

0580

# IGCSE Maths

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Algebra 3

Revision

SP-20 | M-20 | S-20 | W-19 | W-20

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# Functions

1(a)  $f(x) = 4x + 3$        $g(x) = 5x - 4$

$$fg(x) = 20x + p$$

find the value of  $p$ .

---[2]

Solution:  $fg(x) = f(g(x)) = f(5x - 4)$

$$= 4(5x - 4) + 3$$

$$[\because f(x) = 4x + 3]$$

$$= 20x - 16 + 3$$

$$\therefore fg(x) = 20x - 13 \quad \text{---(1)}$$

Given  $fg(x) = 20x + p \quad \text{---(2)}$

from (1) and (2)  $p = \underline{\underline{-13}}$  ✓

(b)  $h(x) = \frac{5x - 1}{3}$

Find  $h^{-1}(x)$ .

---[3]

Solution:  $h(x) = \frac{5x - 1}{3} = y$  (let)

$$\Rightarrow 3y = 5x - 1$$

Interchange  $x$  and  $y$

$$\Rightarrow 3x = 5y - 1$$

$$\Rightarrow 5y = 3x + 1$$

$$\Rightarrow y = \frac{3x + 1}{5}$$

$$\Rightarrow h^{-1}(x) = \frac{3x + 1}{5} \quad \checkmark$$

[S-20/21/Q14]

2  $f(x) = 3x - 5$        $g(x) = 2^x$

(a) Find  $fg(3)$

---[2]

Solution:  $fg(3) = f(g(3))$        $\because g(x) = 2^x$

$$= f(2^3) = f(8)$$

$$= 3 \times 8 - 5$$

$$\because f(x) = 3x - 5$$

$$= 24 - 5 = 19 \quad \checkmark$$

(b) Find  $f^{-1}(x)$ .

---[2]

Given  $f(x) = 3x - 5 = y$  (let)

Interchange  $x$  &  $y \Rightarrow 3y - 5 = x$

$$y = \frac{x + 5}{3}$$

[W-19/21/Q24]

$$\Rightarrow f^{-1}(x) = \underline{\underline{\frac{x + 5}{3}}} \quad \checkmark$$

3.  $f(x) = 2x+1$      $g(x) = x^2+4$      $h(x) = 2^x$
- (a) Solve the equation  $f(x) = g(1)$     ...[2]
- (b) Find  $f^{-1}(x)$     ...[2]
- (c) Find  $gf(x)$  in its simplest form.    ...[3]
- (d) Solve the equation  $h^{-1}(x) = 0.5$     ...[1]
- (e)  $\frac{1}{h(x)} = 2^{kx}$ ,  
Write down the value of  $k$ .    [SP-20/04/Q7] --[1]

Solution: (a) Solve  $f(x) = g(1)$   
 $\Rightarrow 2x+1 = 1^2+4$      $\because g(x) = x^2+4$   
 $\Rightarrow 2x = 4 \Rightarrow \underline{x=2}$  ✓

(b)  $f(x) = 2x+1 = y$  (let)  
Interchange  $x$  and  $y$   
 $2y+1 = x$   
 $\Rightarrow y = \frac{x-1}{2} \Rightarrow \underline{f^{-1}(x) = \frac{x-1}{2}}$  ✓

(c)  $gf(x) = g(f(x))$   
 $= g(2x+1)$   
 $= (2x+1)^2+4$      $\because g(x) = x^2+4$   
 $= 4x^2+4x+1+4$   
 $= \underline{4x^2+4x+5}$  ✓

(d) Solve the equation  $h^{-1}(x) = 0.5$   
 $\Rightarrow x = h(0.5)$   
 $= 2^{0.5}$      $\because h(x) = 2^x$   
 $= 2^{\frac{1}{2}} = \underline{\sqrt{2}}$  ✓

(e)  $\frac{1}{h(x)} = 2^{kx}$   
 $\Rightarrow \frac{1}{2^x} = 2^{kx}$   
 $\Rightarrow 2^{-x} = 2^{kx} \Rightarrow kx = -x$   
 $\Rightarrow \underline{k = -1}$  ✓

4.  $f(x) = 4x - 1$        $g(x) = x^2$        $h(x) = 3^{-x}$

(a) Find in its simplest form:

(i)  $f(x-3)$       ... [1]

(ii)  $g(5x)$       ... [1]

(b) Find  $f^{-1}(x)$       ... [1]

(c) Find the value of  $h(h(1))$ , correct to 4 significant figures.      ... [3]

(d) (i) Show that  $g(3x-2) - h(-3)$  can be written as  $9x^2 - 12x - 23$       ... [2]

(ii) Use the quadratic formula to solve  $9x^2 - 12x - 23 = 0$       ... [4]

(e) Find  $x$  when  $f(61) = h(x)$       [M-20/42/Q10] ... [2]

Solution (a) (i)  $f(x-3) = 4(x-3) - 1$        $\because f(x) = 4x - 1$   
 $= 4x - 12 - 1 = 4x - 13$  ✓

(ii)  $g(5x) = (5x)^2$        $\because g(x) = x^2$   
 $= 25x^2$  ✓

(b)  $f(x) = 4x - 1 = y$  (let)  
 Interchange  $x$  and  $y \Rightarrow 4y - 1 = x$   
 $\Rightarrow y = \frac{x+1}{4} \Rightarrow f^{-1}(x) = \frac{x+1}{4}$  ✓

(c)  $h(h(1)) = h(h(1)) = h(3^{-1})$        $\because h(x) = 3^{-x}$   
 $= h\left(\frac{1}{3}\right) = 3^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{3}} = \frac{1}{1.44225} = 0.6933$  ✓

(d) (i)  $g(3x-2) - h(-3) = (3x-2)^2 - 3^{-(-3)}$   
 $= 9x^2 - 12x + 4 - 27 = 9x^2 - 12x - 23$  ✓

(ii)  $9x^2 - 12x - 23 = 0$        $\begin{cases} a=9, b=-12, c=-23 \\ b^2 - 4ac = (-12)^2 - 4 \times 9 \times (-23) = 972 \end{cases}$   
 $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{12 \pm \sqrt{972}}{2 \times 9}$   
 $= \frac{12 \pm 31.17}{18}$   
 $= \frac{43.17}{18}, -\frac{19.17}{18}$   
 $= 2.34; -1.07$  ✓

(e)  $f(61) = h(x)$   
 $\Rightarrow 4 \times 61 - 1 = 3^{-x}$   
 $\Rightarrow 3^{-x} = 243 = 3^5$   
 $\Rightarrow -x = 5$   
 $\Rightarrow x = -5$  ✓

5.  $f(x) = 3x+2$        $g(x) = x^2+1$        $h(x) = 4^x$
- (a) Find  $h(4)$       --- [1]
- (b) Find  $fg(1)$       --- [2]
- (c) Find  $gf(x)$  in the form  $ax^2+bx+c$       --- [3]
- (d) Find  $x$  when  $f(x) = g(7)$       --- [2]
- (e) Find  $f^{-1}(x)$       --- [2]
- (f) Find  $\frac{g(x)}{f(x)} + x$  Giving your answer as a single fraction in terms of  $x$
- (g) Find  $x$  when  $h^{-1}(x) = 2$       --- [3]

[S-20/42/26]

Solution (a)  $h(4) = 4^4 = 256 \checkmark$       ( $\because h(x) = 4^x$ )

(b)  $fg(1) = f(g(1))$       ( $\because g(x) = x^2+1$ )  
 $= f(1^2+1) = f(2)$   
 $= 3 \times 2 + 2 = 8 \checkmark$       ( $\because f(x) = 3x+2$ )

(c)  $gf(x) = g(f(x)) = g(3x+2)$   
 $= (3x+2)^2 + 1 = 9x^2 + 12x + 4 + 1$   
 $= 9x^2 + 12x + 5 \checkmark$

(d)  $f(x) = g(7) \Rightarrow 3x+2 = 7^2+1$   
 $\Rightarrow 3x+2 = 50 \Rightarrow 3x = 48$   
 $\Rightarrow x = 16 \checkmark$

(e)  $f(x) = 3x+2 = y$  (let)

Interchange  $x$  and  $y \Rightarrow 3y+2 = x$

$\Rightarrow y = \frac{x-2}{3}$

$\Rightarrow f^{-1}(x) = \frac{x-2}{3} \checkmark$

(f)  $\frac{g(x)}{f(x)} + x = \frac{x^2+1}{3x+2} + x$   
 $= \frac{x^2+1 + x(3x+2)}{3x+2} = \frac{x^2+1 + 3x^2+2x}{3x+2}$   
 $= \frac{4x^2+2x+1}{3x+2} \checkmark$

(g)  $h^{-1}(x) = 2 \Rightarrow h(2) = x$   
 $\Rightarrow 4^2 = x$        $\because h(x) = 4^x$   
 $\Rightarrow x = 16 \checkmark$

6.  $f(x) = 7x - 4$        $g(x) = \frac{2x}{x-3}, x \neq 3$ ,       $h(x) = x^2$
- (a) Find  $g(6)$       --- [1]
- (b) Find  $fg(4)$       --- [2]
- (c) Find  $fh(x)$       --- [1]
- (d) Find  $\frac{f(x)}{2} + g(x)$ , Give your answer as a single fraction [3]
- (e) Find the value of  $x$  when  $f(x+2) = -11$       --- [2]
- (f) Find the value of  $p$  that satisfy  $h(p) = p$ .      --- [2]

[5-20/43/Q11]

Solution (a)  $g(6) = \frac{2 \times 6}{6-3} = \frac{12}{3} = \underline{4} \checkmark$  ( $\because g(x) = \frac{2x}{x-3}$ )

(b)  $fg(4) = f(g(4)) = f\left(\frac{2 \times 4}{4-3}\right) = f(8)$   
 $= 7 \times 8 - 4$  ( $\because f(x) = 7x - 4$ )  
 $= 56 - 4 = \underline{52} \checkmark$

(c)  $fh(x) = f(x^2) = \underline{7x^2 - 4} \checkmark$

(d)  $\frac{f(x)}{2} + g(x) = \frac{7x-4}{2} + \frac{2x}{x-3}$   
 $= \frac{(7x-4)(x-3) + 2x \times 2}{2(x-3)}$   
 $= \frac{7x^2 - 21x - 4x + 12 + 4x}{2(x-3)} = \underline{\frac{7x^2 - 21x + 12}{2(x-3)}} \checkmark$

(e)  $f(x+2) = -11$   
 $\Rightarrow 7(x+2) - 4 = -11 \Rightarrow 7x + 14 - 4 = -11$   
 $\Rightarrow 7x = -11 - 10$   
 $\Rightarrow x = \frac{-21}{7} = -3$   
 $\therefore \underline{x = -3} \checkmark$

(f)  $h(p) = p$   
 $\Rightarrow p^2 = p$  ( $\because h(x) = x^2$ )  
 $\Rightarrow p^2 - p = 0$   
 $\Rightarrow p(p-1) = 0$   
 $\Rightarrow p = 0$  or  $p - 1 = 0$   
 $\Rightarrow \underline{p = 0}$  or  $\underline{p = 1} \checkmark$

7.  $f(x) = 7 - 2x$      $g(x) = \frac{10}{x}, x \neq 0$      $h(x) = 27^x$

(a) Find (i)  $f(-3)$  --- [1]

(ii)  $hg(30)$  --- [2]

(iii)  $f^{-1}(x)$  --- [2]

(b) Solve  $g(2x+1) = 4$  --- [3]

(c) Simplify, giving your answer as a single fraction:  $\frac{1}{f(x)} + g(x)$  --- [3]

(d) Find  $h^{-1}(19683)$  --- [1]

[W-19|42|Q7]

Solve (a) (i)  $f(-3) = 7 - 2(-3)$     ( $\because f(x) = 7 - 2x$ )  
 $= 7 + 6 = \underline{13}$  ✓

(ii)  $hg(30)$   
 $= h\left(\frac{10}{30}\right) = h\left(\frac{1}{3}\right)$     ( $\because g(x) = \frac{10}{x}$ )

$= 27^{\frac{1}{3}}$     ( $\because h(x) = 27^x$ )  
 $= (3^3)^{\frac{1}{3}} = 3^{3 \times \frac{1}{3}} = \underline{3}$  ✓

(b)  $g(2x+1) = 4$

$\Rightarrow \frac{10}{2x+1} = 4 \Rightarrow 4(2x+1) = 10$

$\Rightarrow 2x+1 = \frac{10}{4} \Rightarrow 2x = \frac{5}{2} - 1 = \frac{3}{2}$

$\Rightarrow x = \frac{3}{4} = \underline{0.75}$  ✓

(c)  $\frac{1}{f(x)} + g(x) = \frac{1}{7-2x} + \frac{10}{x}$

$= \frac{x + 10(7-2x)}{x(7-2x)} = \frac{x + 70 - 20x}{x(7-2x)}$

$= \frac{70 - 19x}{x(7-2x)}$  ✓

(d)  $h^{-1}(19683) = x$  (let)

$\Rightarrow h(x) = 19683$

$\Rightarrow 27^x = 19683$

$\Rightarrow 27^x = 27^3$

$\Rightarrow x = 3$

$\therefore h^{-1}(19683) = \underline{3}$  ✓

8.  $f(x) = 2x - 3$        $g(x) = 9 - x^2$        $h(x) = 3^x$
- (a) Find (i)  $f(4)$       --[1]  
(ii)  $hg(3)$       --[2]  
(iii)  $g(2x)$  in its simplest form.      ---[1]  
(iv)  $fg(x)$  in its simplest form.      --[2]
- (b) Find  $f^{-1}(x)$       --[2]
- (c) Find  $x$  when  $5f(x) = 3$       --[2]
- (d) Solve the equation  $gf(x) = -16$       ---[4]
- (e) Find  $x$  when  $h^{-1}(x) = -2$       --[1]

W-19/43/Q9

Solution: (a) (i)  $f(4) = 2 \times 4 - 3 = 5 \checkmark$       ( $\because f(x) = 2x - 3$ )  
(ii)  $hg(3) = h(9 - 3^2)$       ( $\because g(x) = 9 - x^2$ )  
 $= h(0) = 3^0 = 1 \checkmark$

(iii)  $g(2x) = 9 - (2x)^2$   
 $= 9 - 4x^2 \checkmark$

(iv)  $fg(x) = f(g(x)) = f(9 - x^2) = 2(9 - x^2) - 3$   
 $= 18 - 2x^2 - 3 = 15 - 2x^2 \checkmark$

(b)  $f(x) = 2x - 3 = y$  (let)

Interchange  $x$  and  $y \Rightarrow 2y - 3 = x$

$\Rightarrow y = \frac{x+3}{2} \Rightarrow f^{-1}(x) = \frac{x+3}{2} \checkmark$

(c)  $5f(x) = 3 \Rightarrow 5(2x - 3) = 3$

$\Rightarrow 10x - 15 = 3 \Rightarrow 10x = 18 \Rightarrow x = \frac{18}{10} = 1.8 \checkmark$

(d)  $gf(x) = -16 \Rightarrow g[2x - 3] = -16$

$\Rightarrow 9 - (2x - 3)^2 = -16$

$\Rightarrow 9 - (4x^2 - 12x + 9) = -16 \Rightarrow 9 - 4x^2 + 12x - 9 + 16 = 0$

$\Rightarrow 4x^2 - 12x - 16 = 0$

$\Rightarrow x^2 - 3x - 4 = 0 \Rightarrow x(x - 4) + (x - 4) = 0$

$\Rightarrow (x - 4)(x + 1) = 0$

$\Rightarrow x = 4$  and  $-1 \checkmark$

(e)  $h^{-1}(x) = -2 \Rightarrow h(-2) = x$

$\Rightarrow 3^{-2} = x \Rightarrow x = \frac{1}{3^2} = \frac{1}{9}$

$\therefore x = \frac{1}{9} \checkmark$



9(a)  $f(x) = 3x^2 + a$ , where  $a$  is an integer,  $f(-2) = 19$ .  
Find the value of  $a$ . ---[2]

(b)  $g(x) = 2x + 7$        $h(x) = 3x - 8$   
(i) Find  $gh(x)$  in its simplest form. ---[2]  
(ii) Find  $g^{-1}(x)$  ---[2]

W-20/21/Q17

Solution (a)  $f(x) = 3x^2 + a$   
 $\therefore f(-2) = 3(-2)^2 + a = 19$  (Given  $f(-2) = 19$ )  
 $\Rightarrow 12 + a = 19 \Rightarrow a = 7 \checkmark$

(b)  $g(x) = 2x + 7$  and  $h(x) = 3x - 8$

(i)  $gh(x) = g[h(x)] = g(3x - 8)$   
 $= 2(3x - 8) + 7$   
 $= 6x - 16 + 7 = 6x - 9 \checkmark$

(ii)  $g(x) = 2x + 7$  let  $y = 2x + 7$

Interchange  $x$  and  $y \Rightarrow x = 2y + 7$

$\Rightarrow y = \frac{x - 7}{2}$

$\Rightarrow g^{-1}(x) = \frac{x - 7}{2} \checkmark$

10.  $f(x) = 4 - 3x$ ;  $g(x) = x^2 + x$ ;  $h(x) = 3^x$

(a) Find  $fh(2)$  ---[2]      (c) Simplify, (i)  $f(1 - 2x)$  ---[2]

(b) Find  $f^{-1}(x)$  ---[2]      (ii)  $gf(x) - 9g(x)$  ---[4]

W-20/43/Q.10

(d)  $\frac{1}{k^x} = 9^{kx}$

Find the value of  $k$ . ---[2]

Solution (a)  $fh(2) = f[h(2)] = f[3^2] = f(9)$   
 $= 4 - 3 \times 9 = 4 - 27 = -23 \checkmark$

(b) Find  $f^{-1}(x)$ , let  $y = f(x) = 4 - 3x$

interchange  $x$  and  $y$

$\Rightarrow x = 4 - 3y$

$\Rightarrow y = \frac{4 - x}{3}$

$\therefore f^{-1}(x) = \frac{4 - x}{3} \checkmark$

(c)(i)  $gf(x) - 9g(x)$

$= g[4 - 3x] - 9(x^2 + x)$

$= (4 - 3x)^2 + (4 - 3x) - 9(x^2 + x)$

$= 16 + 9x^2 - 24x + 4 - 3x - 9x^2 - 9x$

$= 20 - 36x = 4(5 - 9x) \checkmark$

(ii)  $\frac{1}{k^x} = 9^{kx} \Rightarrow \frac{1}{3^x} = (3^2)^{kx}$

$= 3^{-x} = 3^{2kx}$

$\Rightarrow -x = 2kx$

$\Rightarrow 2kx + x = 0 \Rightarrow x[2k + 1] = 0$

$2k + 1 = 0 \quad [x \neq 0]$

$\Rightarrow k = -\frac{1}{2} \checkmark$

(c)(i)  $f(1 - 2x) = 4 - 3(1 - 2x)$  [ $\because f(x) = 4 - 3x$ ]

$= 4 - 3 + 6x =$

$= 1 + 6x \checkmark$



11  $f(x) = x^2 + 1$     $g(x) = 1 - 2x$     $h(x) = \frac{1}{x}, x \neq 0$     $j(x) = 5^x$

(a) Find the value of

(i)  $f(3)$    --- [1]

(ii)  $gf(3)$    --- [1]

(b) Find  $g^{-1}(x)$    --- [2]

(c) Find  $x$  when  $h(x) = 2$    --- [1]

(d) Find  $g(x) \cdot g(x) - gg(x)$ , giving your answer in the form  $ax^2 + bx + c$ .   --- [4]

(e) Find  $hh(x)$ , giving your answer in its simplest form.   --- [1]

(f) Find  $j(5)$    --- [1]

(g) Find  $x$  when  $j^{-1}(x) = 2$    --- [1]

(h)  $j(x) = hg(-12)$   
Find the value of  $x$ .   --- [2]

W-20/42/Q10

Solution (a) (i)  $f(3) = 3^2 + 1 = 10 \checkmark$

(a) (ii)  $gf(3) = g(f(3)) = g(10)$   
 $= 1 - 2 \times 10 = -19 \checkmark$

(b)  $g(x) = 1 - 2x = y$  (let)  
Interchanging  $x$  &  $y \Rightarrow x = 1 - 2y$   
 $\Rightarrow y = \frac{1-x}{2}$   
or  $g^{-1}(x) = \frac{1-x}{2} \checkmark$

(c)  $h(x) = 2 \Rightarrow \frac{1}{x} = 2 \Rightarrow x = \frac{1}{2} \checkmark$

(d)  $g(x) \cdot g(x) - gg(x)$   
 $= (1-2x)(1-2x) - g(1-2x)$   
 $= (1-2x)^2 - [1-2(1-2x)]$   
 $= 1 + 4x^2 - 4x - (1-2+4x)$   
 $= 4x^2 + 1 - 4x + 1 - 4x$   
 $= 4x^2 - 8x + 2 \checkmark$

(e)  $h[h(x)] = h\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}} = x \checkmark$

(f)  $j(5) = 5^5 = 3125 \checkmark$

(g)  $j^{-1}(x) = 2$   
 $\Rightarrow x = j(2)$   
 $= 5^2 = 25 \checkmark$

(h)  $j(x) = hg(-12)$   
 $\Rightarrow 5^x = h[g(-12)]$   
 $= h[1-2 \times (-12)]$   
 $5^x = h(25)$   
 $5^x = \frac{1}{25} \left( = \frac{1}{5^2} \right)$   
 $5^x = 5^{-2}$   
 $\Rightarrow x = -2 \checkmark$

# Derived Functions

1. A curve has equation  $y = x^3 - 6x^2 + 16$
- (a) Find the coordinates of the two turning points. ---[6]
- (b) Determine whether each of the turning points is a maximum or a minimum. Give reasons for your answers. ---[3]

SP-20/04/Q11

Solution:  $y = x^3 - 6x^2 + 16$  — (1)  $\left[ \because \frac{d}{dx} x^n = nx^{n-1} \right]$   
 $\frac{dy}{dx} = 3x^2 - 6 \times 2x + 0$   
 $\frac{dy}{dx} = 3x^2 - 12x$  — (2)  
or  $\frac{dy}{dx} = 3x(x-4)$

for any turning point  $\frac{dy}{dx} = 0$

from (2)  $3x(x-4) = 0$

$\Rightarrow x = 0$  or  $x = 4$

from (1)  $x = 0, y = 16, (0, 16) \checkmark$

$x = 4, y = 4^3 - 6 \times 4^2 + 16 = -16 \Rightarrow (4, -16) \checkmark$

$\therefore$  Turning points are  $(0, 16)$  and  $(4, -16)$   $\checkmark$

Now to check the max/min at turning points.

(i) at  $(0, 16)$  from (2) First derivative Test  
Take  $-1 < 0 \Rightarrow \left(\frac{dy}{dx}\right)_{x=-1} = 3(-1)(-1-4) = +15 > 0$

and  $1 > 0 \Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = 3 \times 1(1-4) = -9 < 0$

$\frac{dy}{dx}$  changes sign at  $(0, 16)$  from + to -ve  $\Rightarrow$  Max at  $(0, 16)$ .

(ii) at  $(4, -16)$  or at  $x = 4$  from (2)

Take  $3 < 4 \Rightarrow \left(\frac{dy}{dx}\right)_{x=3} = 3 \times 3 \times (3-4) = -9 < 0$

and  $5 > 4 \Rightarrow \left(\frac{dy}{dx}\right)_{x=5} = 3 \times 5 \times (5-4) = 15 > 0$

$\therefore \frac{dy}{dx}$  changes sign from -ve to + at  $(4, -16)$  has Minimum.

Alternatively:  
Second derivative test:

diff (2)  
 $\frac{d^2y}{dx^2} = 6x - 12$  — (3)

$\left(\frac{d^2y}{dx^2}\right)_{x=0} = 0 - 12 < 0$

Max at  $(0, 16) \checkmark$

Again: from (3)

$\left(\frac{d^2y}{dx^2}\right)_{x=4} = 6 \times 4 - 12 = 12 > 0$

$\therefore$  Min. at  $(4, -16)$

2. A curve has equation  $y = x^3 - 3x + 4$

(a) Work out the coordinates of the two stationary points. ---[5]

(b) Determine whether each stationary point is a maximum or a minimum, Give reason for your answers. ---[3]

[M-20/42/Q11]

Solution:  $y = x^3 - 3x + 4$  ——— (1)

( $\because \frac{d}{dx} x^n = n x^{n-1}$ )

$\frac{dy}{dx} = 3x^2 - 3$  ——— (2)

diff (2)  $\frac{d^2y}{dx^2} = 6x$  ——— (3) ( $\because \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right)$ )

(a) For stationary points  $\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 3 = 0$  form (2)

$3(x^2 - 1) = 0 \Rightarrow 3(x-1)(x+1) = 0$

$\Rightarrow x = 1$  or  $x = -1$

from (1)  $x = 1, y = 1 - 3 + 4 = 2 \Rightarrow (1, 2)$

$x = -1, y = -1 + 3 + 4 = 6 \Rightarrow (-1, 6)$

$\therefore$  Stationary points are  $(1, 2)$  and  $(-1, 6)$  ✓

(b) To Check Max/Min. Use second derivative test:

at  $(1, 2)$  from (3)

$\left( \frac{d^2y}{dx^2} \right)_{x=1} = 6 \times 1 = 6 > 0 \therefore$  Minimum at  $(1, 2)$  ✓

and at  $(-1, 6)$

$\left( \frac{d^2y}{dx^2} \right)_{x=-1} = 6 \times (-1) = -6 < 0 \therefore$  Max. at  $(-1, 6)$  ✓

3. A curve has the equation  $y = x^3 + 8x^2 + 5x$

(i) Work out the coordinates of the two turning points. --- [6]

(ii) Determine whether each of the turning points is a maximum or a minimum. Give reasons for your answers. --- [3]

[5-20/41/210(6)]

Solution:  $y = x^3 + 8x^2 + 5x$  — (1)

diff.  $\frac{dy}{dx} = 3x^2 + 16x + 5$  — (2)

diff (2)  $\frac{d^2y}{dx^2} = 6x + 16$  — (3)

(i) For any turning point  $\frac{dy}{dx} = 0$

from (1)  $\Rightarrow 3x^2 + 16x + 5 = 0$

$3x^2 + 15x + x + 5 = 0$

$3x(x+5) + 1(x+5) = 0$

$(x+5)(3x+1) = 0$

$\Rightarrow (x+5) = 0 \text{ or } 3x+1 = 0 \Rightarrow x = -5, x = -\frac{1}{3}$

from (1)  $x = -5, y = (-5)^3 + 8(-5)^2 + 5(-5) = 50, (-5, 50) \checkmark$

$x = -\frac{1}{3}, y = (-\frac{1}{3})^3 + 8(-\frac{1}{3})^2 + 5(-\frac{1}{3}) = -\frac{22}{27}, (-\frac{1}{3}, -\frac{22}{27})$

$\therefore$  Two turning points are  $(-5, 50)$  and  $(-\frac{1}{3}, -\frac{22}{27}) \checkmark$

(ii) To check the Max/Min.

at  $(-5, 50)$  from (3)  $(\frac{d^2y}{dx^2})_{x=-5} = 6(-5) + 16 = -14 < 0$  Max.  $\checkmark$

at  $(-\frac{1}{3}, -\frac{22}{27}) \rightarrow (\frac{d^2y}{dx^2})_{x=-\frac{1}{3}} = 6(-\frac{1}{3}) + 16 = 14 > 0$  Min  $\checkmark$

$\therefore (-5, 50)$  is Maximum.  $\checkmark$

$(-\frac{1}{3}, -\frac{22}{27})$  is Minimum  $\checkmark$

4 (a)  $y = x^4 - 4x^3$

(i) Find the value of  $y$  when  $x = -1$ . ---[2]

(ii) Find two stationary points on the graph of  $y = x^4 - 4x^3$  ---[6]

(b)  $y = x^p + 2x^q$

$\frac{dy}{dx} = 11x^{10} + 10x^4$ , where  $\frac{dy}{dx}$  is the derived function.

Find the value of  $p$  and the value of  $q$ . ---[2]

[S-20/42/Q10]

Solution:  $y = x^4 - 4x^3$  ——— (1)

(i) for  $x = -1$ ,  $y = (-1)^4 - 4(-1)^3 = 1 - 4(-1) = \underline{5}$  ✓

(ii) diff (1)

$\frac{dy}{dx} = 4x^3 - 12x^2$  ——— (2)

for any turning point  $\frac{dy}{dx} = 0$

∴ from (2)  $4x^3 - 12x^2 = 0$

$\Rightarrow 4x^2(x-3) = 0 \Rightarrow x = 0, x = 3$

from (1)  $x = 0, y = 0$  and  $x = 3, y = 3^4 - 4 \times 3^3 = 81 - 108 = -27$

∴ the two turning points are (0,0) and (3,-27) ✓

(b)  $y = x^p + 2x^q$  ——— (3)

diff (3)  $\frac{dy}{dx} = p x^{p-1} + 2q x^{q-1}$  ——— (4)

and given  $\frac{dy}{dx} = 11x^{10} + 10x^4$  ——— (5)

Comparing (4) and (5)  $\Rightarrow p = 11$  ✓

and  $q-1 = 4$  (or  $2q = 10$ )

$q = 5$  ✓

∴  $p = 11$  and  $q = 5$  ✓

5. A curve has equation  $y = 4x^3 - 3x + 3$

(i) Find the coordinates of the two stationary points. --- [5]

(ii) Determine whether each of the stationary points is a maximum or a minimum. Give reasons for your answers. --- [3]

[S-20/43/Q12(a)]

Solution:  $y = 4x^3 - 3x + 3$  ——— ①

diff ①  $\frac{dy}{dx} = 12x^2 - 3$  ——— ②

diff ②  $\frac{d^2y}{dx^2} = 24x$  ——— ③

(i) for any stationary point  $\frac{dy}{dx} = 0$

$$\text{from ②} \Rightarrow 12x^2 - 3 = 0$$

$$3(4x^2 - 1) = 0$$

$$3(2x-1)(2x+1) = 0$$

$$\Rightarrow x = \frac{1}{2} \text{ or } x = -\frac{1}{2}$$

from ①  $x = \frac{1}{2}$ ,  $y = 4\left(\frac{1}{2}\right)^3 - 3 \times \frac{1}{2} + 3 = 2$   $\therefore \left(\frac{1}{2}, 2\right) \checkmark$

and  $x = -\frac{1}{2}$ ,  $y = 4\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right) + 3 = 4$   $\therefore \left(-\frac{1}{2}, 4\right) \checkmark$

$\therefore$  turning points are:  $\left(\frac{1}{2}, 2\right)$  and  $\left(-\frac{1}{2}, 4\right) \checkmark$

(ii) To check Max/Min.

at  $\left(\frac{1}{2}, 2\right)$  from ③  $\left(\frac{d^2y}{dx^2}\right)_{x=\frac{1}{2}} = 24 \times \frac{1}{2} = 12 > 0 \therefore \text{Min}$

and at  $\left(-\frac{1}{2}, 4\right)$  from ③  $\left(\frac{d^2y}{dx^2}\right)_{x=-\frac{1}{2}} = 24 \times \left(-\frac{1}{2}\right) = -12 < 0 \text{ Max}$

$\therefore \left(\frac{1}{2}, 2\right)$  is a Minimum  $\checkmark$

and  $\left(-\frac{1}{2}, 4\right)$  is a Maximum  $\checkmark$

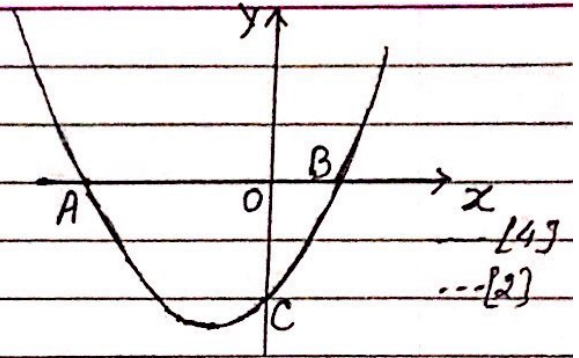


6. The diagram shows a sketch of the curve  $y = x^2 + 3x - 4$ .

(i) Find the coordinates of the points A, B and C.

(ii) Differentiate  $x^2 + 3x - 4$

(iii) Find the equation of the tangent to the curve at the point (2, 6).



[4]  
[2]  
[3]  
[W-20/41/Q 10(a)]

Solution (a) Curve:  $y = x^2 + 3x - 4$  — ①

Curve intersects x-axis at A, B, for  $y = 0$

from ①  $x^2 + 3x - 4 = 0$

$$x^2 + 4x - x - 4 = 0$$

$$x(x+4) - 1(x+4) = 0$$

$$\Rightarrow (x+4)(x-1) = 0 \Rightarrow x = -4, x = 1$$

$$\therefore A(-4, 0), B(1, 0) \checkmark$$

The curve intersects the y-axis at C for  $x = 0$

from ①  $y = 0 + 0 - 4 \Rightarrow$

$$\Rightarrow C(0, -4) \checkmark$$

(b)  $\frac{d}{dx}(x^2 + 3x - 4) = 2x + 3 \checkmark$

(c)  $\frac{dy}{dx} = 2x + 3$

$(\frac{dy}{dx})_{(2, 6)} = 2 \times 2 + 3 = 7$  Gradient of the tangent at (2, 6)

$\therefore$  Equation of tangent to the curve at (2, 6)

$$y - 6 = 7(x - 2)$$

$$\text{or } y = 7x - 8 \checkmark$$

7 (a) Differentiate  $6 + 4x - x^2$  --- [2]

(b) Find the coordinates of the turning point of the graph  $y = 6 + 4x - x^2$  --- [2]

[W-20/22/21]

Solution (a)  $\frac{d}{dx}(6 + 4x - x^2) = 4 - 2x \checkmark$

(b)  $y = 6 + 4x - x^2$  — ①

$\frac{dy}{dx} = 4 - 2x = 0$  for turning point

$\Rightarrow x = 2$ , from ①  $y = 10$

$\therefore$  Turning point (2, 10)





8. Calculate the gradient of  $y = 24 + 5x - x^2$  at  $x = -1.5$  ... [3]

W-20/43/Q7(a)(iii)

Solution:

$$y = 24 + 5x - x^2$$

$$\frac{dy}{dx} = 5 - 2x$$

$$\therefore \text{Gradient at } x = -1.5 \text{ is } \left. \frac{dy}{dx} \right|_{x=-1.5} = 5 - 2(-1.5) = \underline{8} \checkmark$$