Date 06.02.21

0606

Additional Maths

Differentiation Revision

SP-20	M-20	S-20	W-20

Suresh Goel
(Former Director)
Alliance World School,
Noida Delhi-NCR
INDIA

(+91 9810444804)

Differentiation Revision

1 Variables x and y are related by the equation $y = x\sqrt{x}$

(a) Find $\frac{dy}{dx}$ ...[2]

(b) Hence find the approximate change in x when y increases from 8 by the small amount 0.015. --[3]

[SP-20|01|Q2]

Solution (a) $y = x\sqrt{x} = x^{3/2}$ —— ①

$$\frac{dy}{dx} = \frac{3}{2}x^{3/2-1} = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x} \checkmark \text{ —— ②}$$

(b) from ① at $y=8 \Rightarrow x^{3/2} = 8 = 4^{3/2} \Rightarrow x = 4$

Let the small change in x is δx

$$\Rightarrow \frac{\delta y}{\delta x} \approx \left(\frac{dy}{dx}\right)_{x=4} \Rightarrow \frac{0.015}{\delta x} = \frac{3}{2}\sqrt{4} = 3 \quad [\delta y = 0.015] \quad \left[\because \frac{dy}{dx} = \frac{3}{2}\sqrt{x}\right]$$

$$\Rightarrow \delta x = \frac{0.015}{3} = 0.005 \checkmark$$

---

2. Find the equation of normal to the curve $y = \frac{2x-1}{\sqrt{x^2+5}}$ at the point where $x=2$. Give your answer in the form $ax+by=c$, where a, b and c are integers. --[8]

[SP-20|01|Q6]

Solution:

$$y = \frac{(2x-1)}{\sqrt{x^2+5}} \text{ —— ①} \qquad \left[y = \frac{u}{v}, \quad \frac{dy}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2} \right.$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+5} \times 2 - (2x-1) \times \frac{1}{2\sqrt{x^2+5}} \times 2x}{x^2+5} \qquad \left[\frac{d}{dx}\sqrt{u} = \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx} \right.$$

Chain Rule

$$= \frac{2(x^2+5) - x(2x-1)}{(x^2+5)^{3/2}}$$

$$= \frac{10+x}{(x^2+5)^{3/2}} \Rightarrow \left(\frac{dy}{dx}\right)_{x=2} = \frac{12}{27} = \frac{4}{9} \checkmark$$

gradient of tangent $= 4/9$

$\therefore$ gradient of Normal $= -\frac{9}{4} \checkmark$

at $x=2$, $y = \frac{3}{3} = 1$ from ①

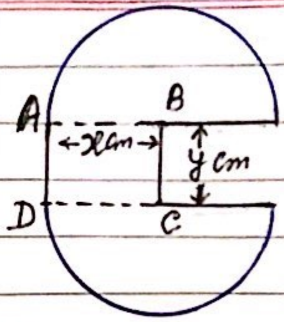
$\therefore$ Equation of normal to the curve at $(2,1)$, Grad $= -\frac{9}{4}$

$$y - 1 = -\frac{9}{4}(x-2) \Rightarrow 4(y-1) = -9x+18 \Rightarrow 9x+4y = 22$$

Equation of Normal $9x+4y = 22 \checkmark$

Scanned with CamScanner

3 The diagram shows a badge, made of thin sheet metal, consisting of two semi-circular pieces, Centres B and C, each of radius x cm. They are attached to each other by a rectangular piece of thin sheet metal, ABCD, such that AB and CD are radii of the semicircular pieces and $AD = BC = y$ cm.



(a) Given that the area of the badge is 20cm^2, show that the perimeter, P cm, of the badge is given by $P = 2x + \frac{40}{x}$ ---[4]

(b) Given that x can vary, find the minimum value of P, justifying that this value is a minimum. ---[5]

SP-20/01/Q7

Solution (a) Area of the badge = 2× area of semicircle + area of rectangle ABCD

$$= \pi x^2 + xy = 40 \text{ given}$$

$$\Rightarrow y = \frac{(40-\pi x^2)}{x} \quad \text{——— ①}$$

Perimeter of the badge,

$$P = 2\pi x + 2x + 2y$$

$$= 2\pi x + 2x + 2\left(\frac{40-\pi x^2}{x}\right) \quad \text{from ①}$$

$$= \cancel{2\pi x} + 2x + \frac{40}{x} - \cancel{2\pi x}$$

$$\therefore \quad P = \underline{2x + \frac{40}{x}} \checkmark \quad \text{——— ②}$$

(b) $P = 2x + 40x^{-1}$

$$\frac{dP}{dx} = 2 - 40x^{-2} \quad \text{——— ③}$$

$$\frac{d^2P}{dx^2} = 80x^{-3} = \frac{80}{x^3} \quad \text{——— ④}$$

Now for P be Max/Min $\frac{dP}{dx} = 0 \Rightarrow 2 - \frac{40}{x^2} = 0$ from ③

$$\Rightarrow 2x^2 = 40 \Rightarrow x^2 = 20 \Rightarrow x = \sqrt{20} = 2\sqrt{5} \checkmark$$

Now from ④ $\left(\frac{d^2P}{dx^2}\right)_{x=2\sqrt{5}} = \frac{80}{(2\sqrt{5})^3} > 0$ hence Min.

$\therefore$ P is Minimum at $\underline{x = 2\sqrt{5}} \checkmark$

and Minimum value of $P = 2\times 2\sqrt{5} + \frac{40}{2\sqrt{5}} = 4\sqrt{5} + 4\sqrt{5} = 8\sqrt{5}$

$$= \underline{17.89\text{ cm}}$$

4. The tangent to the curve $y = \ln(3x^2-4) - \frac{x^3}{6}$, at a point $x=2$, meets the y-axis at the point P. Find the exact coordinates of P. ---[6]
[M-20|12|Q4]

Solution: $y = \ln(3x^2-4) - \frac{x^3}{6}$ —— ①

$\frac{dy}{dx} = \frac{6x}{3x^2-4} - \frac{x^2}{2}$

$\left(\frac{dy}{dx}\right)_{x=2} = \frac{12}{12-4} - \frac{4}{2} = \frac{3}{2} - 2 = -\frac{1}{2}$

for ① at $x=2$, $y = \ln(3\times2^2-4) - \frac{2^3}{6} = \ln 8 - \frac{4}{3}$ ✓

∴ The equation of tangent at $(2, \ln 8 - \frac{4}{3})$, $m = -\frac{1}{2}$ is

$y - (\ln 8 - \frac{4}{3}) = -\frac{1}{2}(x-2)$ —— ②

tangent② meets the y-axis at $x = 0$ in ②

$\Rightarrow y - \ln 8 + \frac{4}{3} = 1 \Rightarrow y = (\ln 8 - \frac{1}{3})$

∴ $P(0, \ln 8 - \frac{1}{3})$ ✓

---

5. Variables x and y are such that $y = \frac{e^{3x} \cdot \sin x}{x^2}$, use differentiation to find the approximate change in y as x increases from 0.5 to $0.5+h$, where h is small.
[M-20|22|Q9]

Solution: $y = \frac{e^{3x} \cdot \sin x}{x^2}$

$\frac{dy}{dx} = \frac{x^2 \frac{d}{dx}(e^{3x} \cdot \sin x) - (e^{3x} \cdot \sin x) \cdot \frac{d}{dx} x^2}{(x^2)^2}$

$= \frac{x^2(e^{3x} \cdot \cos x + 3e^{3x} \cdot \sin x) - 2x(e^{3x} \cdot \sin x)}{x^4}$

$= (e^{3x} \cdot \cos x \cdot x^2 + 3x^2 e^{3x} \sin x - 2x e^{3x} \sin x) / x^4$

$= e^{3x}[x^2 \cos x + \sin x(3x^2 - 2x)] / x^4$

$\left(\frac{dy}{dx}\right)_{x=0.5} = e^{1.5}[(0.5)^2 \cos 0.5 + \sin 0.5(3\times0.5^2 - 2\times0.5)] / (0.5)^4$

$= 4.48168[0.21939 - 0.11985] / 0.0625$

$= 7.137$ ✓

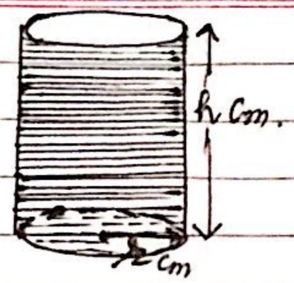
Now $\frac{\delta y}{\delta x} \approx \left(\frac{dy}{dx}\right)_{x=0.5} \Rightarrow \delta y = 7.137 \times \delta x$ $[\delta x = (0.5+h) - 0.5 = h$

$= 7.137h$ ✓

Change in y $= \delta y = \underline{7.137h}$ ✓

Scanned with CamScanner

6. A container is circular cylinder, open at one end, with a base radius of r cm and a height of h cm. The volume of the container is $1000\,cm^3$. Given that r and h can vary and that the total outer surface area of the container has a minimum value, find its value.



[M-20/22/Q 11] --[8]

Solution: Volume of cylinder: $\pi r^2 h = 1000$

$$\Rightarrow h = \frac{1000}{\pi r^2} \quad \text{——— (1)}$$

Now Total surface area,

$$S = \pi r^2 + 2\pi r h \qquad \text{(open at the top)}$$

$$= \pi r^2 + 2\pi r \times \frac{1000}{\pi r^2}$$

$$S = \pi r^2 + \frac{2000}{r} \quad \text{——— (2)}$$

$$S = \pi r^2 + 2000 r^{-1}$$

$$\frac{dS}{dr} = 2\pi r - 2000 r^{-2} = 0 \quad \text{for Minimum "S"}$$

$$\Rightarrow 2\pi r = \frac{2000}{r^2} \Rightarrow r^3 = \frac{1000}{\pi}$$

$$r = \left(\frac{1000}{\pi}\right)^{1/3}$$

$\therefore$ Min. surface area

from (2) $$S = \pi\left(\frac{1000}{\pi}\right)^{2/3} + \frac{2000}{\left(\frac{1000}{\pi}\right)^{1/3}}$$

$$= 146.46 + 292.92$$

$\therefore$ Minimum value of $S = 439.38$ ✓

---

7. The volume, V, of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$. The radius, r cm, of a sphere is increasing at the rate of $0.5\,cm\,s^{-1}$. Find in terms of π, the rate of change of the volume of the sphere when $r = 0.25$

[S-20/12/Q2] --[4]

Solution: $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2 \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Given $r = 0.25 = \frac{1}{4}$ cm and $\frac{dr}{dt} = 0.5\,ms^{-1}$

$$\left(\frac{dV}{dt}\right)_{r=\frac{1}{4}} = 4\pi \times \left(\frac{1}{4}\right)^2 \times 0.5$$

$$= 0.125\pi \ cm^3 s^{-1}$$ ✓

8. The radius, r cm, of circle is increasing at the rate of $5\ cm\,s^{-1}$. Find in terms of π, the rate at which the area of the circle is increasing when $r = 3$. [S-20|11|Q3] -- [4]

Solution: Given $\frac{dr}{dt} = 5$ —— ①

Area of Circle $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

$$\left(\frac{dA}{dt}\right)_{r=3} = 2\pi \times 3 \times 5 \quad \left(\text{from ① } \frac{dr}{dt} = 5\right)$$

$$= 30\pi \checkmark$$

9. Find the equation of tangent to the curve $y = \frac{\ln(3x^2-1)}{(x+2)}$ at the point where $x = 1$. Give your answer in the form $y = mx + c$, where m and c are constants correct to 3 d.p. -- [6]
[S-20| 11|Q5]

Solution: $y = \frac{\ln(3x^2-1)}{(x+2)}$ —— ①

$$\frac{dy}{dx} = \frac{(x+2)\cdot\frac{d}{dx}\ln(3x^2-1) - \ln(3x^2-1)\cdot\frac{d}{dx}(x+2)}{(x+2)^2}$$

$$= \frac{\frac{(x+2)\cdot 6x}{(3x^2-1)} - \ln(3x^2-1)}{(x+2)^2}$$

$$\frac{dy}{dx} = \frac{(x+2)\cdot 6x - (3x^2-1)\ln(3x^2-1)}{(3x^2-1)(x+2)^2} \text{ —— ②}$$

from ① at $x = 1$, $y = \frac{\ln 2}{3}$ ✓

$$\left(\frac{dy}{dx}\right)_{x=1} = \frac{18 - 2\ln 2}{18} = 0.923 \quad (\text{from } ②)$$

$\therefore$ Equation of tangent at $x = 1$, $y = \frac{\ln 2}{3}$

$$y - \frac{\ln 2}{3} = 0.923(x-1)$$

$$\Rightarrow y - 0.231 = 0.923x - 0.923$$

or $y = 0.923x - 0.692$ ✓

10.(a) Find the x-coordinate of the stationary points of the curve $y = e^{3x}(2x+3)^6$ --[6]

(b) A curve has equation $y = f(x)$ and has exactly two stationary points. Given that $f''(x) = 4x-7$, $f'(0.5)=0$ and $f'(3)=0$, use second derivative test to determine the nature of each of the stationary points of this curve. [S-20/21/Q12] --[2]

Solution (a) $y = e^{3x}.(2x+3)^6$ [Product rule of diff]

$\frac{dy}{dx} = e^{3x} \cdot \frac{d}{dx}(2x+3)^6 + (2x+3)^6 \cdot \frac{d}{dx} e^{3x}$

$= e^{3x}.\,6(2x+3)^5 \times 2 + (2x+3)^6 \times e^{3x} \times 3$

$= 3e^{3x}.(2x+3)^5[4+(2x+3)]$

$\frac{dy}{dx} = 3e^{3x}(2x+3)^5(2x+7) = 0$ for stationary points

$\Rightarrow 2x+3=0; \; 2x+7=0$ $[e^{3x} \neq 0]$

$\therefore$ Stationary points at; $x = \underline{-1.5}$ ✓; $x = \underline{-3.5}$ ✓

(b) as $f'(0.5)=0$ and $f'(3)=0$

$\therefore$ The curve $y=f(x)$ has stationary points at $x=0.5$ and $x=3$

To Check the nature of Stationary point

Given $f''(x) = 4x-7$ —— (1)

at $x=0.5 \rightarrow f''(0.5) = 4\times 0.5 - 7 = -5 < 0$

$\therefore$ <u>Max at $x = 0.5$</u> ✓

and at $x=3 \longrightarrow f''(3) = 4\times 3 - 7 = 5 > 0$,

$\therefore$ <u>Min at $x=3$</u> ✓

11 (a) Given that $y=(x^2-1)\sqrt{5x+2}$, show that $\frac{dy}{dx}=\frac{Ax^2+Bx+C}{2\sqrt{5x+2}}$, where A, B and C are integers. ---[5]

(b) Find the coordinates of the stationary point of the curve $y=(x^2-1)\sqrt{5x-2}$, for $x>0$, Give coordinate correct to 2 significant figures. --[3]

(C) determine the nature of this stationary point --[2]

S-20	12	Q 7

Solution (a) $y=(x^2-1)\cdot\sqrt{5x+2}$ —— ①

$$\frac{dy}{dx}=(x^2-1)\frac{d}{dx}\sqrt{5x+2}+\sqrt{(5x+2)}\frac{d}{dx}(x^2-1)$$

$$=(x^2-1)\times\frac{5}{2}(5x+2)^{-\frac{1}{2}}+2x\cdot\sqrt{5x+2}$$

$$=\frac{5(x^2-1)+4x(5x+2)}{2\sqrt{5x+2)}}$$

$$\frac{dy}{dx}=\frac{25x^2+8x-5}{2\sqrt{5x+2}} \quad \text{—— ②}$$

(b) For stationary point $\frac{dy}{dx}=0 \Rightarrow \frac{25x^2+8x-5}{2\sqrt{5x+2}}=0$

$$\Rightarrow 25x^2+8x-5=0$$

$$\Rightarrow x=\frac{-8\pm\sqrt{564}}{50} \qquad \left\{\begin{array}{l} b^2-4ac \\ =64+500 \\ =564 \end{array}\right.$$

$$=\frac{-8\pm 23\cdot 7486}{50}$$

$$=0\cdot 315 \checkmark \qquad (\because x>0)$$

at $x=0\cdot 315$, $y=\left((0\cdot 315)^2-1\right)\sqrt{5\times 0\cdot 315+2}=-0\cdot 9\times\sqrt{3\cdot 575}$
$=-1\cdot 70$

$\therefore$ Coordinates of the stationary point $(0\cdot 315, -1\cdot 70)$

(C) To check the nature of the stationary point at $x=0\cdot 315$

from $\left(\frac{dy}{dx}\right)_{x=0\cdot 3}=\frac{25(0\cdot 3)^2+8(0\cdot 3)-5}{2\sqrt{5\times\cdot 3+2}}=\frac{-0\cdot 35}{+}<0$

and $\left(\frac{dy}{dx}\right)_{x=0\cdot 4}=\frac{25(0\cdot 4)^2+8(0\cdot 4)-5}{2\sqrt{5\times 0\cdot 4+2}}=\frac{2\cdot 2}{+}>0$

The gradient of the curve changes sign from –ve to + as x moves from left to right on the stationary point. Hence there is a Minimum at the stationary point.

12. Variables x and y are such that $y = \sin x + e^{-x}$. Use differtiation to find the approximate change in y as x increases from $\frac{\pi}{4}$ to $\frac{\pi}{4} + h$, where h is small. ---[4]

[S-20/22/Q1]

Solution: $y = \sin x + e^{-x} \Rightarrow \frac{dy}{dx} = \cos x - e^{-x}$

$$\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = \cos\frac{\pi}{4} - e^{-\frac{\pi}{4}} = 0.7071 - 0.4559$$
$$= 0.2512 \quad (1)$$

Now Change y, for a small change in x.

$$\delta y = \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} \times h = \underline{0.251h} \quad \text{from } (1)$$

13 (a) Find the equation of the tangent to the curve $2y = \tan 2x + 7$ at the point where $x = \frac{\pi}{8}$. Give your answer in the form $ax - y = \frac{\pi}{b} + c$, where a, b and c are integers. ---[5]

(b) This tangent intersects the x-axis at P and the y-axis at Q. Find the length of PQ. [S-20/22/Q6] ---[2]

Solution: $2y = \tan 2x + 7 \quad (1)$

at $x = \frac{\pi}{8} \rightarrow 2y = \tan\frac{\pi}{4} + 7 = 8 \Rightarrow y = 4$, Point $\left(\frac{\pi}{8}, 4\right)$ ✓

Differentiating (1)

$$2\frac{dy}{dx} = 2\sec^2 \Rightarrow \frac{dy}{dx} = \sec^2 2x$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{8}} = \sec^2\frac{\pi}{4} = 2 ✓$$

gradient of the tangent $= 2$.

$\therefore$ Equation of tangent at $\left(\frac{\pi}{8}, 4\right)$,

$$y - 4 = 2\left(x - \frac{\pi}{8}\right)$$
$$\Rightarrow \underline{2x - y = \frac{\pi}{4} - 4} ✓ \quad (2)$$

Now tangent (2) intersects x-axis at $y = 0 \Rightarrow 2x = \frac{\pi}{4} - 4$

$\therefore P\left(\frac{\pi}{8} - 2, 0\right)$ ✓ $\qquad x = \left(\frac{\pi}{8} - 2\right)$

and intersects y-axis at $x = 0 \rightarrow 0 - y = \frac{\pi}{4} - 4 \Rightarrow y = \left(4 - \frac{\pi}{4}\right)$

$\therefore Q\left(0, 4 - \frac{\pi}{4}\right)$

$$\therefore PQ = \sqrt{\left(\frac{\pi}{8} - 2\right)^2 + \left(4 - \frac{\pi}{4}\right)^2} = \sqrt{(1.607)^2 + (3.2146)^2}$$
$$= \sqrt{2.5834 + 10.3336}$$
$$PQ = \underline{3.59}$$

14. (a) Given that $y = x\sqrt{x+2}$, show that $\frac{dy}{dx} = \frac{Ax+B}{2\sqrt{x+2}}$, where A and B are constants. ---[5]

(b) Find the exact coordinates of the stationary point of the curve $y = x\sqrt{x+2}$ ---[3]

(c) determine the nature of this stationary point. --- [2]

[S:20/13/Q10]

Solution (a) $y = x \cdot (x+2)^{1/2}$ —— ①

$$\frac{dy}{dx} = x \cdot \frac{d}{dx}(x+2)^{\frac{1}{2}} + (x+2)^{1/2} \cdot \frac{d}{dx}x$$

$$= x \times \frac{1}{2}(x+2)^{-\frac{1}{2}} + (x+2)^{1/2}$$

$$= \frac{x + 2(x+2)}{2\sqrt{x+2}} = \frac{3x+4}{2\sqrt{x+2})}$$ —— ② ✓

(b) For stationary point $\frac{dy}{dx} = 0 \Rightarrow \frac{3x+4}{2\sqrt{x+2}} = 0 \Rightarrow x = -\frac{4}{3}$ ✓

at $x = -\frac{4}{3}$ from ① $y = -\frac{4}{3}\sqrt{\frac{2}{3}} = -\frac{4\sqrt{6}}{9}$ ✓

∴ Stationary point $\left(-\frac{4}{3}, -\frac{4\sqrt{6}}{9}\right)$ ✓

(c) Diff ② $\frac{d^2y}{dx^2} = \frac{2\sqrt{x+2} \times 3 - (3x+4) \times \frac{\cancel{2}\,1}{2\sqrt{x+2}}}{4(x+2)}$

$$= \frac{6(x+2) - (3x+2)}{4(x+2)^{3/2}} = \frac{3x+10}{4(x+2)^{3/2}}$$ —— ③

For nature of stationary point at $x = -4/3$

from ③ $\left(\frac{d^2y}{dx^2}\right)_{x=-\frac{4}{3}} = \frac{6}{4\left(\frac{2}{3}\right)^{3/2}} > 0$

∴ There is a minimum at $x = -4/3$ ✓

15 (a) Differentiate $y = \tan(x+4) - 3\sin x$ with respect to x. ---[2]

(b) Variables x and y are such that $y = \frac{\ln(2x+5)}{2e^{3x}}$. Use differentiation to find the approximate change in y as x increases from 1 to $1+h$, where h is small. --[6]

[S-20/23/Q8]

Solution (a) $y = \tan(x+4) - 3\sin x$

$$\frac{dy}{dx} = \sec^2(x+4) - 3\cos x$$

(b) $y = \frac{\ln(2x+5)}{2e^{3x}} \Rightarrow \frac{dy}{dx} = \frac{2e^{3x}\cdot\frac{d}{dx}\ln(2x+5) - \ln(2x+5)\cdot\frac{d}{dx}2e^{3x}}{(2e^{3x})^2}$

$$= \frac{2e^{3x}\times\frac{2}{(2x+5)} - \ln(2x+5)\times 6e^{3x}}{4e^{6x}}$$

$$= \frac{4e^{3x} - (2x+5)\ln(2x+5)\times 6e^{3x}}{4e^{6x}\cdot(2x+5)}$$

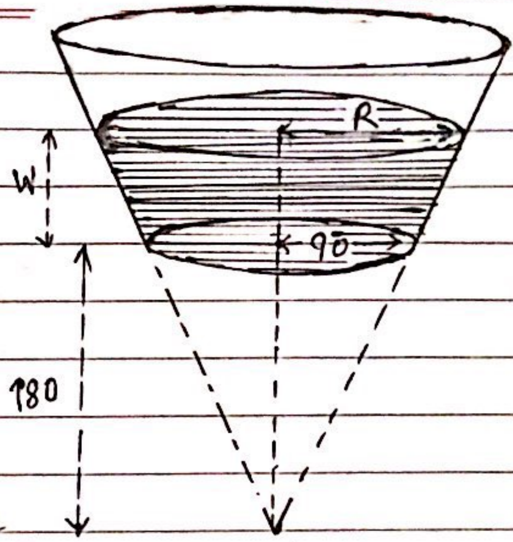
$$= \frac{2e^{3x}[2 - 3\times(2x+5)\ln(2x+5)]}{4e^{6x}(2x+5)}$$

$$= \frac{2 - 3(2x+5)\ln(2x+5)}{2e^{3x}(2x+5)}$$

$$\left(\frac{dy}{dx}\right)_{x=1} = \frac{2 - 3\times 7\cdot\ln 7}{2e^3\times 7} = -0.138 \quad \text{①}$$

∴ Approximate change in y, when x changes from 1 to $1+h$

$$\delta y = \left(\frac{dy}{dx}\right)_{x=1}\times h = -0.138h \quad (\text{from ①})$$



16. In this question all lengths are in centimetres.
The volume, V, of a cone of height h and base radius r is given by $V = \frac{1}{3}\pi r^2 h$.
The diagram shows a large hollow cone from which a smaller cone of height 180 and base radius 90 has been removed. The remainder has been fitted with a circular base of radius 90 to form a container for water. The depth of water in the container is w and the surface of the water is a circle of radius R.

(a) Find an expression for R in terms of w and show that the volume V of the water in the container is given by: $V = \frac{\pi}{12}(w+180)^3 - 486000\pi$. ---[3]

(b) Water is poured into the container at a rate of $10000\ \text{cm}^3\text{s}^{-1}$. Find the rate at which the depth of the water is increasing when $w = 10$.

S-20/23/Q 11 --[4]

Solution (a) $\frac{R}{90} = \frac{w+180}{180} \Rightarrow R = \frac{1}{2}(w+180)$ —— ①

$$V = \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi\left[\frac{1}{2}(w+180)\right]^2(w+180) - \frac{1}{3}\pi \times 90^2 \times 180$$
$$= \frac{1}{3}\pi \times \frac{1}{4}(w+180)^3 - \frac{1}{3}\pi \times 8100 \times 180$$
$$V = \frac{\pi}{12}(w+180)^3 - 486000\pi \checkmark \text{ —— ②}$$

(b) $\frac{dV}{dt} = \frac{dV}{dw} \times \frac{dw}{dt} \Rightarrow \left(\frac{dw}{dt}\right)_{w=10} = \frac{10000}{\left(\frac{dV}{dw}\right)_{w=10}}$ —— ③

diff ②

$$\frac{dV}{dw} = 3 \times \frac{\pi}{12}(w+180)^2 - 0$$
$$\left(\frac{dV}{dw}\right)_{w=10} = \frac{\pi}{4}(190)^2 = 28352.87$$

from ③ $\left(\frac{dw}{dt}\right)_{w=10} = \frac{10,000}{28352.87} = 0.3526(97--)$

$\therefore \frac{dw}{dt} = \underline{0.3527} \checkmark$

17. It is given that $y = \frac{\tan 3x}{\sin x}$

(a) Find the exact value of $\frac{dy}{dx}$, when $x = \frac{\pi}{3}$ --[4]

(b) Hence find the approximate change in y as x increases from $\frac{\pi}{3}$ to $\frac{\pi}{3}+h$ where h is small. --[1]

(c) Given that x increases at the rate of 3 units per seconds, find the corresponding rate of change in y when $x = \frac{\pi}{3}$, giving your answer in its simplest surd form. [W-20/11/Q4] --[2]

Solution (a) $y = \frac{\tan 3x}{\sin x} \Rightarrow \frac{dy}{dx} = \frac{\sin x \cdot \frac{d}{dx}\tan 3x - \tan 3x \cdot \frac{d}{dx}\sin x}{\sin^2 x}$

$$= \frac{\sin x \times 3\sec^2 3x - \tan 3x \cdot \cos x}{\sin^2 x}$$

$$\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{3}} = \frac{3\sin\frac{\pi}{3} \cdot \sec^2\pi - \tan\pi \cdot \cos\pi/3}{\sin^2\frac{\pi}{3}} = \frac{3\frac{\sqrt{3}}{2} - 0}{\frac{3}{4}} = \underline{2\sqrt{3}} \checkmark \text{ —— (1)}$$

(b) Approximate change in $y \rightarrow \delta y = \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{3}} \times \delta x = \underline{2\sqrt{3} \times h} \checkmark$ (from (1))

(c) Given $\frac{dx}{dt} = 3$ —— (2)

$$\frac{dy}{dt} = \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{3}} \times \frac{dx}{dt} \quad \text{(from (1) \& (2))}$$

$$= 2\sqrt{3} \times 3 = \underline{6\sqrt{3}} \checkmark$$

---

18. It is given that $y = \ln(\sin x + 3\cos x)$ for $0 < x < \frac{\pi}{2}$

(a) Find $\frac{dy}{dx}$ ---[3]

(b) Find the value of x for which $\frac{dy}{dx} = -\frac{1}{2}$ ---[3] [W-20/21/Q4]

Solution (a) $y = \ln(\sin x + 3\cos x)$

$$\frac{dy}{dx} = \frac{\cos x - 3\sin x}{\sin x + 3\cos x} \checkmark \text{ —— (1)}$$

(b) Given $\frac{dy}{dx} = -\frac{1}{2} \Rightarrow \frac{\cos x - 3\sin x}{\sin x + 3\cos x} = -\frac{1}{2}$

$\Rightarrow 2\cos x - 6\sin x = -\sin x - 3\cos x \Rightarrow 5\sin x = 5\cos x$

$\Rightarrow \tan x = 1$

$\Rightarrow x = \underline{\frac{\pi}{4}} \checkmark$

19. The equation of a curve is $y = x\sqrt{16-x^2}$ for $0 \leq x \leq 4$

(a) Find the exact coordinates of the stationary point of the curve. --[6]

[W-20/21/Q 11(a)]

Solution: $y = x\sqrt{16-x^2}$ —— ①

$$\frac{dy}{dx} = x \cdot \frac{d}{dx}(16-x^2)^{1/2} + \sqrt{16-x^2} \cdot \frac{d}{dx}x$$

$$= x \times \frac{1}{2}(16-x^2)^{-\frac{1}{2}} \times (-2x) + \sqrt{16-x^2} \times 1$$

$$= \frac{-x^2}{\sqrt{16-x^2}} + \sqrt{16-x^2} \text{ —— ②}$$

for stationary point $\frac{dy}{dx} = 0$

fm ② $\frac{-x^2}{\sqrt{16-x^2}} + \sqrt{16-x^2} = 0$

$$\Rightarrow \sqrt{16-x^2} = \frac{x^2}{\sqrt{16-x^2}}$$

$$\Rightarrow 16 - x^2 = x^2 \Rightarrow x^2 = 8 \Rightarrow x = 2\sqrt{2}$$

from ① at $x = 2\sqrt{2}$, $y = 2\sqrt{2} \cdot \sqrt{8} = 8$

∴ Coordinates of the stationary point are $(2\sqrt{2}, 8)$

20. A curve has equation $y = \frac{\ln(3x^2-5)}{2x+1}$ for $3x^2 > 5$

(a) Find the equation of normal to the curve at the point where $x = \sqrt{2}$ --[6]

(b) Find the approximate change in y as x increases from $\sqrt{2}$ to $\sqrt{2}+h$, where h is small.

[W-20/12/Q7] --[1]

Solution (a) $y = \frac{\ln(3x^2-5)}{(2x+1)}$ —— ①

$$\frac{dy}{dx} = \frac{(2x+1) \times \frac{6x}{(3x^2-5)} - 2 \times \ln(3x^2-5)}{(2x+1)^2}$$

$$= \frac{6x(2x+1) - 2(3x^2-5)\ln(3x^2-5)}{(2x+1)^2 \cdot (3x^2-5)}$$

$$\left(\frac{dy}{dx}\right)_{x=\sqrt{2}} = \frac{6\sqrt{2}(2\sqrt{2}+1) - 0}{(2\sqrt{2}+1)^2 \times 1}$$

$$= \frac{6\sqrt{2}}{(2\sqrt{2}+1)} = \frac{(24-6\sqrt{2})}{7}$$

from ① $x = \sqrt{2} \rightarrow y = 0$

∴ Equation of normal:

$$y - 0 = -\frac{7}{(24-6\sqrt{2})}(x - \sqrt{2})$$

$$\Rightarrow y = -0.451x + 0.638$$

(b)

Approximate change in y when x changes from $\sqrt{2}$

$$\delta y = \left(\frac{dy}{dx}\right)_{x=\sqrt{2}} \times h$$

$$= \frac{(24-6\sqrt{2})}{7} \times h$$

or $= 2.22h$

21. A curve has equation $y = (2x-1)\sqrt{4x+3}$

(a) Show that $\frac{dy}{dx} = \frac{4(Ax+B)}{\sqrt{4x+3}}$, where A and B are constants ---[5]

(b) Hence write down the x-coordinate of the stationary point of the curve, ---[1]

(c) Determine the nature of this stationary point. --[2]

[W-20/12/9]

$$y = (2x-1)\sqrt{4x+3} \quad \text{--- (1)}$$

Solution (a)

$$\frac{dy}{dx} = (2x-1)\frac{d}{dx}(4x+3)^{\frac{1}{2}} + \sqrt{4x+3}\cdot\frac{d}{dx}(2x-1)$$

$$= (2x-1) \times 4 \times \frac{1}{2}(4x+3)^{-\frac{1}{2}} + 2\sqrt{4x+3}$$

$$= \frac{2(2x-1)}{\sqrt{4x+3}} + 2\sqrt{4x+3} = \frac{2(2x-1)+2(4x+3)}{\sqrt{4x+3}}$$

$$= \frac{12x+4}{\sqrt{4x+3}} = \frac{4(3x+1)}{\sqrt{4x+3}} \checkmark$$

(b) Stationary point $\frac{dy}{dx} = 0 \Rightarrow \frac{4(3x+1)}{\sqrt{4x+3}} = 0$

$$\Rightarrow x = -\frac{1}{3} \checkmark$$

(c) To Check the nature of stationary point at $x = -\frac{1}{3} = -0.333$

when $x < -\frac{1}{3}$, $\left(\frac{dy}{dx}\right)_{x=-\frac{1}{2}} = \frac{4\left(\frac{-3}{2}+1\right)}{\sqrt{-2+3}} = -ve$

when $x > -\frac{1}{3}$, $\left(\frac{dy}{dx}\right)_{x=0} = \frac{4}{\sqrt{3}} > 0$

As The $\frac{dy}{dx}$ changes sign from $-ve$ to $+$

$\therefore$ Minimum at $x = -\frac{1}{3} \checkmark$

22 (a) Find the equation of the tangent to the curve $y = x^3 - 6x^2 + 3x + 10$, at the point where $x = 1$ --[4]

(b) Find the coordinates of the point where this tangent meets the curve again. --[5]

W-20/22/Q5

Solution (a) $y = x^3 - 6x^2 + 3x + 10$ —— ①

$\frac{dy}{dx} = 3x^2 - 12x + 3$

$\left(\frac{dy}{dx}\right)_{x=1} = 3 - 12 + 3 = -6$ ✓ and $x = 1, y = 8$ from ①

∴ Equation of tangent at the point $(1, 8)$, $m = -6$

$y - 8 = -6(x - 1)$

or $y = \underline{-6x + 14}$ ✓ —— ②

(b) To find the point of intersection of the curve and tangent again from ① & ②

$-6x + 14 = x^3 - 6x^2 + 3x + 10$

$\Rightarrow x^2 - 6x^2 + 9x - 4 = 0$ —— ③

as the point of contact at $x = 1$ is a solution of ③ $(x-1)$ is a factor

$x - 1 \overline{) x^3 - 6x^2 + 9x - 4} (x^2 - 5x + 4$

$\underline{-x^3 \mp x^2}$

$-5x^2 + 9x$

$\underline{-5x^2 + 5x}$

$4x - 4$

$4x - 4$

$\Rightarrow (x-1)(x^2 - 5x + 4) = 0$

$(x-1)(x-1)(x-4) = 0$

$x = 1^{\times}$; $x = 4$

Now $x = 4$ from ①

$y = 4^3 - 6 \times 4^2 + 3 \times 4 + 10$

$= 64 - 96 + 12 + 10$

$= -10$

∴ $(4, -10)$ is the point where tangent meets the curve again.

23 (a) Given that $y = \frac{e^{2x-3}}{x^2+1}$ find $\frac{dy}{dx}$ --[3]

(b) Hence, given that y is increasing at the rate of 2 units per second, find the exact rate of change of x when $x = 2$ ---[3]

W-20/13/Q2

Solution (a) $y = \frac{e^{2x-3}}{x^2+1}$

$\Rightarrow \frac{dy}{dx} = \frac{(x^2+1) \cdot 2e^{2x-3} - e^{2x-3} \times 2x}{(x^2+1)^2}$

$= \frac{2e^{2x-3}[x^2 + 1 - x]}{(x^2+1)^2}$ --- ①

(b) $\left(\frac{dy}{dx}\right)_{x=2} = \frac{2e[4+1-2]}{25} = \frac{6e}{25}$ ✓ —②

Now at $x = 2$,

$\frac{dy}{dx} \times \frac{dx}{dt} = \frac{dy}{dt}$

$\Rightarrow \frac{6e}{25} \times \frac{dx}{dt} = 2$ from ② $\frac{dy}{dt} = 2$

$\Rightarrow \frac{dx}{dt} = \frac{2 \times 25}{6e} = \underline{\frac{25}{3e}}$ ✓

24. It is given that $y = \ln(1+\sin x)$ for $0 < x < \pi$

(a) Find $\frac{dy}{dx}$ --- [2]

(b) Find the value of $\frac{dy}{dx}$ when $x = \frac{\pi}{6}$, giving your answer in the form $\frac{1}{\sqrt{a}}$, where a is an integer. ---[2]

(c) Find the value of x for $\frac{dy}{dx} = \tan x$ --[5]

[W-20/23/Q4]

Solution (a) $y = \ln(1+\sin x)$ —— ①

$$\frac{dy}{dx} = \frac{1}{1+\sin x} \times \cos x = \frac{\cos x}{1+\sin x} \text{ —— ②}$$

(b) $$\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{6}} = \frac{\cos\frac{\pi}{6}}{1+\sin\frac{\pi}{6}} = \frac{\frac{\sqrt{3}}{2}}{1+\frac{1}{2}} = \frac{\sqrt{3}}{2} \times \frac{2}{3} = \frac{1}{\sqrt{3}}$$ ✓

(c) $$\frac{dy}{dx} = \tan x \Rightarrow \frac{\cos x}{1+\sin x} = \frac{\sin x}{\cos x} \quad \text{from ②}$$

$$\Rightarrow \cos^2 x = \sin x + \sin^2 x$$

$$\Rightarrow 1 - \sin^2 x = \sin x + \sin^2 x$$

or $$2\sin^2 x + \sin x - 1 = 0$$

$$(2\sin x - 1)(\sin x + 1) = 0$$

$$\Rightarrow \sin x = \frac{1}{2}, \quad \sin x = -1 \times$$

$$= \sin\frac{\pi}{6}$$

$0 < x < \pi$

+ (180−α) | Sin x + α

$$x = \frac{\pi}{6}, \ \pi - \frac{\pi}{6}$$

$$x = \frac{\pi}{6} \text{ and } x = \frac{5\pi}{6}$$ ✓

---

25 A curve has equation $y = x\cos x$

(a) Find $\frac{dy}{dx}$ ---[2]

(b) Find the equation of the normal to the curve at the point where $x = \pi$, giving your answer in the form $y = mx + c$ [W-20/23/Q7] --[4]

Solution (a) $y = x\cos x$ —— ①

$$\frac{dy}{dx} = x\frac{d}{dx}\cos x + \cos x \cdot \frac{d}{dx}x$$

$$= -x\sin x + \cos x \text{ —— ②}$$

(b) $x = \pi \Rightarrow y = \pi\cos\pi = -\pi$ ✓ from ①

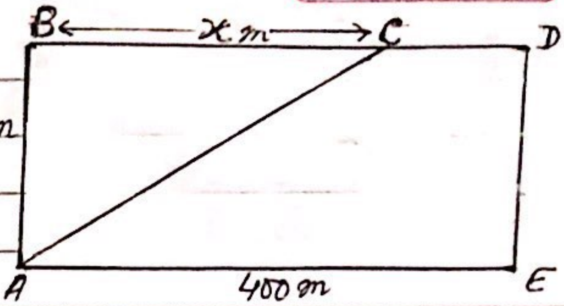
$$\left(\frac{dy}{dx}\right)_{x=\pi} = -\pi\sin\pi + \cos\pi = -1$$

Hence gradient of Normal $= 1$

Equation of normal at $(\pi, -\pi)$

$$y + \pi = 1\cdot(x - \pi)$$

$$\Rightarrow y = x - 2\pi$$ ✓



26\. The rectangle ABCDE represents a ploughed field where AB = 300 m and AE = 400 m. Joseph needs to walk from A to D in the least possible time. He can walk at $0.9\,ms^{-1}$ on the ploughed field and at $1.5\,ms^{-1}$ on any part of the path BCD along the edge of the field. He walks from A to C and then from C to D. The distance BC = x m.

(a) Find in terms of x, the total time, T s, Joseph takes for journey. ---[3]

(b) Given that x can vary, find the value of x for which T is a minimum and hence find the minimum value of T. ---[6]

W-20 | 23 | Q 9

Solution (a) $AC = \sqrt{300^2 + x^2}$ and $CD = (400 - x)$

$$T = \frac{\sqrt{300^2+x^2}}{0.9} + \frac{400-x}{1.5} \quad \text{--- (1)}$$

(b) $$\frac{dT}{dx} = \frac{\frac{1}{2}(300^2+x^2)^{-\frac{1}{2}} \times 2x}{0.9} + \frac{-1}{1.5}$$

$$= \frac{10x}{9\sqrt{300^2+x^2}} - \frac{2}{3}$$

for Min Time $\frac{dT}{dx} = 0 \Rightarrow \frac{10x}{9\sqrt{300^2+x^2}} - \frac{2}{3} = 0$

$$\Rightarrow 5x = 3\sqrt{300^2+x^2}$$

$$\Rightarrow 25x^2 = 9(300^2+x^2)$$

$$\Rightarrow 16x^2 = 300^2 \times 9$$

$$x^2 = \frac{300^2 \times 9}{16} = 50625$$

$$\Rightarrow x = 225\,m$$

from (1) $$\text{Min } T = \frac{\sqrt{300^2+225^2}}{.9} + \frac{400-225}{1.5}$$

$$= 416.66\cdots + 116.66\cdots$$

$$= 533.33\ s$$

$\therefore$ Min $T = \underline{533\ s}$ ✓