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0606

Additional Maths

Functions
Revision

SP-20	M-20	S-20	W-20

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Functions

Revision

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1. Functions g and h are such that: $g(x) = 2 + 4\ln x$ for $x > 0$
$h(x) = x^2 + 4$ for $x > 0$

(a) Find $g^{-1}(x)$, stating its domain and range. ---[4]

(b) Solve $gh(x) = 10$ ---[3]

(c) Solve $g'(x) = h'(x)$ $\boxed{SP\text{-}20|02|Q3}$ --[3]

Solution (a) $g(x) = 2 + 4\ln x$ for $x > 0$

Let $y = 2 + 4\ln x \Rightarrow \ln x = \frac{y-2}{4}$

Interchanging x & y

$\Rightarrow \ln y = \frac{x-2}{4}$

$\Rightarrow y = e^{\left(\frac{x-2}{4}\right)}$

$\Rightarrow g^{-1}(x) = e^{\left(\frac{x-2}{4}\right)}$ Domain is $x \in R$

Range is $y > 0$

(b) To Solve $gh(x) = 10$

$g[x^2+4] = 10$

$2 + 4\ln(x^2+4) = 10$

$\Rightarrow \ln(x^2+4) = 2$

$x^2 + 4 = e^2$

$\Rightarrow x^2 = e^2 - 4$

$x = \sqrt{e^2-4} = \underline{1.84}$ ✓ $x > 0$

(c) $g'(x) = h'(x)$

$\Rightarrow \frac{4}{x} = 2x$

$\Rightarrow 2x^2 = 4$

$\Rightarrow x^2 = 2$

$\Rightarrow \underline{x = \sqrt{2}}$ ✓ for $x > 0$

$$\left\{\begin{array}{l} g(x) = 2 + 4\ln x \\ g'(x) = \frac{4}{x} \\ \text{and } h(x) = x^2 + 4 \\ h'(x) = 2x \end{array}\right.$$

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2 (a) $g(x) = 3 + \frac{1}{x}$ for $x \geq 1$

(i) Find an expression for $g^{-1}(x)$ ---[2]

(ii) Write down the range of g^{-1} --[1]

(iii) Write down the domain of g^{-1} ---[2]

(b) $h(x) = 2\ln(3x-1)$ for $x \geq \frac{2}{3}$

The graph of $y = h(x)$ intersects the line $y = x$ at two distinct points. Sketch the graph of $y = h(x)$ and hence sketch the graph of $y = h^{-1}(x)$ [M-20/22/Q10] --[4]

Solution (a) $g(x) = 3 + \frac{1}{x}$

(i) or $y = 3 + \frac{1}{x}$

Interchange x & y $\Rightarrow$ $x = 3 + \frac{1}{y}$

$\Rightarrow \frac{1}{y} = x - 3$

$\Rightarrow y = \frac{1}{(x-3)}$

$\therefore g^{-1}(x) = \frac{1}{(x-3)}$

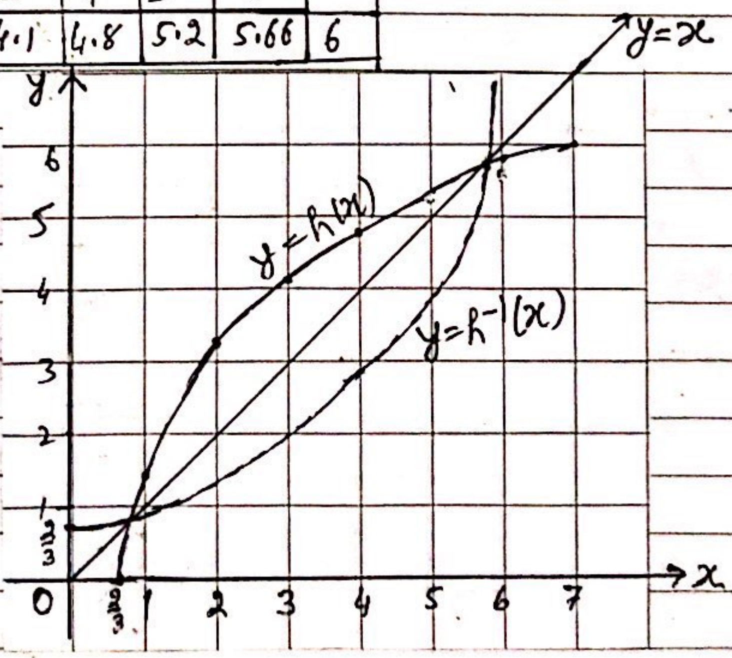
(ii) range of $g^{-1}(x) \rightarrow g^{-1} \geq 1$ or $[1, \infty)$ [$\because$ range of $g^{-1}(x)$ = domain of $g(x)$]

(iii) domain of g^{-1} is $3 < x \leq 4$ or $(3, 4]$ {Domain of $g^{-1}(x)$ is = range of $g(x)$}

(b)

x	2/3	1	2	3	4	5	6	7
$h(x)$	0	1.4	3.2	4.1	4.8	5.2	5.66	6

$h(x) = 2\ln(3x-1)$



The graph of $y = h^{-1}(x)$ is the reflection of $y = h(x)$ in line $y = x$

3. The function f is defined by $f(x)=\ln(2x+1)$ for $x\geqslant 0$

(a) Sketch the graph of $y=f(x)$ and hence sketch the graph of $y=f^{-1}(x)$. --[3]

The function g is defined by $g(x)=(x-4)^2+1$ for $x\leqslant 4$

(b) (i) Find an expression for $g^{-1}(x)$ and state its domain and range. --[4]

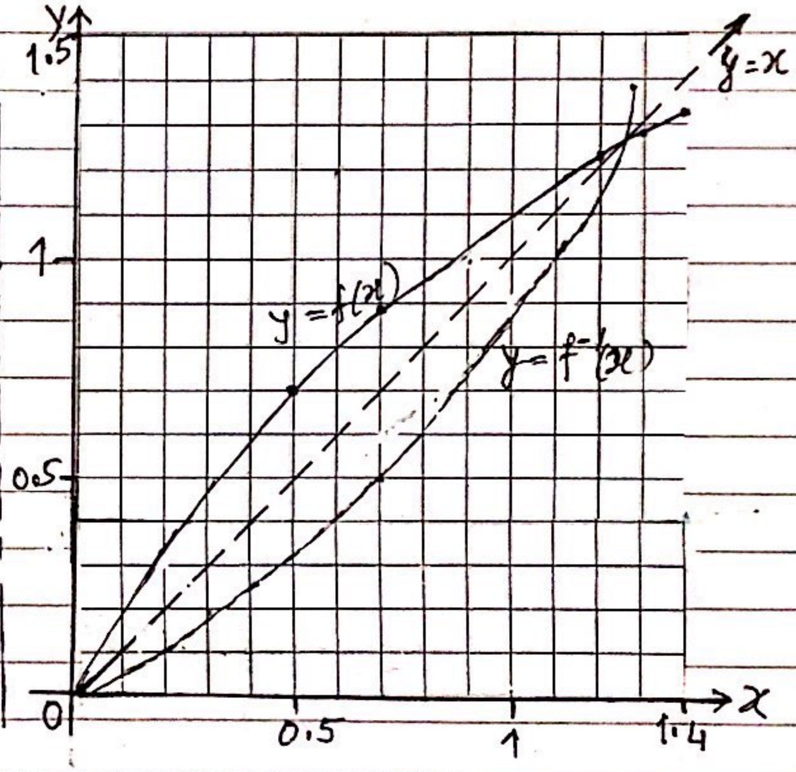
(ii) Find and simplify an expression for $fg(x)$. $\boxed{S\text{-}20 \mid 21 \mid Q11}$ --[2]

(iii) Explain why the function gf does not exist. --[1]

Solution (a) $f(x)=\ln(2x+1)$ for $x\geqslant 0$

x	0	·2	·5	·7	·9	1	1·2	1·3	1·4
$y=f(x)$	0	·33	·7	·88	1	1·16	1·22	1·28	1·33

Graph of $f^{-1}(x)$ is the reflection of $f(x)$ in line $y=x$



(b) $g(x)=(x-4)^2+1,\ x\leqslant 4$

(i) $y=(x-4)^2+1$

Interchange x & y

$x=(y-4)^2+1$

$y-4=\pm\sqrt{x-1}$

$y=4-\sqrt{x-1}$ $x\geqslant 1$

or $g^{-1}(x)=4-\sqrt{x-1}$ ± ?

Domain is $x\geqslant 1$

Range is $x\leqslant 4$

(ii) $fg(x)=f(g(x))=f((x-4)^2+1)=f(x^2-8x+17)$

$= \ln\left(2(x^2-8x+17)+1\right)$

$= \ln(2x^2-16x+35)$

(iii) $gf(x)=g(\ln(2x+1))$ ∵ $\left\{\begin{array}{l} f(x)=\ln(2x+1) \text{ for } x\geqslant 0 \\ \text{for } x=10,\ f(10)=4.39\geqslant 4\end{array}\right.$

$f(10)=4.39$ is in the range of $f(x)$ but $g(4.39)$ is not defined as for $g(x)$, $x\leqslant 4$

or Some of the values in the range of $f(x)$ are outside the domain of $g(x)$

Hence gf does not exist.

4. $f : x \mapsto (2x+3)^2$ for $x > 0$

(a) Find the range of f --[1]

(b) Explain when f has an inverse. --[1]

(c) Find f^{-1} --[3]

(d) Find the domain of f^{-1} --[1]

(e) Given that $g : x \mapsto \ln(x+4)$ for $x>0$, find the exact solution of $fg(x) = 49$ --[3]

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Solution: $f(x) = (2x+3)^2$ for $x>0$

(a) Range of $f(x)$ is $f(x) > 9$ ✓

(b) f is a one-one function ($f(x)$ is increasing function for $x>0$)

(c) $y = (2x+3)^2$

Interchange x & y $\Rightarrow$ $x = (2y+3)^2$

$\Rightarrow 2y+3 = \pm\sqrt{x}$

$\Rightarrow y = \dfrac{\pm\sqrt{x}-3}{2}$

$y = \dfrac{\sqrt{x}-3}{2}$

(±) we take + sign as $x > 9$, $y > 0$, Range of f^{-1}, $y>0$

(d) Domain of f^{-1} = (Range of f) is $x > 9$

(e) $g(x) = \ln(x+4)$ for $x>0$

Now $fg(x) = 49$

$\Rightarrow f(\ln(x+4)) = 49$

$\Rightarrow [2(\ln(x+4))+3]^2 = 49$

$\Rightarrow 2\ln(x+4) + 3 = 7$

only + out of ±7 otherwise $\ln(x+4) < 0$ false.

$\Rightarrow \ln(x+4) = 2$

$\rightarrow x + 4 = e^2$

$x = e^2 - 4$ ✓

5. $f(x) = 3 + e^x$ for $x \in R$

$g(x) = 9x - 5$ for $x \in R$

(a) Find the range of f and of g. --- [2]

(b) Find the exact solution of $f^{-1}(x) = g'(x)$. -- [3]

(c) Find the solution of $g^2(x) = 112$ -- [2]

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Solution: $f(x) = 3 + e^x$ for $x \in R$; $g(x) = 9x - 5$ for $x \in R$

(a) Range of f is $f(x) > 3$ ✓

Range of g is $g \in R$ ✓

(b) $g(x) = 9x - 5$

$g'(x) = 9$ —— ①

and $f(x) = y = 3 + e^x$

Interchange x & y ⇒ $x = 3 + e^y$

$\Rightarrow e^y = x - 3$

$\Rightarrow y = \ln(x-3)$

$\Rightarrow f^{-1}(x) = \ln(x-3)$ —— ②

To solve $f^{-1}(x) = g'(x)$

$\Rightarrow \ln(x-3) = 9$ (from ① & ②)

$\Rightarrow x - 3 = e^9$

$\Rightarrow x = e^9 + 3$ ✓

(c) Solve $g^2(x) = 112$

$\Rightarrow g(g(x)) = 112 \Rightarrow g(9x-5) = 112$

$\Rightarrow 9(9x-5) - 5 = 112$

$\Rightarrow 81x - 45 - 5 = 112$

$81x = 162$

$x = 2$ ✓

6. It is given that $f(x) = 5\ln(2x+3)$ for $x > -\frac{3}{2}$

(a) Write down the range of f. ---[1]

(b) Find f^{-1} and state its domain. ---[3]

(c) Sketch the graph of $y = f(x)$ and the graph of $y = f^{-1}(x)$. Label each curve and state the intercepts on the coordinate axes. --[5]

[W-20/11/Q7]

Solution: $f(x) = 5\ln(2x+3)$ for $x > -\frac{3}{2}$

(a) Range of f, $f \in R$ (R is set of all real numbers)

(b) Given $f(x) = y = 5\ln(2x+3)$

Interchange x & y $\Rightarrow$ $x = 5\ln(2y+3)$

$\Rightarrow \frac{x}{5} = \ln(2y+3)$

$\Rightarrow 2y+3 = e^{x/5}$

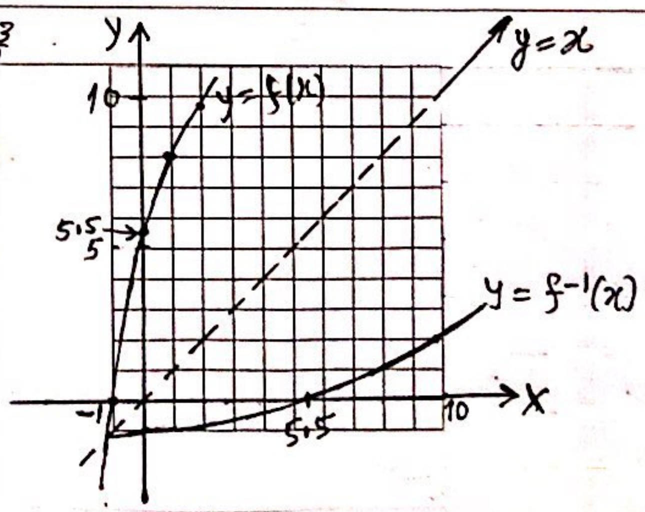
$y = \frac{e^{x/5} - 3}{2}$

$\therefore f^{-1}(x) = \frac{e^{x/5} - 3}{2}$ ✓

Domain of f^{-1} is (the range of $f(x)$)

$\Rightarrow x \in R$

(c) $f(x) = y = 5\ln(2x+3)$ for $x > -\frac{3}{2}$



x	-1.1	-1	0	1	2
y	-1.11	0	5.5	8	9.7

The graph of $y = f(x)$ intersects x-axis at $x = -1$, and y-axis at $y = 5.5$ ($5\ln 3$)

The graph of $y = f^{-1}(x)$ is drawn as the reflection of $f(x)$ in the line $y = x$. $f^{-1}(x)$ intersects x-axis at $x = 5.5$ and y-axis at $y = -1$

7. $f(x) = x^2 + 2x - 3$ for $x \geq -1$

(a) Given that the minimum value of x^2+2x-3 occurs when $x=-1$, explain why $f(x)$ has an inverse. --[1]

(b) Sketch the graph of $y=f(x)$ and the graph of $y=f^{-1}(x)$. Label each graph and state the intersect on the coordinate axes. --[4]

W-20/12/Q6

Solution: $f(x) = x^2+2x-3$ for $x \geq -1$

(a) $f(x) = x^2+2x-3 = (x+1)^2-4$ for $x \geq -1$

$f(x)$ is one-one function (increasing) in the domain $x \geq -1$

$\therefore$ $f(x)$ has an inverse.

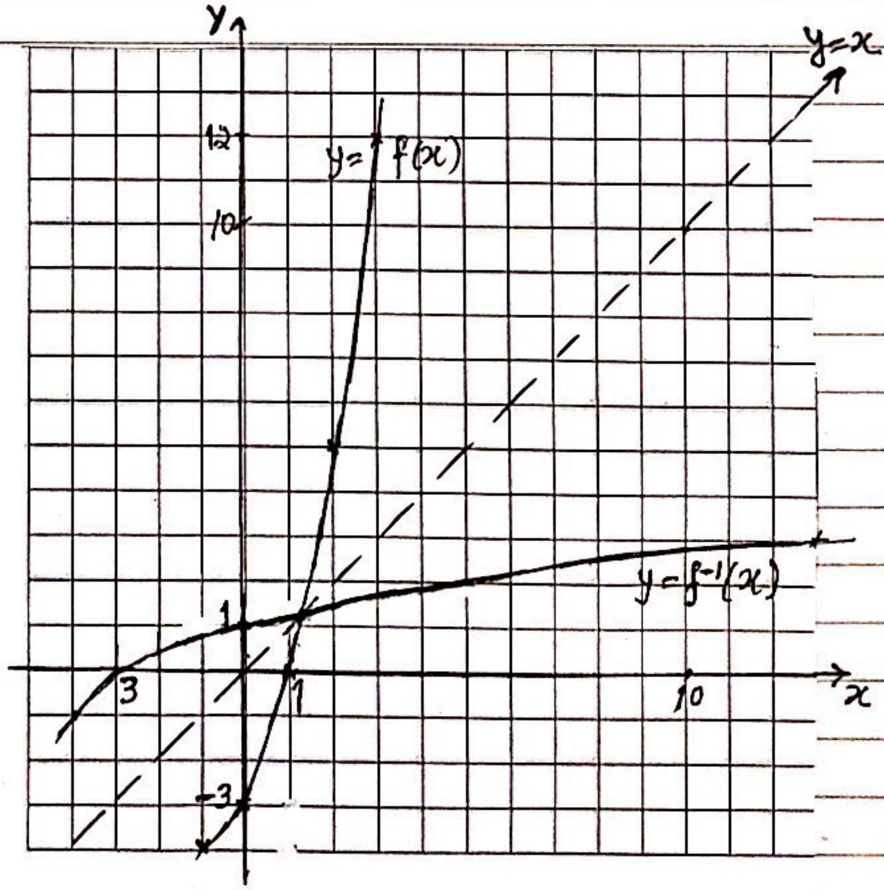
(b) $y = x^2+2x-3 = (x+1)^2-4$

x	-1	0	1	2	3	1.3
y	-4	-3	0	5	12	1.3

The graph of $y=f(x)$
x-intercept $=1$
y-intersept $=-3$

The graph of $y=f^{-1}(x)$ is the reflection of $y=f(x)$ in the line $y=x$.

The x-intrecept of $y=f^{-1}(x) = -3$
and y-intcept $= 1$



8. (a) $f(x) = 4\ln(2x-1)$

(i) Write down the largest possible domain for the function f --[1]

(ii) Find $f^{-1}(x)$ and its domain. --[3]

(b) $g(x) = x+5$ for $x \in R$

$h(x) = \sqrt{2x-3}$ for $x \geq \frac{3}{2}$

Solve $gh(x) = 7$ [W-20/13/Q3] --[3]

Solution; $f(x) = 4\ln(2x-1)$

(a) (i) Domain of $f(x)$ is $x > \frac{1}{2}$ ✓ $(2x-1>0)$

(ii) $f(x) = y = 4\ln(2x-1)$

Interchange x & y $\Rightarrow$ $x = 4\ln(2y-1)$

$\Rightarrow 2y-1 = e^{x/4}$

$y = \frac{1}{2}(1+e^{x/4})$

$\therefore f^{-1}(x) = \frac{1}{2}(1+e^{x/4})$ ✓

Domain of $f^{-1}(x)$ is $x \in R$ ✓ [Domain of $f^{-1}(x)$ is same as range of $f(x)$]

(b) $g(x) = x+5$ for $x \in R$ and $h(x) = \sqrt{2x-3}$ for $x \geq 3/2$

To Solve $g.h(x) = 7$

$\Rightarrow g(\sqrt{2x-3}) = 7$

$\Rightarrow \sqrt{2x-3} + 5 = 7$

$\sqrt{2x-3} = 2$

$2x-3 = 4$

$2x = 7$

$x = \frac{7}{2}$ (or 3.5) ✓