}\pi} x^2 \sin 2x \, dx$ ---[5]

[S-16/32/Q3]

Solution. Consider $\int x^2 \cdot \sin 2x \, dx$

$$= x^2 \cdot \int \sin 2x \, dx - \int\left(\frac{d}{dx}x^2 \cdot \int \sin 2x \, dx\right) dx$$ (using integration by Parts)

$$= x^2 \cdot \left(\frac{-\cos 2x}{2}\right) - \int \cancel{2}x \cdot \left(-\frac{\cos 2x}{\cancel{2}}\right) dx$$

$$= -\frac{1}{2}x^2\cos 2x + \int x\cos 2x \, dx$$ Again apply by Parts

$$= -\frac{1}{2}x^2\cos 2x + x\int \cos 2x \, dx - \int\left(\frac{d}{dx}x \cdot \int \cos 2x \, dx\right) dx$$

$$= -\frac{1}{2}x^2\cos 2x + x\frac{\sin 2x}{2} - \int 1 \times \frac{\sin 2x}{2} \, dx$$

$$= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x - \frac{1}{2}\left(-\frac{\cos 2x}{2}\right)$$

$$= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x.$$

$$\therefore \int_0^{\frac{1}{2}\pi} x^2\sin 2x \, dx = \left[-\frac{1}{2}(\cos 2x)x^2 + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right]_0^{\pi/2}$$

$$= \left(-\frac{1}{2}(-1)\frac{\pi^2}{4} + 0 + \frac{1}{4}(-1)\right) - \left(0 + 0 + \frac{1}{4}\right)$$

$$= \frac{\pi^2}{8} - \frac{1}{2} = \frac{1}{8}(\pi^2 - 4) \checkmark$$

Integration

Example 16. Find the exact value of $\int_0^2 x^2 e^{2-x}\,dx$ ...[6]

[S-15/31/Q9(ii)]

Solution. Consider $\int x^2 e^{2-x}\,dx$ (Using Integration by Parts.)

$$= x^2 \cdot \int e^{2-x}\,dx - \int\left(\frac{d}{dx}x^2 \cdot \int e^{2-x}\,dx\right)dx$$

$$= x^2 \cdot \frac{e^{2-x}}{-1} - \int\left(2x \cdot \frac{e^{2-x}}{-1}\right)dx$$

$$= -x^2 e^{2-x} + 2\int x e^{2-x}\,dx \quad \text{(Again By Parts)}$$

$$= -x^2 e^{2-x} + 2\left[x\int e^{2-x}\,dx - \int\left(\frac{d}{dx}x \cdot \int e^{2-x}\,dx\right)dx\right]$$

$$= -x^2 e^{2-x} + 2\left[x\frac{e^{2-x}}{-1} - \int 1 \times \frac{e^{2-x}}{-1}\,dx\right]$$

$$= -x^2 e^{2-x} + 2\left[-xe^{2-x} + \frac{e^{2-x}}{-1}\right]$$

$$= e^{2-x}\left[-x^2 - 2x - 2\right]$$

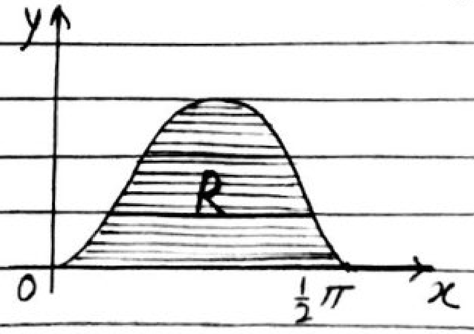
$$\therefore \int_0^2 x e^{2-x}\,dx = \left[e^{2-x}(-x^2-2x-2)\right]_0^2$$

$$= \left[e^0(-4-4-2) - e^2(0-0-2)\right]$$

$$= 1\cdot(-10) + 2e^2 = (2e^2 - 10)$$

Example 17. The diagram shows the curve $y = 5\sin^2 x\cos^3 x$ for $0 \le x \le \frac{1}{2}\pi$. The shaded area R is bounded by the curve and the x-axis ...[4]

Using the substitution $u = \sin x$, find the exact value of R.



Solution: $R = \int_0^{\pi/2} y\,dx$

$$= \int_0^{\pi/2} 5\sin^2 x\cos^3 x\,dx$$

$$= \int_0^{\pi/2} 5\sin^2 x(1-\sin^2 x)\cos x\,dx$$

Put $u = \sin x$

$\frac{du}{dx} = \cos x \Rightarrow du = \cos x\,dx$

adjust the limits

$x = 0,\ u = 0$

$x = \frac{\pi}{2},\ u = 1$

$$= \int_0^1 5u^2(1-u^2)\,du$$

$$= \int_0^1 5(u^2 - u^4)\,du = 5\left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^1$$

$$= 5\left[\frac{1}{3} - \frac{1}{5}\right]$$

$$= \frac{2}{3}$$

(P3)

Integrations

Example 18. Show that $\int_0^{\ln 4} \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx = \ln\left(\frac{8}{5}\right)$ --- [10]

Use the substitution $u = e^x$.

W-14/33/Q10

Solution: $\int_0^{\ln 4} \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx$

Put $u = e^x$ — ①

$\frac{du}{dx} = e^x$

$du = e^x dx$

adjust the limits

from ① $x = 0;\ u = e^0 = 1$

$x = \ln 4;\ u = e^{\ln 4} = 4$

$$= \int_0^{\ln 4} \frac{e^x \cdot e^x dx}{(e^x)^2 + 3e^x + 2}$$

$$= \int_1^4 \frac{u\,du}{u^2+3u+2}$$

$$= \int_1^4 \frac{u}{(u+2)(u+1)}\,du$$

$$= \int_1^4 \left(\frac{2}{u+2} - \frac{1}{u+1}\right) du \quad \text{(using partial fractions)}$$

$$= \left[2\ln(u+2) - \ln(u+1)\right]_1^4$$

$$= (2\ln 6 - \ln 5) - (2\ln 3 - \ln 2)$$

$$= \ln 36 - \ln 5 - \ln 9 + \ln 2$$

$$= \ln\left(\frac{36\times 2}{5\times 9}\right) = \ln\left(\frac{8}{5}\right) \checkmark$$

Example 19. (i) Prove that $\cot\theta + \tan\theta = 2\operatorname{cosec} 2\theta$ --- [3]

(ii) Hence show that $\int_{\frac{\pi}{6}}^{\pi/3} \operatorname{cosec} 2\theta\, d\theta = \frac{1}{2}\ln 3$ -- [4]

W-13/31/Q5

Solution (i) $\cot\theta + \tan\theta = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{2\times 1}{2\times\sin\theta\cos\theta}$$

$$= \frac{2}{\sin 2\theta}$$

$$= 2\operatorname{cosec} 2\theta.$$

(ii) $\int_{\frac{\pi}{6}}^{\pi/3} \operatorname{cosec} 2\theta\, d\theta$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\pi/3} (\tan\theta + \cot\theta)\,d\theta \quad \text{from Part (i)}$$

$$= \frac{1}{2}\left[\ln\sin\theta - \ln\cos\theta\right]_{\frac{\pi}{6}}^{\pi/3}$$

$$= \frac{1}{2}\left[\ln\tan\theta\right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left[\ln\tan\frac{\pi}{3} - \ln\tan\frac{\pi}{6}\right]$$

$$= \frac{1}{2}\left[\ln\sqrt{3} - \ln\frac{1}{\sqrt{3}}\right] = \frac{1}{2}\ln 3 \checkmark$$

(P3)

Integration

§ $\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$

Example 20 Show that, $\int \tan x\,dx = -\log \cos x + c$

Solution: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

let $\cos x = u$
diff
$-\sin x\,dx = du$

$= \int \frac{-1\,du}{u}$

$= -\ln u = -\ln \cos x + c$

Example 21

Show that, $\int \cot x\,dx = \ln \sin x + c$

Solution: $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$

$\sin x = u$
$\cos x\,dx = du$

$\int \frac{1}{u}\,du$

$= \ln u$

$= \ln \sin x + c$

Example 22. $\int \sec x\,dx = \ln(\sec x + \tan x)$

Solution. $\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}\,dx$

Put $\sec x + \tan x = u$
$(\sec x \tan x + \sec^2 x)dx = du$
$\sec x(\sec x + \tan x)dx = du$

$= \int \frac{1}{u}\,du$

$= \ln u$

$= \ln(\sec x + \tan x) + c$