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0606

Additional Maths

Vectors
Revision

SP-20	M-20	S-20	W-20

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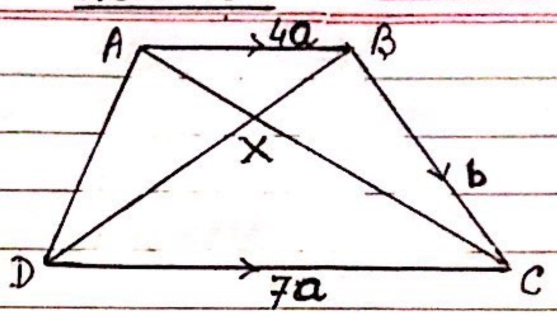
Vectors

Revision

1. In the diagram $\overrightarrow{AB} = 4\mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{DC} = 7\mathbf{a}$.
The lines AC and DB intersect at the point X.
Find in terms of $\mathbf{a}$ and $\mathbf{b}$,



(a) $\overrightarrow{DB}$ --- [1]

(b) $\overrightarrow{DA}$ -- [1]

Given that $\overrightarrow{AX} = \lambda \overrightarrow{AC}$, find in terms of $\mathbf{a}$, $\mathbf{b}$ and λ,

(c) $\overrightarrow{AX}$ -- [1]

(d) $\overrightarrow{DX}$ -- [2]

Given that $\overrightarrow{DX} = \mu \overrightarrow{DB}$

(e) Find the value of λ and μ. SP-20/02/Q9 -- [4]

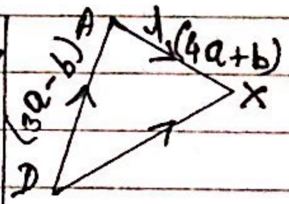
Solution

(a) In ΔDBC, $\overrightarrow{DB} + \overrightarrow{BC} = \overrightarrow{DC} \Rightarrow \overrightarrow{DB} + \mathbf{b} = 7\mathbf{a} \Rightarrow \overrightarrow{DB} = \underline{7\mathbf{a} - \mathbf{b}}$ ✓

(b) $\overrightarrow{DA} + \overrightarrow{AB} = \overrightarrow{DB} \Rightarrow \overrightarrow{DA} = \overrightarrow{DB} - \overrightarrow{AB} = (7\mathbf{a} - \mathbf{b}) - 4\mathbf{a} = \underline{3\mathbf{a} - \mathbf{b}}$ ✓

(c) $\overrightarrow{AX} = \lambda \overrightarrow{AC} = \lambda(\overrightarrow{AB} + \overrightarrow{BC}) = \underline{\lambda(4\mathbf{a} + \mathbf{b})}$ ✓

(d) $\overrightarrow{DX} = \overrightarrow{DA} + \overrightarrow{AX} = (3\mathbf{a} - \mathbf{b}) + \lambda(4\mathbf{a} + \mathbf{b})$



(e) $\overrightarrow{DX} = \mu \overrightarrow{DB}$

$\Rightarrow (3\mathbf{a} - \mathbf{b}) + \lambda(4\mathbf{a} + \mathbf{b}) = \mu(7\mathbf{a} - \mathbf{b})$

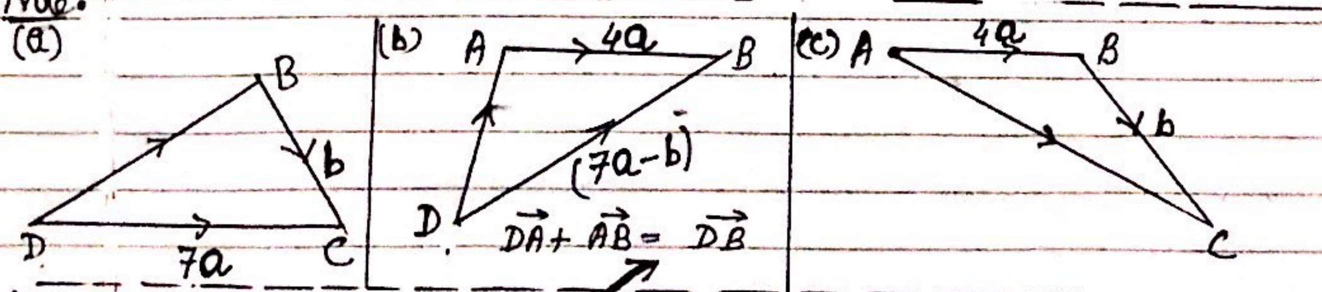
Comparing the coefficients of vectors $\mathbf{a}$ & $\mathbf{b}$

$\Rightarrow \quad 3 + 4\lambda = 7\mu \quad$ —— ①

$-1 + \lambda = -\mu \quad$ —— ②

Solving ① & ②, $\lambda = \frac{4}{11}$ and $\mu = \frac{7}{11}$ ✓

Note:



Note: **Triangle Law of Vector addition:** If two vectors (in magnitude and direction) are denoted by two sides of a triangle, taken in same order, then their vector sum is denoted by the third side in the opposite order.

2. In this question all distances are in km. A ship P sails from a point A, which has position Vector $\begin{pmatrix}0\\0\end{pmatrix}$, with a speed of $52\,\text{Km}h^{-1}$ in the direction of $\begin{pmatrix}-5\\12\end{pmatrix}$.

(a) Find the velocity vector of the ship. ---[1]

(b) Write down the position vector of P at a time t hours after leaving A. --[1]

At the same time that ship P sails from A, a ship Q sails from a point B, which has position vector $\begin{pmatrix}12\\8\end{pmatrix}$, with velocity vector $\begin{pmatrix}-25\\45\end{pmatrix}$ Km h^{-1}. --[1]

(c) Write down the position vector of Q at a time t hours after leaving B.

(d) Using your answers to parts (b) and (c), find the displacement vector $\vec{PQ}$ at time t hours. ---[1]

(e) Hence show that $PQ = \sqrt{34t^2 - 168t + 208}$ --[2]

(f) Find the value of t when P and Q are first 2 Km apart --[2]

[M-20/12/Q8]

Solution (a) Direction of speed of P, $\vec{r} = -5i + 12j$

$|\vec{r}| = \sqrt{(-5)^2 + 12^2} = 13$

Unit vector along $\vec{r} = \frac{1}{13}(-5i+12j)$

Given S/ of P is 52 $\text{Km}h^{-1}$

$\therefore$ Required Vector V along $\vec{r}$ is

$V = \frac{52}{13}(-5i+12j) = (-20i+48j)$

or $= \begin{pmatrix}-20\\48\end{pmatrix}$ ✓

(b) Position Vector of P at time $t = \begin{pmatrix}-20\\48\end{pmatrix} \cdot t$ ✓

(c) Position Vector Q at time t

$= \begin{pmatrix}12\\8\end{pmatrix} + t\begin{pmatrix}-25\\45\end{pmatrix}$ ✓

(d) displacement Vector $\vec{PQ}$;

$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{pmatrix}12\\8\end{pmatrix} + t\begin{pmatrix}-25\\45\end{pmatrix} - \begin{pmatrix}-20\\48\end{pmatrix}t$

$\Rightarrow \vec{PQ} = \begin{pmatrix}12\\8\end{pmatrix} + \begin{pmatrix}-5\\-3\end{pmatrix}t$

$\vec{PQ} = \begin{pmatrix}12-5t\\8-3t\end{pmatrix}$

(e) $|\vec{PQ}| = \sqrt{(12-5t)^2 + (8-3t)^2}$

or $PQ = \sqrt{34t^2 - 168t + 208}$ ✓

(f) when $PQ = 2$

from Part (e)

$PQ = \sqrt{34t^2 - 168t + 208} = 2$

$34t^2 - 168t + 208 = 4$

$\Rightarrow 34t^2 - 168t + 204 = 0$

$\Rightarrow 17t^2 - 84t + 102 = 0$

$b^2 - 4ac$

$= (-84)^2 - 4 \times 17 \times 102$

$= 120$

$t = \frac{84 \pm \sqrt{120}}{34}$

$= \frac{84 \pm 11}{34}$

$= \frac{95}{34}, \frac{73}{34}$

$= 2.79^{\times}, 2.15$ ✓

$\therefore$ P and Q are first 2 Km apart for $t = 2.15$ ✓

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3. The position vectors of three points A, B and C, relative an origin O, are $\begin{pmatrix}-5\\-7\end{pmatrix}$, $\begin{pmatrix}10\\-4\end{pmatrix}$ and $\begin{pmatrix}x\\y\end{pmatrix}$ respectively, given that $\vec{AC} = 4\vec{BC}$, find the vector in the direction of $\vec{OC}$. --- [5]

[M-20/22/Q4]

Solution: Given $\vec{AC} = 4\vec{BC}$ $\quad \left(\because \vec{AC} = \vec{OC} - \vec{OA} \text{ and } \vec{BC} = \vec{OC} - \vec{OB} \right)$

$$\Rightarrow \vec{OC} - \vec{OA} = 4(\vec{OC} - \vec{OB}) \text{ — (1)}$$

$$\Rightarrow \vec{OC} - \vec{OA} = 4\vec{OC} - 4\vec{OB}$$

$$\Rightarrow 3\vec{OC} = 4\vec{OB} - \vec{OA} = 4\cdot\begin{pmatrix}10\\-4\end{pmatrix} - \begin{pmatrix}-5\\-7\end{pmatrix}$$

$$3\vec{OC} = \begin{pmatrix}40\\-16\end{pmatrix} + \begin{pmatrix}5\\7\end{pmatrix}$$

$$\vec{OC} = \frac{1}{3}\begin{pmatrix}45\\-9\end{pmatrix} = \begin{pmatrix}15\\-3\end{pmatrix} \text{ — (2)}$$

$$|\vec{OC}| = \sqrt{15^2 + (-3)^2} = \sqrt{234} \text{ — (3)}$$

$\therefore$ from (2) and (3) Vector along $\vec{OC} = \frac{1}{\sqrt{234}}\begin{pmatrix}15\\-3\end{pmatrix}$ ✓

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4. The vectors **a** and **b** are such that $\mathbf{a} = \alpha \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 12\mathbf{i} + \beta \mathbf{j}$

(a) Find the value of each of the constants α and β such that $4\mathbf{a} - \mathbf{b} = (\alpha+3)\mathbf{i} - 2\mathbf{j}$ --- [3]

(b) Hence find the unit vector in the direction of $\mathbf{b} - 4\mathbf{a}$ -- [2]

[S-20/21/Q5]

Solution (a) Given $4\mathbf{a} - \mathbf{b} = (\alpha+3)\mathbf{i} - 2\mathbf{j}$

$$\Rightarrow 4(\alpha \mathbf{i} + \mathbf{j}) - (12\mathbf{i} + \beta \mathbf{j}) = (\alpha+3)\mathbf{i} - 2\mathbf{j}$$

$$\Rightarrow (4\alpha - 12)\mathbf{i} + (4 - \beta)\mathbf{j} = (\alpha+3)\mathbf{i} - 2\mathbf{j}$$

$$\Rightarrow 4\alpha - 12 = \alpha + 3 \ ; \ 4 - \beta = -2$$

$$\Rightarrow \underline{\alpha = 5} \quad \text{and} \quad \underline{\beta = 6} \ ✓$$

(b) $\mathbf{a} = \alpha \mathbf{i} + \mathbf{j} = 5\mathbf{i} + \mathbf{j}$

$\mathbf{b} = 12\mathbf{i} + \beta \mathbf{j} = 12\mathbf{i} + 6\mathbf{j}$

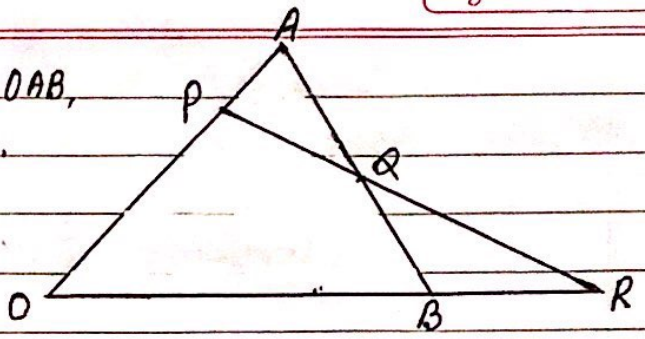
$\therefore \mathbf{b} - 4\mathbf{a} = (12\mathbf{i} + 6\mathbf{j}) - 4(5\mathbf{i} + \mathbf{j}) = (-8\mathbf{i} + 2\mathbf{j})$ — (1)

$|(\mathbf{b} - 4\mathbf{a})| = \sqrt{(-8)^2 + 2^2} = \sqrt{68}$ — (2)

$\therefore$ Unit vector in the direction of $\mathbf{b} - 4\mathbf{a}$ is $= \frac{\mathbf{b} - 4\mathbf{a}}{|\mathbf{b} - 4\mathbf{a}|}$

$$= \frac{-8\mathbf{i} + 2\mathbf{j}}{\sqrt{68}} \ ✓ \text{ from (1) \& (2)}$$

5. The diagram shows a triangle OAB, such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point P lies on OA such that $OP = \frac{3}{4} OA$. The point Q is the mid point of AB.



The lines OB and PQ are extended to meet at the point R.
Find, in terms of **a** and **b**,

(a) $\overrightarrow{AB}$ [S-20/12/Q8] ---[1]

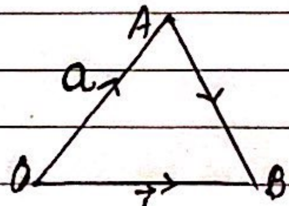
(b) $\overrightarrow{PQ}$. Give your answer in its simplest form. ---[3]

It is given that $n\overrightarrow{PQ} = \overrightarrow{QR}$ and $\overrightarrow{BR} = k\mathbf{b}$, where n and k are positive constants.

(c) Find $\overrightarrow{QR}$ in terms of n, **a** and **b**. ---[1]

(d) Find $\overrightarrow{QR}$ in terms of k, **a** and **b**. ---[2]

(e) Hence find the value of n and of k. ---[3]



Solution (a) $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

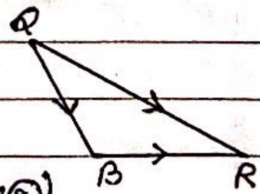
$\Rightarrow a + \overrightarrow{AB} = \overrightarrow{OB} \Rightarrow \overrightarrow{AB} = b - a$ ✓

(b) $\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ}$ $\quad / \ \overrightarrow{PA} = \frac{1}{4}\overrightarrow{OA},\ \overrightarrow{AQ} = \frac{1}{2}\overrightarrow{AB})$

$= \frac{1}{4}a + \frac{1}{2}(b-a) = \left(\frac{1}{2}b - \frac{1}{4}a\right)$ ✓

(c) $\overrightarrow{QR} = n\,\overrightarrow{PQ}$

$= n\left(\frac{1}{2}b - \frac{1}{4}a\right)$ — ① (from part (b))



(d) $\overrightarrow{QR} = \overrightarrow{QB} + \overrightarrow{BR} = \frac{1}{2}\overrightarrow{AB} + kb$

$= \frac{1}{2}(b-a) + kb$ — ②

(e) $\overrightarrow{QR} = n\left(\frac{1}{2}b - \frac{1}{4}a\right) = \frac{1}{2}(b-a) + kb$ (from ① & ②)

Equaling the coefficients of a & b

$\Rightarrow -\frac{n}{4} = -\frac{1}{2} \Rightarrow n = 2$ ✓ and $\frac{n}{2} = \frac{1}{2} + k$

or $\frac{2}{2} = \frac{1}{2} + k \quad (\because n = 2)$

$1 = \frac{1}{2} + k$

$\Rightarrow k = \frac{1}{2}$ ✓

$\therefore$ $n = 2$ and $k = \frac{1}{2}$ ✓

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6(a) Find the unit vector in the direction of $\begin{pmatrix}5\\-12\end{pmatrix}$ ---[1]

(b) Given that $\begin{pmatrix}4\\1\end{pmatrix} + k\begin{pmatrix}-2\\3\end{pmatrix} = r\begin{pmatrix}-10\\5\end{pmatrix}$, find the value of each of the constants k and r. ---[3]

(C) Relative to an origin O, the points A, B and C have position vectors p, 3q−p and 9q−5p respectively.

(i) Find $\vec{AB}$ in terms of p and q. --[1]

(ii) Find $\vec{AC}$ in terms of p and q. --[1]

(iii) Explain why A, B and C all lie in a straight line. --[1]

(iv) Find the ratio AB : BC --[1]

Solution (a) Let $a = \begin{pmatrix}5\\-12\end{pmatrix} \Rightarrow |a| = \sqrt{5^2+(-12)^2} = 13$

$\therefore$ unit vector in the direction of $\begin{pmatrix}5\\-12\end{pmatrix} = \frac{1}{13}\begin{pmatrix}5\\-12\end{pmatrix}$

(b) $\begin{pmatrix}4\\1\end{pmatrix} + k\begin{pmatrix}-2\\3\end{pmatrix} = r\begin{pmatrix}-10\\5\end{pmatrix} \Rightarrow \begin{pmatrix}4-2k\\1+3k\end{pmatrix} = \begin{pmatrix}-10r\\5r\end{pmatrix}$

$\Rightarrow \begin{cases}4-2k = -10r\\ \text{and } 1+3k = 5r\end{cases}$

$\Rightarrow \begin{cases}2k-10r = 4\\ 3k-5r = -1\end{cases}$ Solving we get $r = -\frac{7}{10}$ and $k = -\frac{3}{2}$

(C) (i) $\vec{AB} = \vec{OB} - \vec{OA} = (3q-p)-p = 3q-2p$ —— ①

(ii) $\vec{AC} = \vec{OC} - \vec{OA} = (9q-5p)-p = 9q-6p$ —— ②

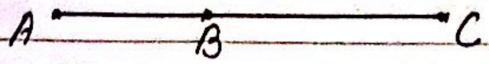
(iii) for ② $\vec{AC} = 9q-6p = 3(3q-2p) = 3\vec{AB}$ from ①

$\Rightarrow \vec{AC} = 3\vec{AB} \Rightarrow \vec{AC}$ is parallel to $\vec{AB}$

and the initial point A is common in both

$\therefore$ The three points A, B and C lie in a straight line.

(iv)



from Part (iii) $\vec{AC} = 3\vec{AB} \Rightarrow AC = 3AB$ —— ③

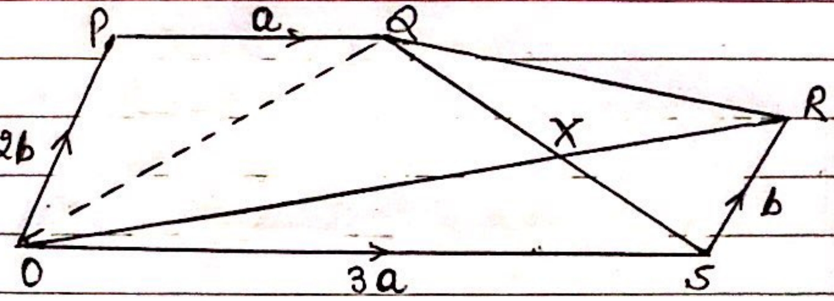
A, B, C lie in line $\rightarrow AB + BC = AC$ —— ④

from ③ & ④ $AB + BC = 3AB$

$\Rightarrow BC = 2AB$

$\Rightarrow \frac{AB}{BC} = \frac{1}{2}$

7. In the diagram,
$\vec{OP} = 2b$,
$\vec{OS} = 3a$,
$\vec{SR} = b$,
$\vec{PQ} = a$,



The lines OR and QS intersect at X.

(a) Find $\vec{OQ}$ in terms of a and b. ---[1]

(b) Find $\vec{QS}$ in terms of a and b ---[1]

(c) Given that $\vec{QX} = \mu \vec{QS}$, find $\vec{OX}$ in terms of a, b and μ, ---[1]

(d) Given that $\vec{OX} = \lambda \vec{OR}$, find $\vec{OX}$ in terms of a, b and λ --[1]

(e) Find the value of λ and μ --[3]

(f) Find the value of $\frac{QX}{XS}$ --[1]

(g) Find the value of $\frac{OR}{OX}$ --[1]

W-20/22/Q9

Solution (a) $\vec{OQ} = \vec{OP} + \vec{PQ} = (2b + a)$ — ①

(b) $\vec{OQ} + \vec{QS} = \vec{OS}$

$\Rightarrow \vec{QS} = \vec{OS} - \vec{OQ}$

$= 3a - (2b + a)$

$= (2a - 2b)$ — ②

(c) $\vec{QX} = \mu \vec{QS}$ — ③

$\vec{OQ} + \vec{QX} = \vec{OX}$

$\Rightarrow \vec{OX} = (2b + a) + \mu QS$

$= (2b + a) + \mu(2a - 2b)$ from ①②&③

(d) $\vec{OR} = \vec{OS} + \vec{SR} = 3a + b$ — ④

Given $\vec{OX} = \lambda \vec{OR}$

$\vec{OX} = \lambda(3a + b)$ from ④

(e) from (c) and (d) parts

$\vec{OX} = (2b + a) + \mu(2a - 2b) = \lambda(3a + b)$

$\Rightarrow 3\lambda = 1 + 2\mu$ — ⑤

and $\lambda = 2 - 2\mu$ — ⑥

Solving ⑤ & ⑥ $\lambda = \frac{3}{4}$ and $\mu = \frac{5}{8}$

(f) from Part ©

$\frac{QX}{QS} = \mu = \frac{5}{8}$

$\Rightarrow \frac{QX}{QX + SX} = \frac{5}{8}$

$\Rightarrow \frac{QX + SX}{QX} = \frac{8}{5}$

$\Rightarrow 1 + \frac{SX}{QX} = \frac{8}{5} \Rightarrow \frac{SX}{QS} = \frac{3}{5}$

$\Rightarrow \frac{QS}{SX} = \frac{5}{3}$

(g) $\frac{OR}{OX} = \frac{4}{3}$

$\because \vec{OX} = \lambda \vec{OR}$

$\frac{OX}{OR} = \lambda = \frac{3}{4}$

$\Rightarrow \frac{OR}{OX} = \frac{4}{3}$

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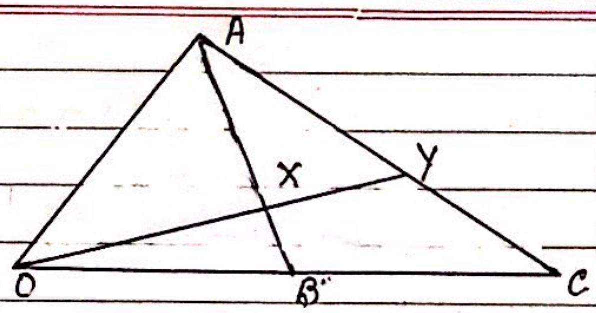
8\. The diagram show the triangle OAC. The point B is the mid point of OC. The point Y lies on AC, such that OY intersects AB at the point X where AX : XB = 3 : 1. It is given that $\vec{OA} = a$ and $\vec{OB} = b$



(a) Find OX in terms of a and b, giving your answer in its simplest form. --[3]

(b) Find $\vec{AC}$ in terms of a and b. --[1]

(c) Given that $\vec{OY} = h\,\vec{OX}$, Find $\vec{AY}$ in terms of a, b and h. --[1]

(d) Given that $\vec{AY} = m\,\vec{AC}$, find the value of h and m. --[4]

W-20/13/Q9

Solution. (a) $\vec{OX} = \vec{OA} + \vec{AX}$ $\quad (AX : XB = 3:1 \Rightarrow AX = \frac{3}{4}AB)$

$= \vec{OA} + \frac{3}{4}\vec{AB}$ $\quad (\vec{OA} + \vec{AB} = \vec{OB} = \vec{AB} + a = b \Rightarrow \vec{AB} = b - a)$

$= a + \frac{3}{4}(b - a) = \frac{a}{4} + \frac{3}{4}b$

(b) $\vec{OA} + \vec{AC} = \vec{OC}$

$a + \vec{AC} = 2b$ $\quad (\because$ B is mid point of OC $\to \vec{OC} = 2\vec{OB} = 2b)$

$\Rightarrow \vec{AC} = (2b - a)$

(c) $\vec{OA} + \vec{AY} = \vec{OY} = h\,\vec{OX}$ $\quad$ (Given $\vec{OY} = h\,\vec{OX}$)

$a + \vec{AY} = h\left(\frac{a}{4} + \frac{3}{4}b\right)$ $\quad$ from Part (a)

$\Rightarrow \vec{AY} = h\left(\frac{a}{4} + \frac{3}{4}b\right) - a$ —— (1)

(d) $\vec{AY} = m\,\vec{AC} = m(2b - a)$ —— (2) $\quad$ (from Part (b) $\vec{AC} = (2b - a)$)

$\therefore m(2b - a) = h\left(\frac{a}{4} + \frac{3}{4}b\right) - a$ $\quad$ (from (1) & (2))

Comparing the coeff. of a & b,

$\Rightarrow -1 + \frac{h}{4} = -m$ —— (3) and $\frac{3}{4}h = 2m$ —— (4)

Solving (3) & (4) $\quad h = \frac{8}{5}, \quad m = \frac{3}{5}$