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0606

Addional Maths

Integration Revision

SP-20	M-20	S-20	W-20

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Integration

Revision

1 (a) Giving your answer in its simplest form, find the exact value of

(i) $\int_{0.2}^{1} e^{5x-1}\, dx$ --- [4]

(ii) $\int_{1}^{2} \left(x+\frac{1}{x^2}\right)^2 dx$ --- [5]

(b) $\int \sin\frac{x}{6}\, dx$ --- [2]

[SP-20/01/Q8]

Solution (a) (i) $\int_{0.2}^{1} e^{5x-1}\, dx = \frac{1}{5}\left[e^{5x-1}\right]_{0.2}^{1} = \frac{1}{5}\left[e^{4} - e^{0}\right]$

$= \frac{1}{5}\left[e^{4}-1\right]$ ✓

(a)(ii) $\int_{1}^{2} \left(x+\frac{1}{x^2}\right)^2 dx$

$= \int_{1}^{2} \left(x^2 + \frac{2}{x} + x^{-4}\right) dx$

$= \left[\frac{x^3}{3} + 2\ln x + \frac{x^{-3}}{-3}\right]_{1}^{2}$

$= \left[\left(\frac{8}{3} + 2\ln 2 - \frac{1}{24}\right) - \left(\frac{1}{3} + 2\times 0 - \frac{1}{3}\right)\right]$

$= 2\ln 2 + \frac{21}{8}$ ✓

(b) $\int \sin\frac{x}{6}\, dx$

$= -6\cos\left(\frac{x}{6}\right) + c$

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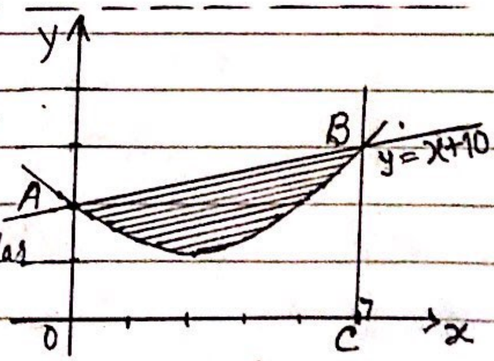
2. The graph of $y = x^2 - 4x + 10$ cuts the y-axis at point A. The graph of $y = x^2 - 4x + 10$ and $y = x + 10$ intersect one another at the points A and B. The line BC is perpendicular to the x-axis. Calculate the area of the shaded region enclosed by the the curve and the line AB.



[SP-20/01/Q11] --- [8]

Solution: Curve: $y = x^2 - 4x + 10$ — ①

Line: $y = x + 10$ — ②

Solving ① and ② $x^2 - 4x + 10 = x + 10$

$x^2 - 5x = 0 \Rightarrow x(x-5) = 0$

$x = 0, \quad x = 5$

from ② $y = 10, \quad y = 15$

$A(0,10), B(5,15)$ ✓

Area between the curve and x-axis: for $x=0$ to $x=5$

$\int_0^5 (x^2 - 4x + 10)\, dx$

$= \left[\frac{x^3}{3} - 2x^2 + 10x\right]_0^5 = \frac{125}{3}$ — ③

Area of Trapezium OABC

$= \frac{1}{2}(10+15)\times 5 = \frac{125}{2}$ — ④

∴ Area of Shaded Region, (from ③ & ④)

$= \frac{125}{2} - \frac{125}{3} = \frac{125}{6}$ ✓

3. Given that $\int_1^a \left(\frac{2}{2x+3} + \frac{3}{3x-1} - \frac{1}{x}\right)dx = \ln 2.4$ and that $a > 1$, find the value of a. [M-20 | 12 | Q 11] --[7]

Solution: $\int_1^a \left(\frac{2}{(2x+3)} + \frac{3}{3x-1} - \frac{1}{x}\right) dx = \ln 2.4$ (Given)

$\Rightarrow \left[\frac{\cancel{2}\ln(2x+3)}{\cancel{2}} + \frac{\cancel{3}\ln(3x-1)}{\cancel{3}} - \ln x\right]_1^a = \ln 2.4$

$\Rightarrow \left(\ln(2a+3) + \ln(3a-1) - \ln a\right) - \left(\ln 5 + \ln 2 - \ln 1\right) = \ln 2.4$

$\Rightarrow \ln \frac{(2a+3)(3a-1)}{a} - \ln 10 = \ln 2.4$

$\Rightarrow \ln \frac{(6a^2+7a-3)}{10a} = \ln 2.4$

$\Rightarrow 6a^2+7a-3 = 2.4 \times 10a$ $\quad (\because 2.4 \times 10a = 24a)$

$\Rightarrow 6a^2 - 17a - 3 = 0$

$6a^2 - 18a + a - 3 = 0$

$6a(a-3) + 1(a-3) = 0 \Rightarrow (6a+1)(a-3) = 0$

$\Rightarrow a = 3$ ✓, $a = -\frac{1}{6}$ ✗ $(\because a > 1)$

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4 (a) Show that $\frac{3}{(2x-3)} + \frac{3}{(2x+3)}$ can be written as $\frac{12x}{(4x^2-9)}$ --[2]

(b) Hence find $\int \frac{12x}{4x^2-9} dx$, giving your answer as a single logarithm and an arbitrary constant. --[3]

(c) Given that $\int_2^a \frac{12x}{4x^2-9} dx = \ln 5\sqrt{5}$, where $a > 2$, find the exact value of a. [S-20 | 11 | Q 8] --[4]

Solution (a) $\frac{3}{(2x-3)} + \frac{3}{(2x+3)} = \frac{3(2x+3)+3(2x-3)}{(2x-3)(2x+3)}$

$= \frac{6x+9+6x-9}{(2x)^2-3^2}$

$= \frac{12x}{4x^2-9}$ ✓ —— ①

(b) $\int \frac{12x}{4x^2-9} dx = \int \left(\frac{3}{2x-3} + \frac{3}{2x+3}\right) dx$ from ①

$= 3\frac{\ln(2x-3)}{2} + 3\frac{\ln(2x+3)}{2} + c$

$= \frac{3}{2}\ln(2x-3)(2x+3) + c$

$= \frac{3}{2}\ln(4x^2-9) + c = \ln(4x^2-9)^{3/2} + c$ ✓

(c) Given $\int_2^a \frac{12x}{4x^2-9} dx = \ln 5\sqrt{5}$

$\Rightarrow \left[\ln(4x^2-9)^{3/2} + c\right]_2^a = \ln(5)^{3/2}$ [from Part (b)]

$\Rightarrow \ln(4a^2-9)^{3/2} - \ln(7)^{3/2} = \ln 5^{3/2}$

$\Rightarrow \ln(4a^2-9)^{3/2} = \ln(35)^{3/2}$

$\Rightarrow 4a^2 - 9 = 35$

$4a^2 = 44$

$a^2 = 11$

$a = \sqrt{11}$ ✓ $(a > 2)$

5. A curve is such that $\frac{d^2y}{dx^2} = 5\cos 2x$. This curve has a gradient of $\frac{3}{4}$ at the point $\left(-\frac{\pi}{12}, \frac{5\pi}{4}\right)$. Find the equation of the curve. --[8]

[S-20 | 11 | Q 11]

Solution: Given $\frac{d^2y}{dx^2} = 5\cos 2x$

$\Rightarrow \frac{dy}{dx} = \int 5\cos 2x\,dx$

$\rightarrow \frac{dy}{dx} = \frac{5}{2}\sin 2x + c$ —— ①

$\left(\frac{dy}{dx}\right)_{x=-\frac{\pi}{12}} = \frac{5}{2}\sin\left(-\frac{\pi}{6}\right) + c = \frac{3}{4}$ Given

$\Rightarrow \frac{5}{2}\times\left(-\frac{1}{2}\right) + c = \frac{3}{4}$

$\Rightarrow c = \frac{3}{4} + \frac{5}{4} = 2$ ✓

from ① $\frac{dy}{dx} = \frac{5}{2}\sin 2x + 2$ ⇗

$y = \int\left(\frac{5}{2}\sin 2x + 2\right)dx$

$y = \frac{5}{2}\left(\frac{-\cos 2x}{2}\right) + 2x + d$ —— ②

Curve ② passes through $\left(-\frac{\pi}{12}, \frac{5\pi}{4}\right)$

$\Rightarrow \frac{5\pi}{4} = -\frac{5}{4}\cos\left(-\frac{\pi}{6}\right) - \frac{\pi}{6} + d$

$\Rightarrow d = \frac{5\pi}{4} + \frac{\pi}{6} + \frac{5}{4}\times\frac{\sqrt{3}}{2} = \frac{17\pi}{12} + \frac{5\sqrt{3}}{8}$

∴ from ② Equⁿ of the curve

$y = -\frac{5}{4}\cos 2x + 2x + \frac{17\pi}{12} + \frac{5\sqrt{3}}{8}$

or

$y = -\frac{5}{4}\cos 2x + 2x + 5.53$ ✓

6. The gradient of the normal to a curve at the point (x, y) is given by $\frac{x}{(x+1)}$;

(a) Given that the curve passes through the point $(1, 4)$, show that its equation is $y = 5 - \ln x - x$. --[5]

(b) Find the form $y = mx + c$, the equation of the tangent to the curve at the point where $x = 3$. [W-20 | 21 | Q 10] --[3]

Solution (a) Gradient of the normal

$-\frac{dx}{dy} = \frac{x}{x+1} \Rightarrow \frac{dy}{dx} = -\frac{x+1}{x}$ —— ①

$\Rightarrow y = -\int\left(1 + \frac{1}{x}\right)dx$

$y = -\ln x - x + c$ —— ②

Passes through a point $(1, 4)$

from ① $\Rightarrow 4 = -\ln 1 - 1 + c$

$\Rightarrow c = 5$

∴ Equation of the curve from ①

$y = -\ln x - x + 5$ ✓ —— ③

(b) $y = 5 - \ln x - x$ —— ③

at $x = 3$, $y = 5 - \ln 3 - 3$

$y = (2 - \ln 3)$

from ① $\left(\frac{dy}{dx}\right)_{x=3} = -\left(\frac{3+1}{3}\right) = -\frac{4}{3}$

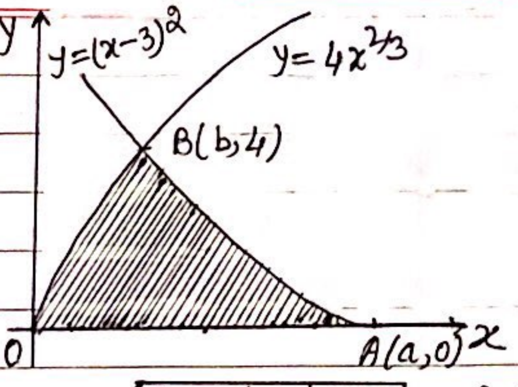
∴ Equation of tangent at $x = 3$

$y - (2 - \ln 3) = -\frac{4}{3}(x - 3)$

$y = -\frac{4}{3}x + 4 + 2 - \ln 3$

$y = -1.33x + 4.9$ ✓

7 The diagram shows part of the graph of $y = 4x^{2/3}$ and $y = (x-3)^2$. The graph of $y=(x-3)^2$ meets the x-axis at the point $A(a,0)$ and the two graphs intersect at the point $B(b,4)$.



(a) Find value of a and of b. ---[2]

(b) Find the area of the shaded region. [S-20/21/Q10] ---[5]

Solution (a) Curve: $y = 4x^{2/3}$ --- (i)

Passes through $(b,4) \Rightarrow 4 = 4b^{2/3} \Rightarrow b^{2/3} = 1 \Rightarrow \underline{b=1}$ ✓

Second curve: $y=(x-3)^2$ meets X-axis at $(a,0)$

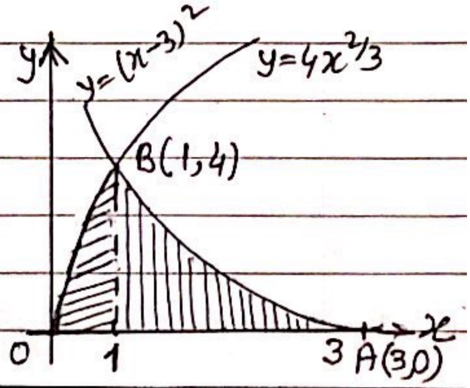
$\Rightarrow 0 = (a-3)^2 \Rightarrow \underline{a=3}$ ✓

(b) Shade Area $= \int_0^1 4x^{2/3}\,dx + \int_1^3 (x-3)^2\,dx$

$= \left[4\frac{x^{5/3}}{5/3}\right]_0^1 + \left[\frac{(x-3)^3}{3}\right]_1^3$

$= \frac{12}{5} \times 1^{2/3} + \frac{1}{3}[0-(-2)^3]$

$= \frac{12}{5} + \frac{1}{3}\times 8 = \frac{76}{15} = \underline{5.07}$ ✓



8. Giving your answer in its simplest form, find the exact value of,

(a) $\int_0^4 \frac{10}{5x+2}\,dx$ ---[4]

(b) $\int_0^{\ln 2} \left(e^{4x+2}\right)^2 dx$ [S-20/22/Q7] ---[5]

Solution (a) $\int_0^4 \frac{10}{(5x+2)}\,dx$

$= \left[\frac{10\ln(5x+2)}{5}\right]_0^4$

$= 2[\ln 22 - \ln 2]$

$= 2\left[\ln\frac{22}{2}\right]$

$= \ln 11^2$

$= \underline{\ln 121}$

(b) $\int_0^{\ln 2} \left(e^{4x+2}\right)^2 dx$

$= \int_0^{\ln 2} e^{8x+4}\,dx$

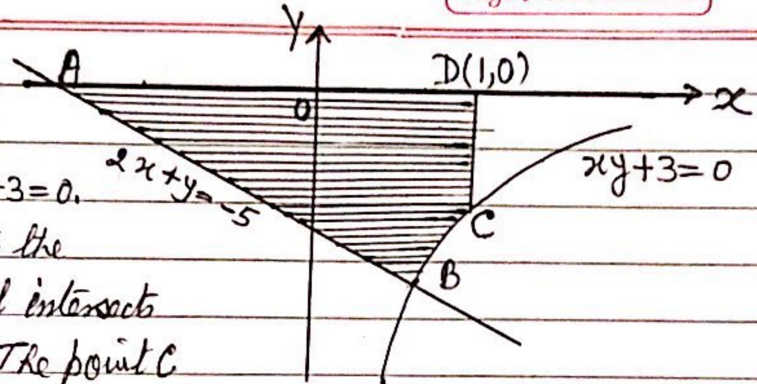
$= \left[\frac{1}{8}e^{8x+4}\right]_0^{\ln 2}$

$= \frac{1}{8}\left[e^{8\ln 2}\cdot e^4 - e^4\right]$

$= \frac{1}{8}e^4\left[e^{\ln 2^8} - \right]$

$= \frac{1}{8}e^4[256-1] = \underline{\frac{255}{8}e^4}$ ✓

9. The diagram show the straight line $2x+y=-5$ and part of the curve $xy+3=0$. The straight line intersect the x-axis at the point A and intersect the curve at the point B. The point C lies on the curve. The point D has coordinates $(1,0)$. The line CD is parallel to the y-axis.



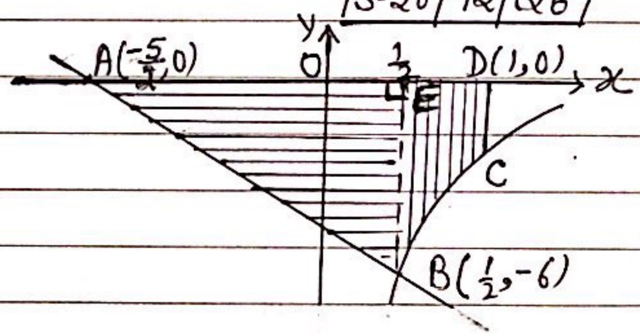
(a) Find the coordinates of each of the points A and B. ---[3]

(b) Find the area of the shaded region, giving your answer in the form $p+\ln q$, where p and q are positive integers. ---[6]

[S-20/12/Q6]

Solution: line: $2x+y=-5$ --- (1)

Curve: $xy+3=0$ --- (2)



(a) Line intersects X-axis at A, $y=0 \Rightarrow 2x+0=-5$

$x=-5/2$

$\therefore A(-\frac{5}{2},0)$ ✓

for point B, solving (1) & (2)

$x(-5-2x)+3=0$

$\Rightarrow 2x^2+5x-3=0$

$(2x-1)(x+3)=0$

$\Rightarrow x=\frac{1}{2},\ \cancel{x=-3}$ (see fig)

$y=-6 \quad \therefore B(\frac{1}{2},-6)$ ✓

(b) Shaded Area = ar ΔABE + Area between the curve and X-axis

$= \frac{1}{2}\left(\frac{1}{2}+\frac{5}{2}\right)\times 6 + \int_{\frac{1}{2}}^{1} -\frac{3}{x}\,dx$

$\left\{ \begin{array}{l} x=\frac{1}{2} \text{ to } x=1 \\ \text{for curve } xy=-3 \\ \Rightarrow y=-\frac{3}{x} \text{ from (2)} \end{array} \right.$

$= 9 + \left[-3\ln x\right]_{1/2}^{1}$

$= 9 + \left[-3\ln 1-(-3\ln\frac{1}{2})\right]$

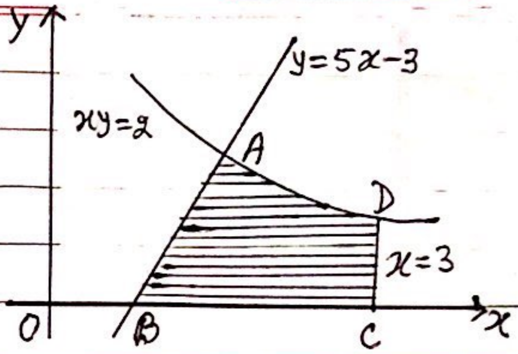
[Area below X-axis is -ve value. So taking modulus]

$= 9 + \left[0+3\ln\frac{1}{2}\right]$

$= 9 + |-3\ln 2|$

$= 9 + \ln 2^3 = \underline{9+\ln 8}$ ✓

10. The diagram shows part of the curve $xy = 2$ intersecting the straight line $y = 5x - 3$ at a point A. The straight line meets the x-axis at the point B. The point C lies on x-axis and the point D lies on the curve such that the line CD has equation $x = 3$. Find the exact area of the shaded region, giving your answer in the form $p + \ln q$, where p and q are constants. [S-20/13/Q9] ---[8]



Solution Curve; $xy = 2$ or $y = \frac{2}{x}$ —— ①

line ; $y = 5x - 3$ —— ②

solving ① & ② $\frac{2}{x} = 5x - 3$

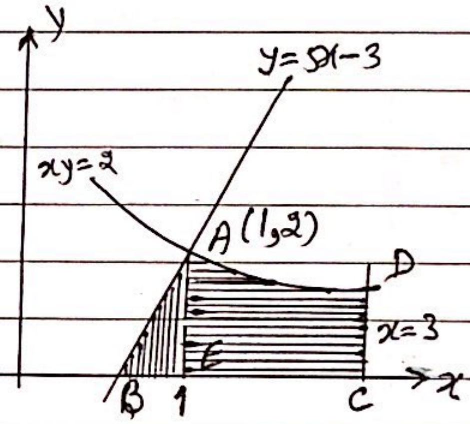
$\Rightarrow x(5x-3) = 2$

$5x^2 - 3x - 2 = 0$

$(x-1)(5x+2) = 0$

$x = 1$, $x = -\frac{2}{5}$ ✗ (see fig)

$y = 2$ from ② i.e. $A(1,2)$



The required shaded area

$=$ area of $\Delta ABE +$ area under the curve ① (for $x = 1$ to $x = 3$)

$= \frac{1}{2} \times AE \times BE + \int_1^3 \frac{2}{x}\,dx$ (AE ⊥ x axis)

$= \frac{1}{2} \times 2 \times \frac{3}{5} + 2\left[\ln x\right]_1^3$

$= \frac{2}{5} + 2\left[\ln 3 - \ln 1\right]$

$= \frac{2}{5} + \ln 3^2$ ($\ln 1 = 0$)

$= \frac{2}{5} + \ln 9$ ✓

$AE = 2$

line $y = 5x - 3$ intersects x-axis at $y = 0$, $0 = 5x - 3$

$x = 3/5$

$B\left(\frac{3}{5}, 0\right)$

11 (a) (i) Given that $f(x) = \frac{1}{\cos x}$, show that $f'(x) = \tan x \sec x$ --[3]
(ii) Hence find $\int (3\tan x \sec x - \sqrt[4]{e^{3x}})\,dx$ --[3]
(b) Given that $\int_2^5 \frac{p}{px+10}\,dx = \ln 2$, find the value of positive constant p.
[S-20/23/Q12] ---[5]

Solution:
(a) (i) $f(x) = \frac{1}{\cos x} \Rightarrow f'(x) = \frac{\cos x \cdot \frac{d}{dx}1 - 1 \times \frac{d}{dx}\cos x}{\cos^2 x}$

$$= \frac{\cos x \times 0 - (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x \times 1}{\cos x \times \cos x} = \underline{\tan x \cdot \sec x} \quad \text{—①}$$

(ii) To Find $\int (3\tan x \sec x - \sqrt[4]{e^{3x}})\,dx$

$$= 3\int \tan x \sec x\,dx - \int e^{\frac{3x}{4}}\,dx$$

$$= 3 \times \frac{1}{\cos x} - \frac{e^{3x/4}}{\frac{3}{4}} + c \qquad \left[\because \frac{d}{dx}\frac{1}{\cos x} = \tan x \sec x \text{ from ①}\right]$$

$$= \underline{3\sec x - \frac{4}{3}e^{\frac{3x}{4}} + c}$$

(b) $\int_2^5 \frac{p}{px+10}\,dx = \ln 2$

$$\Rightarrow \left[p \cdot \frac{\ln(px+10)}{p}\right]_2^5 = \ln 2$$

$$\Rightarrow \ln(5p+10) - \ln(2p+10) = \ln 2$$

$$\Rightarrow \ln\left(\frac{5p+10}{2p+10}\right) = \ln 2$$

$$\Rightarrow \frac{5p+10}{2p+10} = 2$$

$$\Rightarrow 5p+10 = 2(2p+20)$$

$$\Rightarrow \underline{p = 10}$$

12.(a) Given that $\int_1^a \left(\frac{1}{x} - \frac{1}{2x+3}\right)dx = \ln 3$, where $a>0$, find the exact value of a, giving your answer in simplest surd form. --[6]

(b) Find the exact value of $\int_0^{\frac{\pi}{3}} \left(\sin\left(2x+\frac{\pi}{3}\right) - 1 + \cos 2x\right)dx$ --[5]

[W-20/11/Q9]

Solution(a) $\int_1^a \left(\frac{1}{x} - \frac{1}{2x+3}\right)dx = \ln 3$

$\Rightarrow \left[\ln x - \frac{1}{2}\ln(2x+3)\right]_1^a = \ln 3$

$\Rightarrow \ln a - \frac{1}{2}\ln(2a+3) - \ln 1 + \frac{1}{2}\ln 5 = \ln 3$

$\ln a + \ln\left(\frac{5}{2a+3}\right)^{1/2} = \ln 3$ $\qquad$ $(\ln 1 = 0)$

$\Rightarrow \ln a \cdot \sqrt{\frac{5}{2a+3}} = \ln 3$

$\Rightarrow a\sqrt{\frac{5}{2a+3}} = 3$

$\Rightarrow a^2 \times \frac{5}{2a+3} = 9$

$\Rightarrow 5a^2 = 18a + 27$

$\Rightarrow 5a^2 - 18a - 27 = 0$

$a = \frac{18 \pm \sqrt{864}}{2\times 5}$ $\qquad$ $b^2 - 4ac = 18^2 - 4\times 5\times 27 = 864$

$= \frac{18 \pm 12\sqrt{6}}{2\times 5}$

$= \frac{9+6\sqrt{6}}{5}$ ✓ $\quad (a>0)$

(b) $\int_0^{\frac{\pi}{3}} \left(\sin\left(2x+\frac{\pi}{3}\right) - 1 + \cos 2x\right)dx$

$= \left[-\frac{1}{2}\cos\left(2x+\frac{\pi}{3}\right) + \frac{1}{2}\sin 2x - x\right]_0^{\pi/3}$

$= \left(-\frac{1}{2}\cos\pi + \frac{1}{2}\sin\frac{2\pi}{3} - \frac{\pi}{3}\right) - \left(-\frac{1}{2}\cos\frac{\pi}{3}\right)$

$= -\frac{1}{2}\times(-1) + \frac{1}{2}\times\frac{\sqrt{3}}{2} - \frac{\pi}{3} + \frac{1}{4}$

$= \frac{3}{4} + \frac{\sqrt{3}}{4} - \frac{\pi}{3}$ ✓

13. Find $\frac{d}{dx}(16-x^2)^{3/2}$ and hence evaluate the area enclosed by the curve $y = x\sqrt{16-x^2}$ and the lines $y=0$, $x=1$ and $x=3$ --[5]

[W-20 | 21 | Q 11(b)]

Solution: $\frac{d}{dx}(16-x^2)^{3/2} = \frac{3}{2}(16-x)^{\frac{1}{2}} \times (-2x) = -3x(16-x)^{1/2}$ ✓ —— ①

Now Area $= \int_1^3 y\,dx = \int_1^3 x(16-x^2)^{1/2}\,dx$

$= \left[-\frac{1}{3}(16-x^2)^{3/2}\right]_1^3$ from ①

$= -\frac{1}{3}\left[7^{3/2} - 15^{3/2}\right] = \underline{13.2}$ ✓

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14. Find the exact value of $\int_2^4 \frac{(x+1)^2}{x^2}\,dx$ ---[6]

[W-20 | 22 | Q 6]

Solution: $\int_2^4 \frac{(x+1)^2}{x^2}\,dx$

$= \int_2^4 \frac{(x^2+2x+1)\,dx}{x^2}$

$= \int_2^4 \left(1 + \frac{2}{x} + x^{-2}\right) dx$

$= \left[x + 2\ln x - \frac{1}{x}\right]_2^4$

$= \left(4 + 2\ln 4 - \frac{1}{4}\right) - \left(2 + 2\ln 2 - \frac{1}{2}\right)$

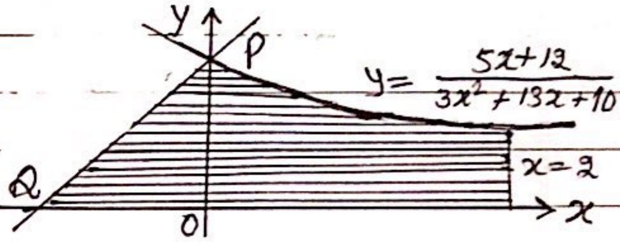
$= 4 - 2 + 2\ln 2^2 - 2\ln 2 + \frac{1}{2} - \frac{1}{4}$

$= \frac{9}{4} + 4\ln 2 - 2\ln 2$

$= \underline{\frac{9}{4} + 2\ln 2}$ ✓

15. (a) Show that $\frac{1}{(x+1)} + \frac{2}{3x+10}$ can be written as $\frac{5x+12}{3x^2+13x+10}$ --- [1]

(b) The diagram shows part of the curve $y = \frac{5x+12}{3x^2+13x+10}$, the line $x = 2$, and a straight line of gradient 1. The curve intersects the y-axis at the point P. The line of gradient 1 passes through P and intersects the x-axis at the point Q. Find the area of the shaded region, giving your answer in the form, $a + \frac{2}{3}\ln(b\sqrt{3})$, where a and b are constants. --- [9]



[W-20 | 13 | Q10]

Solution (a) $\frac{1}{(x+1)} + \frac{2}{(3x+10)} = \frac{3x+10+2(x+1)}{(x+1)(3x+10)} = \frac{3x+10+2x+2}{3x^2+3x+10x+10}$

$= \frac{5x+12}{3x^2+13x+10}$ —— (1) ✓

(b) Curve; $y = \frac{5x+12}{3x^2+13x+10}$ —— (2)

intersects y-axis ⟶ for $x=0 \Rightarrow y = \frac{12}{10} \Rightarrow P(0, 6/5)$ ✓

Eqn of line PQ ⟶ $y - \frac{6}{5} = 1(x-0) \Rightarrow y = x + \frac{6}{5}$ —— (3)

line intersects x-axis for $y=0 \Rightarrow x = -\frac{6}{5}$ ⟶ $Q(-\frac{6}{5}, 0)$

Area of the shaded region = area of Δ POQ + area under the curve (for $x=0$ to $x=2$)

$= \frac{1}{2} \times \frac{6}{5} \times \frac{6}{5} + \int_0^2 \frac{5x+12}{3x^2+13x+10}\, dx$

$= \frac{18}{25} + \int_0^2 \left(\frac{1}{x+1} + \frac{2}{3x+10}\right) dx$

$= \frac{18}{25} + \left[\ln(x+1) + \frac{2}{3}\ln(3x+10)\right]_0^2$

$= \frac{18}{25} + \left[\left(\ln 3 + \frac{2}{3}\ln 16\right) - \left(\ln 1 + \frac{2}{3}\ln 10\right)\right]$

$= \frac{18}{25} + \frac{2}{3}(\ln 16 - \ln 10) + \frac{2}{3}\ln 3^{3/2}$

$= \frac{18}{25} + \frac{2}{3}\ln\frac{16}{10} + \frac{2}{3}\ln 3\sqrt{3}$

$= \frac{18}{25} + \frac{2}{3}\left[\ln\frac{8}{5} + \ln 3\sqrt{3}\right) = \frac{18}{25} + \frac{2}{3}\ln\left(\frac{24\sqrt{3}}{5}\right)$ ✓

16 A curve has equation $y = x\cos x$

(a) Find $\frac{dy}{dx}$ --[2]

(c) Using your answer to part (a), find the exact value of $\int_0^{\frac{\pi}{6}} x\sin x\,dx$ --[5]

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Solution (a) $y = x\cos x$

$$\frac{dy}{dx} = x \cdot \frac{d}{dx}\cos x + \cos x \frac{d}{dx} x$$

$$= -x\sin x + \cos x)$$

(b) $\int_0^{\frac{\pi}{6}} x\sin x\,dx$ —— ①

Now from Part (a) $\int(-x\sin x + \cos x)dx = x\cos x$

$$\Rightarrow -\int x\sin x\,dx + \int \cos x\,dx = x\cos x$$

$$\Rightarrow -\int x\sin x\,dx + \sin x = x\cos x$$

$$\Rightarrow \int x\sin x\,dx = \sin x - x\cos x \text{ —— ②}$$

from ① & ②

$$\int_0^{\frac{\pi}{6}} x\sin x\,dx = \left[\sin x - x\cos x\right]_0^{\frac{\pi}{6}}$$

$$= \left(\sin\frac{\pi}{6} - \frac{\pi}{6}\cos\frac{\pi}{6}\right) - (\sin 0 - 0)$$

$$= \frac{1}{2} - \frac{\pi}{6} \times \frac{\sqrt{3}}{2}$$

$$= \frac{1}{2} - \frac{\pi\sqrt{3}}{12}$$