classmate

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P-1

Pure Maths-1

Integration Notes

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Integration

Integration (as a reverse process of differentiation):

1. (i) $y = x^7 \Rightarrow \frac{dy}{dx} = 7x^6 \Rightarrow \int 7x^6\,dx = x^7$ ✓

if $y = \frac{1}{7}x^7 \Rightarrow \frac{dy}{dx} = x^6 \Rightarrow y = \int x^6\,dx = \frac{x^7}{7}$ ✓

(ii) $y = \frac{1}{6}x^6 \Rightarrow \frac{dy}{dx} = \frac{1}{6}\times 6x^5 = x^5$

we say $y = \int x^5\,dx = \frac{x^6}{6}$ ✓

(iii) Given $\frac{dy}{dx} = x^5$

$y = \int x^5\,dx = \frac{x^6}{6}$ ✓ $\left[\because \frac{d}{dx}\frac{1}{6}x^6 = x^5\right]$

(iv) $\frac{dy}{dx} = x \Rightarrow y = \int x\,dx = \frac{x^2}{2}$ ✓ $\left[\because \frac{d}{dx}\frac{1}{2}x^2 = x\right.$

(v) $\frac{dy}{dx} = 1 \Rightarrow y = \int 1\,dx = x$ ✓ $\left[\because \frac{d}{dx}x = 1.\right.$

2. (i) $y = x^3 + 7 \Rightarrow \frac{dy}{dx} = 3x^2 \Rightarrow \int 3x^2\,dx = x^3 + 7$

(ii) $y = x^3 - 5 \Rightarrow \frac{dy}{dx} = 3x^2 \Rightarrow \int 3x^2\,dx = x^3 - 5$?

(iii) $y = x^3 + 1 \Rightarrow \frac{dy}{dx} = 3x^2 \Rightarrow \int 3x^2\,dx = x^3 + 1$

Note: $\therefore$ In general $\int 3x^2\,dx = x^3 + c$ ✓

here c is called the constant of integration.

§ **Indefinite Integral:** Given $\frac{dy}{dx} = x^n$

Then $y = \int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

Example: (i) $\int x^8\,dx = \frac{x^{8+1}}{8+1} + c = \frac{x^9}{9} + c$ ✓

(ii) $\int 5\,dx = 5x + c \quad \left(\because \frac{d}{dx}5x = 5\right)$

§ Given $\boxed{f'(x) = x^n \Rightarrow f(x) = \frac{x^{n+1}}{n+1} + c}$

Example 1. Find $f(x)$ for the following:

(i) $f'(x) = 5x^4 \Rightarrow f(x) = 5.\int x^4 dx = 5 \cdot \frac{x^{4+1}}{4+1} + c$

$= 5\frac{x^5}{5} + c = \underline{x^5 + c}$ ✓

(ii) $f'(x) = 7x^5 + 9x^4 - 5$

$\Rightarrow f(x) = \int (7x^5 - 9x^4 + 5)\, dx$

$= 7 \cdot \frac{x^6}{6} - 9\frac{x^5}{5} + 5x + c$ ✓

(iii) $f'(x) = \sqrt{x}$

$f(x) = \int x^{\frac{1}{2}} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{x^{3/2}}{3/2} + c$

$= \underline{\frac{2}{3} x^{3/2} + c}$ ✓

Example 2: Find y.

(i) $\frac{dy}{dx} = x^3 + \sqrt{x} \Rightarrow y = \int (x^3 + x^{\frac{1}{2}})\, dx$

$= \frac{x^4}{4} + \frac{x^{3/2}}{3/2} + c = \underline{\frac{1}{4}x^4 + \frac{2}{3}x^{3/2} + c}$ ✓

(ii) $\frac{dy}{dx} = x^2(2x+3)$

$y = \int x^2(2x+3)\, dx = \int (2x^3 + 3x^2)\, dx$

$= 2 \times \frac{x^4}{4} + 3 \cdot \frac{x^3}{3} + c$

$= \underline{\frac{1}{2}x^4 + x^3 + c}$ ✓

(iii) $\frac{dy}{dx} = \frac{x^4 - 2x + 5}{2x^3}$

$\Rightarrow y = \int \left(\frac{x^4}{2x^3} - \frac{2x}{2x^3} + \frac{5}{2x^3} \right) dx$

$= \int \left(\frac{1}{2}x - x^{-2} + \frac{5}{2}x^{-3} \right) dx$

$= \frac{1}{2} \cdot \frac{x^2}{2} - \frac{x^{-1}}{-1} + \frac{5}{2} \cdot \frac{x^{-2}}{-2} + c$

$= \underline{\frac{x^2}{4} + \frac{1}{x} - \frac{5}{4x^2} + c}$ ✓

Example 3. Find,

(i) $\int \frac{7}{x\sqrt{x}}\,dx = \int \frac{7}{x^{3/2}}\,dx = 7\cdot\int x^{-\frac{3}{2}}\,dx = 7\times \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + C$

$= 7\times(-2)\times x^{-\frac{1}{2}} + C$

$= -\frac{14}{\sqrt{x}} + C$ ✓

(ii) $\int \frac{(1-\sqrt{x})^2}{\sqrt{x}}\,dx = \int\left(\frac{1-2\sqrt{x}+x}{\sqrt{x}}\right)dx = \int\left(\frac{1-2x^{1/2}+x}{x^{1/2}}\right)dx$

$= \int\left(x^{-\frac{1}{2}} - 2 + x^{1/2}\right)dx$

$= \frac{x^{1/2}}{\frac{1}{2}} - 2x + \frac{x^{3/2}}{\frac{3}{2}} + C = 2\sqrt{x} - 2x + \frac{2}{3}x^{3/2} + C$

§ Finding the constant of Integration:

Example 4. A curve is such that $\frac{dy}{dx} = \frac{6}{x^2}$ and $(2,9)$ is a point on the curve. Find the equation of the curve. ---[3]

[S-13/12/Q1]

Solution: $\frac{dy}{dx} = \frac{6}{x^2} \Rightarrow y = \int 6x^{-2}\,dx$

$= 6\cdot\frac{x^{-2+1}}{-2+1} + C = -6x^{-1} + C$

$\Rightarrow y = -\frac{6}{x} + C$ ——— (1)

Passes through $(2,9) \Rightarrow 9 = -\frac{6}{2} + C \Rightarrow C = 12$

$\therefore$ The equation of the curve is $y = -\frac{6}{x} + 12$ ✓

Example 5: A curve has equation $y = f(x)$.
It is given that $f'(x) = x^{-3/2} + 1$ and that $f(4) = 5$.
Find $f(x)$. [W-13/13/Q2] ---[4]

Solution: $f'(x) = x^{-3/2} + 1$

$\therefore f(x) = \int (x^{-3/2} + 1)\,dx$

$= \frac{x^{-1/2}}{-\frac{1}{2}} + x + C$

$f(x) = -\frac{2}{\sqrt{x}} + x + C$ ——— (1)

Now given $f(4) = 5$

from (1) $f(4) = -\frac{2}{\sqrt{4}} + 4 + C = 5$

$\Rightarrow -1 + 4 + C = 5 \Rightarrow C = 2$ ✓

$\therefore$ from (1) The equation the curve.

$f(x) = -\frac{2}{\sqrt{x}} + x + 2$ ✓

§ $$\boxed{\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a\cdot(n+1)} + c}$$

as $\frac{d}{dx}(ax+b)^{n+1} = (n+1)(ax+b)^n \times a$

$$\Rightarrow \frac{d}{dx}\frac{1}{a}\frac{(ax+b)^{n+1}}{(n+1)} = \frac{1}{a}\frac{(n+1)(ax+b)^n \times a}{(n+1)} = (ax+b)^n$$

$$\Rightarrow \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$ ✓

Example 6 Find,

(i) $\int (5x-8)^7 dx = \frac{(5x-8)^{7+1}}{5\times(7+1)} + c = \frac{(5x-8)^8}{40} + c$ ✓

(ii) $\int (2-3x)^5 dx = \frac{(2-3x)^{5+1}}{-3\times(5+1)} = \frac{(2-3x)^6}{-18} + c = -\frac{(2-3x)^6}{18} + c$ ✓

(iii) $\int \frac{7}{(2x+3)^5} dx = \int 7(2x+3)^{-5} dx = 7\times\frac{(2x+3)^{-5+1}}{2\times(-5+1)} + c$

$$= \frac{7(2x+3)^{-4}}{-8} + c = \frac{-7}{8(2x+3)^4} + c$$ ✓

Example 7: A curve is such that $\frac{dy}{dx} = \frac{8}{(5-2x)^2}$, Given that the curve passes through $(2,7)$. Find equation of the curve. ---[4] $\boxed{S\text{-}16 / 12 / Q2}$

Solution: $\frac{dy}{dx} = \frac{8}{(5-2x)^2} \Rightarrow y = 8.\int (5-2x)^{-2} dx$

$$= \frac{8\times(5-2x)^{-2+1}}{(-2)\times(-2+1)} = \frac{8(5-2x)^{-1}}{2} + c$$

$$\therefore y = \frac{4}{(5-2x)} + c \quad \text{—— (1)}$$

curve passes through $(2,7)$

from (1) $7 = \frac{4}{(5-2\times2)} + c \Rightarrow c = 7-4 = 3$

$\therefore$ from (1) Eqn of curve: $y = \frac{4}{(5-2x)} + 3$ ✓

Example 8. A curve is such that $\frac{dy}{dx} = 6x^2 + \frac{k}{x^3}$ and passes through the point $P(1,9)$. The gradient of the curve at P is 2.

(i) Find the value of k. --- [1]

(ii) Find the equation of the curve. | S-16 | 13 | Q3 | --- [4]

Solution:

(i) $\frac{dy}{dx} = 6x^2 + \frac{k}{x^3}$ --- ①

Gradient at $P(1,9) \Rightarrow \left(\frac{dy}{dx}\right)_{(1,9)} = 6\times1^2 + \frac{k}{1^3} = 2$ (Given)

$\Rightarrow \quad k = -4$ ✓ --- ②

(ii) $\frac{dy}{dx} = 6x^2 - \frac{4}{x^3}$ ($k = -4$ from ②)

$\Rightarrow \quad y = \int (6x^2 - 4x^{-3})\,dx = \frac{6x^3}{3} - 4\frac{x^{-2}}{-2} + c$

or $\quad y = 2x^3 + \frac{2}{x^2} + c$ --- ③

Given the curve ③ passes through $P(1,9)$

$\Rightarrow \quad 9 = 2\times1^3 + \frac{2}{1^2} + c \Rightarrow c = 5$

$\therefore$ from ③ the equation of the curve: $y = 2x^3 + \frac{2}{x^2} + 5$ ✓

Example 9. A curve has equation $y = f(x)$.
It is given that $f'(x) = \frac{1}{\sqrt{x+6}} + \frac{6}{x^2}$ and that $f(3) = 1$.
Find $f(x)$. --- [5]

| W-13 | 11 | Q2 |

Solution: $f'(x) = \frac{1}{\sqrt{x+6}} + \frac{6}{x^2}$

$f(x) = \int \left((x+6)^{-\frac{1}{2}} + 6x^{-2} \right) dx$

$= \frac{(x+6)^{1/2}}{\frac{1}{2}} + 6\frac{x^{-1}}{-1} + c$

$f(x) = 2\sqrt{x+6} - \frac{6}{x} + c$ — ①

$f(3) = 1$

$\Rightarrow \quad 1 = 2\sqrt{3+6} - \frac{6}{3} + c \Rightarrow c = -3$

Hence from ① Equation of the curve.

$y = 2\sqrt{x+6} - \frac{6}{x} - 3$ ✓

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Example 10: A curve is such that $\frac{dy}{dx} = \frac{2}{a}x^{-\frac{1}{2}} + ax^{-3/2}$, where a is a positive constant.
The point $A(a^2, 3)$ lies on the curve.
Find in term of a.
(i) the equation of the tangent to the curve at A, Simplify your answer. ---[3]
(ii) the equation of the curve. --[4]
(iii) It is given that $B(16, 8)$ also lies on the curve, find the value of a and using this value, find the distance AB. --[5]

| W-16 | 13 | Q10 |

Solution: $\frac{dy}{dx} = \frac{2}{a}x^{-\frac{1}{2}} + ax^{-3/2}$

(i)

Gradient at A, $\left(\frac{dy}{dx}\right)_{(a^2,3)} = \frac{2}{a}(a^2)^{-\frac{1}{2}} + a(a^2)^{-3/2}$

$= \frac{2}{a^2} + \frac{1}{a^2} = \frac{3}{a^2}$

$\therefore$ Equation of tangent at $A(a^2, 3)$

$y - 3 = \frac{3}{a^2}(x - a^2)$

$y - 3 = \frac{3x}{a^2} - 3$

or $y = \frac{3}{a^2}x$ ✓

(ii) $\frac{dy}{dx} = \frac{2}{a}x^{-\frac{1}{2}} + ax^{-3/2}$

Hence equation of the curve is

$y = \int\left(\frac{2}{a}x^{-\frac{1}{2}} + ax^{-3/2}\right)dx$

$= \frac{2}{a} \times \frac{x^{1/2}}{1/2} + a\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c$

or $y = \frac{4}{a}x^{1/2} - 2ax^{-1/2} + c$ --(1)

Given that the curve passes through $A(a^2, 3) \Rightarrow 3 = \frac{4}{a}(a^2)^{\frac{1}{2}} - 2a(a^2)^{-\frac{1}{2}} + c$

$\Rightarrow 3 = 4 - 2 + c \Rightarrow c = 1$

Hence from (1) Eqn of the curve is $y = \frac{4}{a}x^{\frac{1}{2}} - 2ax^{-\frac{1}{2}} + 1$ --(2) ✓

(iii) $B(16, 8)$ lies on the curve (2)

$8 = \frac{4}{a}\sqrt{16} - 2a(16)^{-\frac{1}{2}} + 1$

$\Rightarrow 8 = \frac{16}{a} - \frac{a}{2} + 1$

$\Rightarrow$

$a^2 + 14a - 32 = 0$

$(a+16)(a-2) = 0$

$a = 2$, $a = -16$ ✗ ($\because a > 0$ Given)

$\therefore A(a^2, 3) = (2^2, 3) = (4, 3)$

and $B(16, 8)$

$\therefore AB = \sqrt{(16-4)^2 + (8-3)^2}$

$= \sqrt{144 + 25}$

$\therefore AB = 13$ ✓

Example 11. The curve $y = f(x)$ has a stationary point at $(2, 10)$ and it is given that $f''(x) = \frac{12}{x^3}$

(i) Find $f(x)$ --- [6]

(ii) Find the coordinates of the other stationary point. --- [2]

(iii) Find the nature of each of the stationary point. -- [2]

|W-15|12|Q9|

Solution: $f''(x) = \frac{12}{x^3}$ --- ①

(i) $f'(x) = \int 12x^{-3}\,dx$

$= 12 \times \frac{x^{-2}}{-2} + C$

or $f'(x) = -\frac{6}{x^2} + C$ --- ②

Curve has a stationary point at $(2, 10) \Rightarrow f'(2) = 0$

from ② $f'(2) = -\frac{6}{2^2} + C = 0 \Rightarrow C = \frac{3}{2}$

$\therefore$ from ② $f'(x) = -\frac{6}{x^2} + \frac{3}{2}$ --- ③

Now $f(x) = \int \left(-\frac{6}{x^2} + \frac{3}{2}\right) dx$

$= -6\frac{x^{-2+1}}{(-2+1)} + \frac{3}{2}x + k$

or $f(x) = \frac{6}{x} + \frac{3}{2}x + k$ --- ④

Curve passes through $(2, 10)$

from ④ $\Rightarrow 10 = \frac{6}{2} + \frac{3}{2} \times 2 + k \Rightarrow k = k = 4$

from ④

$f(x) = \frac{6}{x} + \frac{3}{2}x + 4$ --- ⑤ ✓

(ii) For any stationary point $f'(x) = 0$

from ③ $f'(x) = -\frac{6}{x^2} + \frac{3}{2} = 0$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

$\therefore$ other stationary point $(-2, -2)$

$\left[\because \text{from ⑤ } x = -2,\ f(x) = \frac{6}{-2} + \frac{3}{2} \times (-2) + 4 = -2 \text{ or } y = -2\right]$

(iii) To check the nature of stationary points. from ①

at $(2, 10)$, $f''(2) = \frac{12}{2^3} > 0$ Min

$\therefore$ Min at $(2, 10)$ ✓

again at $(-2, -2)$ from ①

$f''(-2) = \frac{12}{(-2)^3} = \frac{12}{-8} < 0$

$\therefore$ Max. at $(-2, -2)$ ✓

12. A curve is such that $\frac{dy}{dx} = kx^2 - 12x + 5$, where k is a constant. Given that the curve passes through the points $(1, -3)$ and $(3, 11)$. Find the equation of the curve.

Solution: $\frac{dy}{dx} = kx^2 - 12x + 5$

$y = \int (kx^2 - 12x + 5)\,dx$

or $y = k\frac{x^3}{3} - 12\frac{x^2}{2} + 5x + C$

or $y = \frac{k}{3}x^3 - 6x^2 + 5x + C$ --- ①

Curve passes through the point $(1, -3)$

from ① $-3 = \frac{k}{3} - 6 + 5 + C$

$\Rightarrow \frac{k}{3} + C = -2$ ——— ②

$(3, 11)$ also lies on the curve ①

$\therefore 11 = \frac{k}{3} \times 27 - 6 \times 9 + 15 + C$

$\Rightarrow 9k + C = 50$ ——— ③

Solving ② and ③

$C = -4$ and $k = 6$

Hence from ① The equation of the curve is:

$y = \frac{6}{3}x^3 - 6x^2 + 5x - 4$

or $y = 2x^3 - x^2 + 5x - 4$ ✓

13. A curve is such that $\frac{dy}{dx} = kx + 3$, where k is a constant. The gradient of the normal to the curve at the point $(1, -2)$ is $-\frac{1}{7}$. Find the equation of the curve.

Solution: $\frac{dy}{dx} = kx + 3$ ——— ①

gradient of the tangent to the curve at $(1, -2)$, $\left(\frac{dy}{dx}\right)_{x=1} = k + 3$ ——— ②

Now as the gradient of the normal is given $= -\frac{1}{7}$

$\Rightarrow$ Gradient of tangent $= 7$ ——— ③

from ② & ③ $k + 3 = 7$

$\Rightarrow k = 4$ ✓

$\therefore$ from ①

$\frac{dy}{dx} = 4x + 3$ ——— ②

$y = \int (4x + 3)\,dx$

$y = 4 \times \frac{x^2}{2} + 3x + C$

or $y = 2x^2 + 3x + C$ --- ③

Curve passes through a point $(1, -2)$

from ②

$-2 = 2 \times 1^2 + 3 \times 1 + C$

$\Rightarrow C = -7$

from ③ the equation of the curve is

$y = 2x^2 + 3x - 7$ ✓

§ Integration as reverse process of differentiation:

(For Some Complicated functions)

If $\frac{d}{dx}(g(x)) = f(x)$

Then $\int f(x)\,dx = g(x) + C$

Example 14 (i) Show that $\frac{d}{dx}(5x^2-7)^6 = 60x(5x^2-7)^5$ (Chain rule)

(ii) Hence find $\int 2x(5x^2-7)^5\,dx$

Solution (i) $\frac{d}{dx}(5x^2-7)^6 = 6(5x^2-7)^5 \times 10x$

$= 60x(5x^2-7)^5$ ------ ①

(ii) $\int 2x(5x^2-7)^5\,dx$

$= \frac{1}{30}\int 60x(5x^2-7)^5\,dx$

$= \frac{1}{30}(5x^2-7)^6 + C$ ✓ (from ①)

Example 15 (i) Differentiate $\frac{1}{4x^2-3x+1}$ with respect to x.

(ii) Hence find $\int \frac{16x-6}{(4x^2-3x+1)^2}\,dx$

Solution (i) $\frac{d}{dx}\left(\frac{1}{4x^2-3x+1}\right) = \frac{d}{dx}(4x^2-3x+1)^{-1}$

$= -(4x^2-3x+1)^{-2} \times (8x-3)$

$= \frac{-(8x-3)}{(4x^2-3x+1)^2}$ ---- ①

(ii) $\int \frac{16x-6}{(4x^2-3x+1)^2}\,dx$

$= \int \frac{2(8x-3)}{(4x^2-3x+1)^2}\,dx = -2\int \frac{-(8x-3)}{(4x^2-3x+1)^2}\,dx$ (from ①)

$= -2 \times \frac{1}{(4x^2-3x+1)} + C$

$= \frac{-2}{(4x^2-3x+1)} + C$ ✓

§ Definite Integral:

Consider $\int x^2\,dx = \frac{1}{3}x^3 + c$ is indefinite integral of x^2

Now if we integrate x^2 between the limits $x=3$ and $x=4$

$$\int_3^4 x^2\,dx = \left[\frac{1}{3}x^3 + c\right]_3^4$$
$$= \left(\frac{1}{3}\times 4^3 + c\right) - \left(\frac{1}{3}\times 3^3 + c\right)$$
$$= \frac{1}{3}(64-27) \quad (\text{here "c" cancells})$$
$$= \frac{37}{3}$$

$$\therefore \int_3^4 x^2\,dx = \left[\frac{1}{3}x^3\right]_3^4 = \left(\frac{1}{3}\times 4^3\right) - \left(\frac{1}{3}\times 3^3\right) = \frac{1}{3}\left(4^3 - 3^3\right)$$
$$= \frac{1}{3}\times(64-27)$$
$$= \frac{37}{3} \quad \left(\text{or } 12\tfrac{1}{3}\right)$$

$$\boxed{\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)}$$

Example 16. Find $\int \frac{2}{\sqrt{(5x-6)}}\,dx$ and hence evaluate $\int_2^3 \frac{2}{\sqrt{(5x-6)}}\,dx$

[W-13 | 12 | Q 3]

Solution: $\int \frac{2}{\sqrt{(5x-6)}}\,dx = \int 2\,(5x-6)^{-\frac{1}{2}}\,dx$

$$= 2\times\frac{(5x-6)^{1/2}}{5\times 1/2} = \frac{4}{5}\sqrt{(5x-6)}$$

hence $\int_2^3 \frac{2}{\sqrt{(5x-6)}}\,dx = \left[\frac{4}{5}\sqrt{(5x-6)}\right]_2^3$

$$= \frac{4}{5}\left[\sqrt{5\times3-6} - \sqrt{5\times2-6}\right]$$
$$= \frac{4}{5}\left[\sqrt{9} - \sqrt{4}\right]$$
$$= \frac{4}{5}(3-2) = \frac{4}{5}$$

§ Improper integrals:

1. $\int_1^{\infty} \frac{1}{x^2}\,dx$

2. $\int_{-\infty}^{3} \frac{1}{x^2}\,dx$

In these definite integral we replace ∞ by p then then take the limit $p \to \infty$ (or $p \to -\infty$)

1. $\int_1^{\infty} \frac{1}{x^2}\,dx = \int_1^{p} x^{-2}\,dx$

$= \left[\frac{x^{-1}}{-1}\right]_1^p = \left[-\frac{1}{x}\right]_1^p$

$= \left(-\frac{1}{p}\right) - \left(-\frac{1}{1}\right)$

$= 1 - \frac{1}{p}$ $\quad$ [Now as $p \to \infty$, $\frac{1}{p} \to 0$

$= 1 - 0 = \underline{1}$ ✓

2. $\int_{-\infty}^{-3} \frac{1}{x^2}\,dx = \int_p^{-3} x^{-2}\,dx$

$= \left[\frac{x^{-1}}{-1}\right]_p^{-3} = \left[-\frac{1}{x}\right]_p^{-3}$

$= \left(-\frac{1}{-3}\right) - \left(-\frac{1}{p}\right)$ $\quad$ [as $p \to -\infty$, $\frac{1}{p} \to 0$

$= +\frac{1}{3} + 0$

$= \underline{\frac{1}{3}}$ ✓

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3. $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ [here $\frac{1}{\sqrt{x}}$ is not defined at $x = 0$]

$= \int_a^1 x^{-\frac{1}{2}}\,dx = \left[\frac{x^{1/2}}{1/2}\right]_a^1 = \left[2\sqrt{x}\right]_a^1 =$

$= 2\sqrt{1} - 2\sqrt{a}$ $\quad$ [$a \to 0$, $2\sqrt{a} \to 0$

$= 2 - 0$

$= \underline{2}$ ✓

Example 17. Evaluate, if it exists. $\int_0^2 \frac{1}{x^2} dx$

Solution: $\int_0^2 \frac{1}{x^2} dx$ [$\frac{1}{x^2}$ is not defined at $x=0$]

$$= \int_a^2 x^{-2} dx = \left[\frac{x^{-1}}{-1}\right]_a^2 = \left[-\frac{1}{x}\right]_a^2$$

$$= \left(-\frac{1}{2}\right) - \left(-\frac{1}{a}\right)$$

$$= \frac{1}{a} - \frac{1}{2}$$ [as $a \to 0$, $\frac{1}{a} \to \infty$ ✓

$\therefore \int_0^2 \frac{1}{x^2} dx$ does not exists.

Example 18: Find $\int_2^5 \frac{1}{\sqrt{x-2}} dx$

Solution:

$$\int_2^5 \frac{1}{\sqrt{x-2}} dx = \int_t^5 (x-2)^{-\frac{1}{2}}$$ [as the function $\frac{1}{\sqrt{x-2}}$ is not defined at $x=2$

$$= \left[\frac{(x-2)^{1/2}}{1/2}\right]_t^5$$

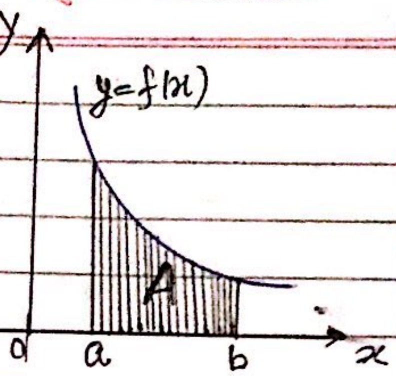
$$= \left[2\sqrt{x-2}\right]_t^5$$

$$= \left(2\sqrt{5-2}\right) - \left(2\sqrt{t-2}\right)$$

$$= 2\sqrt{3} - 0$$ [as $t \to 2$, $t-2 \to 0$, $2\sqrt{t-2} \to 0$]

$$= 2\sqrt{3}$$ ✓

§ Area under a Curve:



Given a curve $y=f(x)$

Area under a curve $y=f(x)$, above X-axis and between the ordinates $x=a$ and $x=b$ (i,e, between the lines parallel to Y-axis)

$$A=\int_a^b y\,dx$$

$$=\int_a^b f(x)\,dx \qquad [\,y=f(x)$$

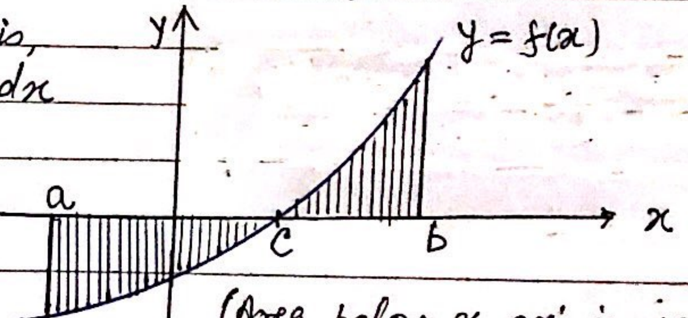
Note; If the equation of curve is given as an implicit function then y in terms of x.

Note: If the area is below X-axis (Partially or wholly).

Let the curve intersects X-axis at $x=c$.

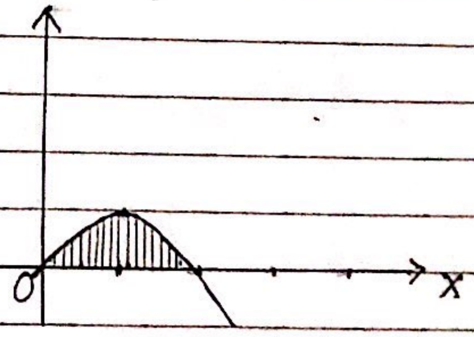
Then the shaded area is,

$$A=\int_a^b y\,dx=\left|\int_a^c y\,dx\right|+\int_c^b y\,dx$$



(Area below x-axis is −ve)

Example 19: Find the area of the region bounded by $y=2x-x^2$ and x-axis.



Solution: $y=2x-x^2$ --- ①

or $y=x(2-x)$

For the intersection of curve ① and X-axis $x(2-x)=0$

$\Rightarrow x=0,\ x=2$

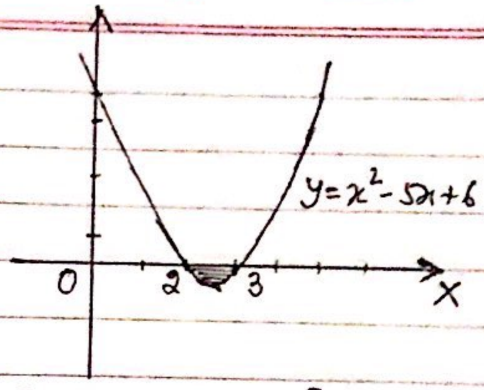
$\therefore$ Required Area $=\int_0^2 y\,dx$

$$=\int_0^2 (2x-x^2)\,dx$$

$$=\left[\cancel{2}\cdot\frac{x^2}{\cancel{2}}-\frac{x^3}{3}\right]_0^2=\left(4-\frac{8}{3}\right)-(0-0)$$

$$=\frac{4}{3}\text{ sq units (or } 1\tfrac{1}{3})$$

Example 20. Find the area of the region bounded by $y = x^2 - 5x + 6$ and x-axis.



Solution: $y = x^2 - 5x + 6$ --- ①

curve ① intersects X-axis at

$$x^2 - 5x + 6 = 0$$

$$(x-2)(x-3) = 0 \Rightarrow x = 2,\ x = 3$$

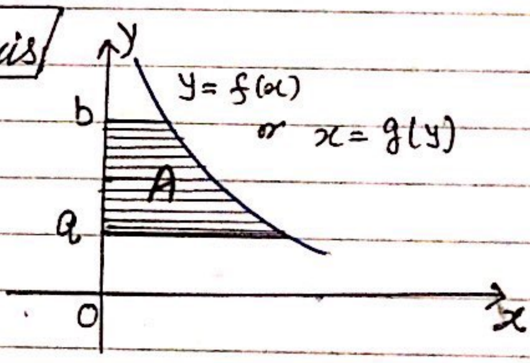
$$\therefore A = \int_2^3 y\,dx = \int_2^3 (x^2 - 5x + 6)\,dx = \left[\frac{x^3}{3} - 5\cdot\frac{x^2}{2} + 6x\right]_2^3$$

$$= \left(9 - \frac{45}{2} + 18\right) - \left(\frac{8}{3} - 10 + 12\right) = -\frac{1}{6}$$

$$\therefore A = \left|-\frac{1}{6}\right| = \frac{1}{6}$$ ✓ (Area below X-axis)

---

§ $\boxed{\text{Area enclosed by a curve and Y-axis}}$



Given the equation of a curve,

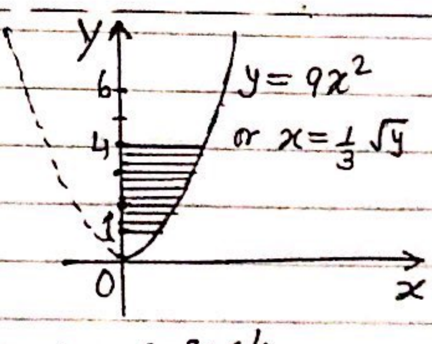
$y = f(x) \longrightarrow$ Express $x = g(y)$

Area between the curve $x = g(y)$, Y-axis and the lines $y = a$ and $y = b$ (lines parallel to x-axis)

$$A = \int_a^b x\,dy$$

$$= \int_a^b g(y)\,dy.$$

Example 21. Given equation of a curve $y = 9x^2$. Find the area bounded by Y-axis and line $y = 1$ and $y = 4$ and the curve in the first quadrant.



Solution: Given curve $y = 9x^2 \Rightarrow x = \frac{1}{3}\sqrt{y}$ ...①

$$A = \int_1^4 x\,dy = \int_1^4 \frac{1}{3}\sqrt{y}\,dy$$

$$= \frac{1}{3}\int_1^4 y^{1/2}\,dy$$

$$= \frac{1}{3}\left[\frac{y^{3/2}}{3/2}\right]_1^4$$

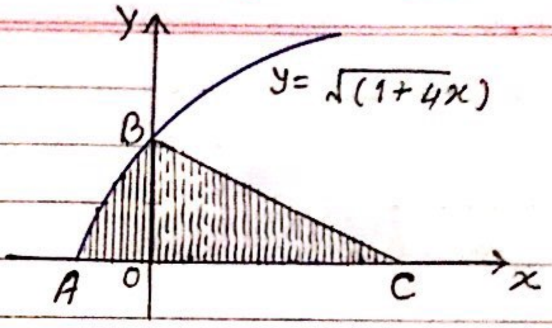
$$A = \frac{1}{3}\times\frac{2}{3}\left[y^{3/2}\right]_1^4$$

$$= \frac{2}{9}\left[4^{3/2} - 1^{3/2}\right]$$

$$= \frac{2}{9}(8-1)$$

$$= \frac{14}{9} \text{ sq unit } \left(\text{or } 1\frac{5}{14}\right)$$

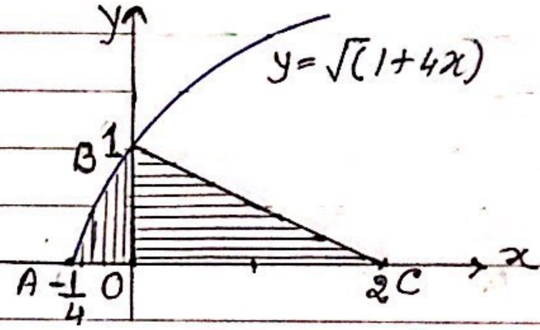
Example 22: The diagram shows the curve $y = \sqrt{(1+4x)}$, which intersects the x-axis at A and Y-axis at B. The normal to the curve at B meets the x-axis at C.



(i) Find equation of BC. ---[5]

(ii) Find the area of the shaded region. [S-13/13/Q1] ---[5]

Solution: Curve: $y = \sqrt{(1+4x)}$ ---- (1)



(i) curve intersects X-axis at A, $y=0$

from (1) $0 = \sqrt{1+4x} \Rightarrow x = -\frac{1}{4}$

$A(-\frac{1}{4}, 0)$ ✓

At the point B, $x=0$ from (1)

$y = \sqrt{1+0} = 1$, $B(0,1)$ ✓

diff (1)

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+4x}} \times 4 \quad \text{--- (2)}$$

Gradient of the tangent at $B(0,1)$

from (2) $\left(\frac{dy}{dx}\right)_{(0,1)} = \frac{4}{2\sqrt{1+0}} = 2$ ✓

$\therefore$ Gradient of the normal at $B = -\frac{1}{2}$

$\therefore$ Equation of normal to the curve at $B(0,1)$, $y - 1 = -\frac{1}{2}(x-0)$

Normal $BC \Rightarrow y = -\frac{1}{2}$ —— (3)

(ii) Normal (3) intersects X-axis at C,

for $y=0$, from (3) $0 = -\frac{1}{2}x + 1 \Rightarrow x = 2$

$C(2,0)$

(ii)

Shaded Area = Area under the curve and x-axis from A and O. + Area of ΔBOC —(4)

Area of $\Delta BOC = \frac{1}{2} \times OB \times OC$

$= \frac{1}{2} \times 1 \times 2$

$A_1 = 1$ sq unit —(5) ✓

Area under curve and X-axis, $x = -\frac{1}{4}$, to $x = 0$

$$A_2 = \int_{-\frac{1}{4}}^{0} y\,dx$$

$$= \int_{-\frac{1}{4}}^{0} (1+4x)^{1/2}\,dx$$

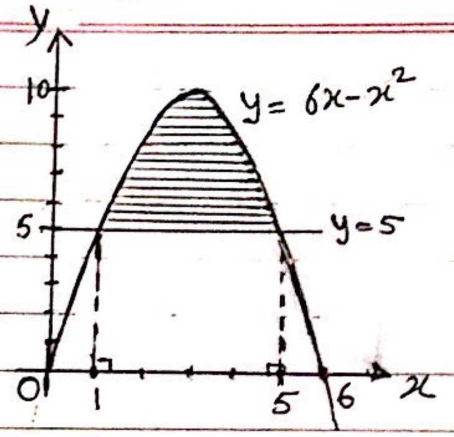
$$= \left[\frac{(1+4x)^{3/2}}{\frac{3}{2} \times 4}\right]_{-\frac{1}{4}}^{0}$$

$A_2 = \frac{1}{6}[(1) - (0)] = \frac{1}{6}$ ✓ —(6)

$\therefore$ Area $= A_1 + A_2 = 1 + \frac{1}{6} = \frac{7}{6}$ sq units

Example 23. The diagram shows the curve, $y = 6x - x^2$ and the line $y = 5$. Find the area of the shaded region. ----[6]

[S-10 | 11 | Q4]



Solution: Curve: $y = 6x - x^2$ ---- ①

line: $y = 5$ --- ②

for the intersection of line and curve from ① and ② $6x - x^2 = 5 \Rightarrow x^2 - 6x + 5 = 0$

$(x-1)(x-5) = 0 \Rightarrow x = 1, x = 5$

Shaded area = Area under the curve and above X-axis for $x=1$ to $x=5$ − Area of the rectangle — ③

A_1 = Area under the curve from $x=1$ to $x=5$ is $\int_1^5 y\,dx$

from ① $= \int_1^5 (6x - x^2)\,dx$

$= \left[3x^2 - \frac{x^3}{3}\right]_1^5$

$= \left(75 - \frac{125}{3}\right) - \left(3 - \frac{1}{3}\right)$

$A_1 = 72 - \frac{124}{3} = \frac{92}{3}$ ✓

A_2 = Area of rectangle

$= 5 \times (5-1) = 20$ — ⑤

∴ Area of the shaded region

$A = A_1 - A_2 =$

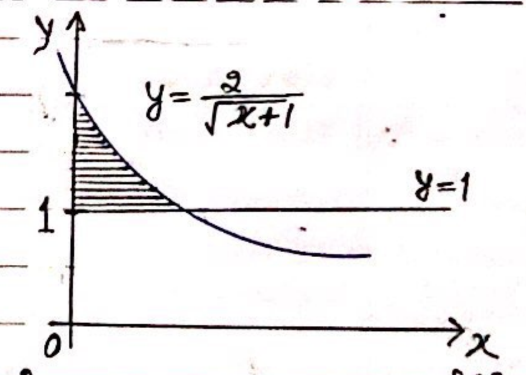
$= \frac{92}{3} - 20 = \frac{32}{3} = 10\frac{2}{3}$ sq units ✓

Example 24: The diagram shows the line $y = 1$ and part of the curve $y = \frac{2}{\sqrt{x+1}}$.

Find the area of the shaded region.

[S-12 | 11 | Q11]

---[6]



Solution: Curve: $y = \frac{2}{\sqrt{(x+1)}}$ .. ① $\Rightarrow \sqrt{x+1} = \frac{2}{y}$

$\Rightarrow x + 1 = \frac{4}{y^2}$

$\Rightarrow x = \frac{4}{y^2} - 1$ --- ②

curve intersects Y-axis at $x=0$

from ① $y = \frac{2}{\sqrt{1}} = 2$ ✓

∴ Shaded area is bounded by the curve, Y-axis and lines $y=1$ & $y=2$

$A = \int_1^2 x\,dy = \int_1^2 \left(\frac{4}{y^2} - 1\right) dy$

$A = \int_1^2 (4y^{-2} - 1)\,dy$

$= \left[\frac{4y^{-1}}{-1} - y\right]_1^2$

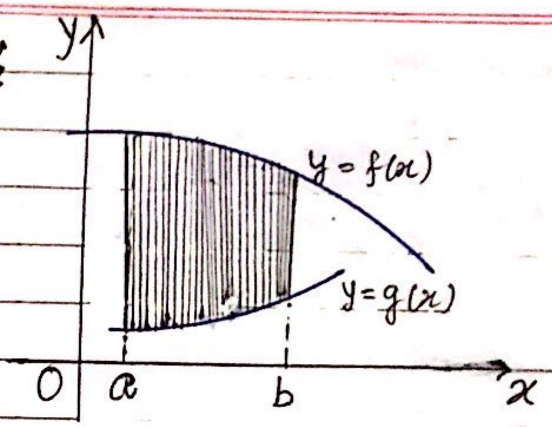
$= \left[-\frac{4}{y} - y\right]_1^2$

$= \left(-\frac{4}{2} - 2\right) - \left(-\frac{4}{1} - 1\right)$

$= -4 + 5$

$A = 1$ Sq. Unit ✓

§ Area bounded by two curves:

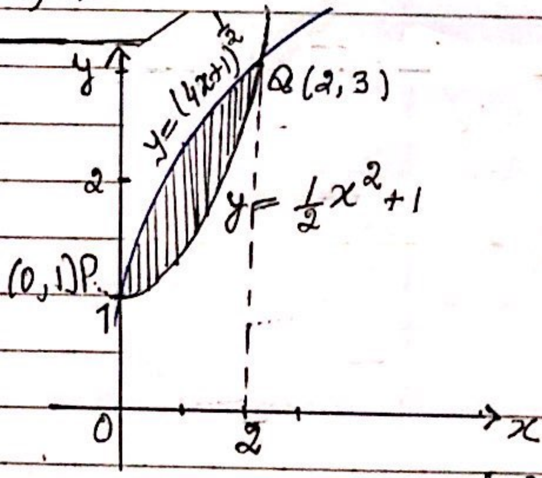


Given two curves: $y = f(x)$
and $y = g(x)$
Area bounded by the two curve and the ordinates $x = a$ and $x = b$.

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

or

$$\int_a^b (f(x) - g(x))\,dx \checkmark$$



Example 25: The diagram shows part of the curves $y = (4x+1)^{1/2}$ and $y = \frac{1}{2}x^2+1$, Intersecting at points $P(0,1)$ and $Q(2,3)$.
Find the shaded area.

[W-14 | 11 | Q 11] ---[6]

Solution: Curve $y = (4x+1)^{1/2}$ ---- ①
Curve $y = \frac{1}{2}x^2+1$ ---- ②
Shaded area = Area under the curve ① − area under curve 2 and between $x = 0$ and $x = 2$, above X-axis
$= A_1 - A_2$ --- ③

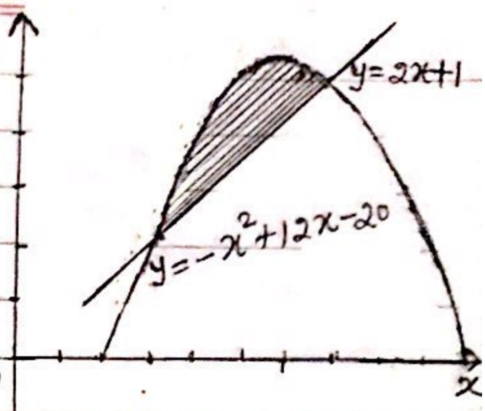
$$\text{Area } A_1 = \int_0^2 (4x+1)^{1/2}\,dx$$
$$= \left[\frac{(4x+1)^{3/2}}{\frac{3}{2}\times 4}\right]_0^2$$
$$= \frac{1}{6}[27 - 1]$$
$$= \frac{1}{6}\times 26 = \frac{13}{3} \quad \text{--- ④}$$

$$\text{Area } A_2 = \int_0^2 \left(\frac{1}{2}x^2+1\right)dx$$
$$= \left[\frac{1}{2}\times\frac{x^3}{3} + x\right]_0^2$$
$$= \left(\frac{8}{6}+2\right) - (0) = \frac{10}{3} \quad \text{--- ⑤}$$

from ④ and ⑤ in ③
Shaded Area = $A_1 - A_2$
$= \frac{13}{3} - \frac{10}{3} = \frac{3}{3}$
$= 1$ Sq. unit.

Example 26: The diagram shows the curve, $y = -x^2 + 12x - 20$ and the line $y = 2x + 1$. Find the area of the shaded region. ---[8]

S-14/13 Q10



Solution: Curve : $y = -x^2 + 12x - 20$ ---- ①

Line : $y = 2x + 1$ --- ②

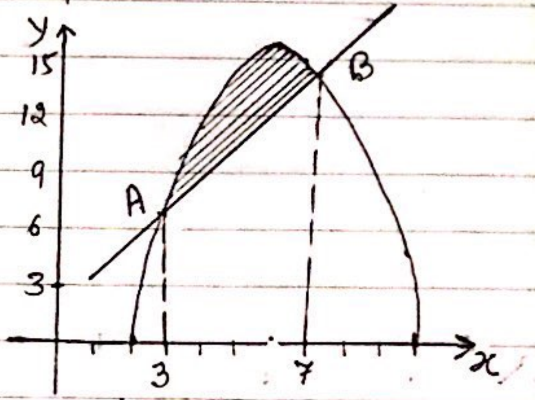
To find the points of intersections of the curve and the line from ① & ②

$$-x^2 + 12x - 20 = 2x + 1$$

$$\Rightarrow \quad x^2 - 10x + 21 = 0$$

$$(x-3)(x-7) = 0$$

$$x = 3,\ 7$$



Shaded area = Area under the curve − area under the line between $x = 3$ and $x = 7$, above x-axis

$$= \int_3^7 (-x^2 + 12x - 20)\,dx - \int_3^7 (2x+1)\,dx$$

$$= \int_3^7 (-x^2 + 12x - 20 - 2x - 1)\,dx$$

$$= \int_3^7 (-x^2 + 10x - 21)\,dx$$

$$= \left[-\frac{x^3}{3} + 5x^2 - 21x \right]_3^7$$

$$= \left(-\frac{343}{3} + 245 - 147\right) - (-9 + 45 - 63)$$

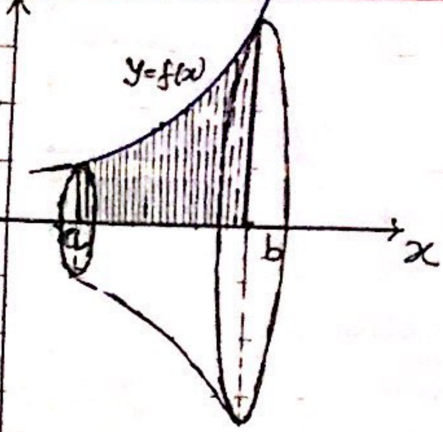
$$= -\frac{49}{3} - (-27)$$

$$= 27 - \frac{49}{3} = \frac{32}{3} \checkmark$$

$\therefore$ Shaded area $= 10\frac{2}{3}$ sq. units.

§ Volume of Revolution:

Case I Volume of the solid formed by rotating through 360°, shaded region under the curve $y = f(x)$ above x-axis and between $x = a$, $x = b$. about x-axis.

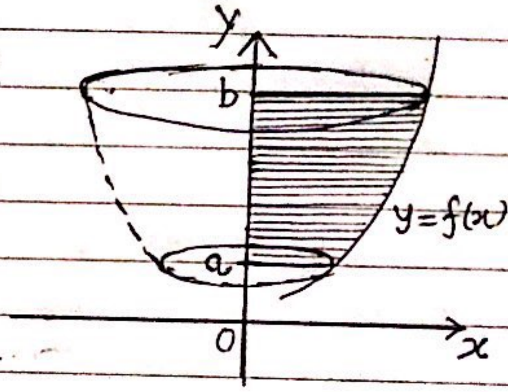


$$V = \int_a^b \pi y^2 \, dx$$

$$V = \int_a^b \pi \, [f(x)]^2 \, dx \quad \text{for } y = f(x).$$

§ Case II.

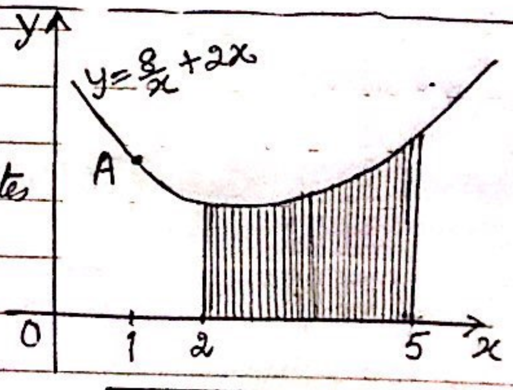
Volume obtained by rotating the shaded area about y-axis through 360°.
$y = f(x)$ between $y = a$ and $y = b$
$y = f(x) \longrightarrow x = g(y)$



$$V = \int_a^b \pi x^2 \, dy$$

$$V = \int_a^b \pi \, [g(y)]^2 \, dy$$

Example 27; The diagram shows part of the curve $y = \frac{8}{x} + 2x$ and three points A, B and C on the curve with x-coordinates 1, 2 and 5 respectively.
Find the volume obtained when the shaded region is rotated through 360° about the x-axis.



[W-13/12/Q9] --- [6]

Solution:

$$y = \frac{8}{x} + 2x \quad \text{--- (1)}$$

$$V = \int_2^5 \pi y^2 \, dx$$

$$= \pi \int_0^2 \left(\frac{8}{x} + 2x\right)^2 dx$$

$$= \pi \int_0^2 \left(\frac{64}{x^2} + 4x^2 + 32\right) dx$$

$$V = \pi \int_2^5 (64x^{-2} + 4x^2 + 32) \, dx$$

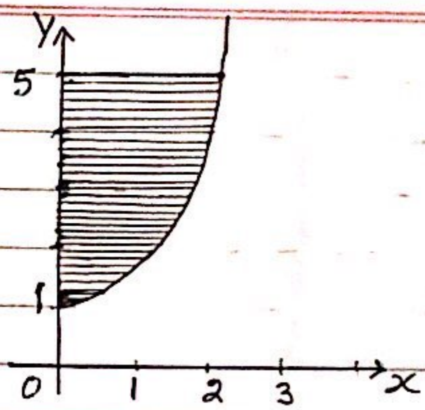
$$= \pi \left[-\frac{64}{x} + 4 \cdot \frac{x^3}{3} + 32x\right]_2^5$$

$$= \pi \left[\left(-\frac{64}{5} + \frac{500}{3} + 160\right) - \left(-32 + \frac{32}{3} + 64\right)\right]$$

$$= \pi \, (271.2)$$

$\therefore V = 271\pi$ ✓ cubic units

Example 28: The diagram shows part of the curve $y = x^2 + 1$



Find the volume obtained when the shaded region is rotated through 360° about the y-axis. ---- [4]

[W-14/12/Q1]

Solution: $y = x^2 + 1 \Rightarrow x = \sqrt{y-1}$ --- ①

Volume, when the shaded region is rotated about y-axis through 360°, between $y=1$ and $y=5$

$$V = \int_1^5 \pi x^2 \, d$$

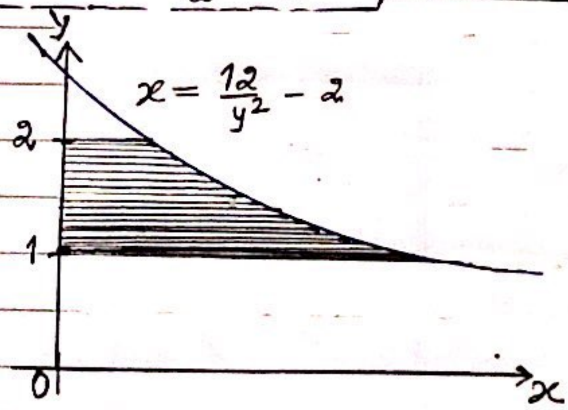
$$V = \pi \int_1^5 (\sqrt{y-1})^2 dy$$

$$= \pi \int_1^5 (y-1)\, dy$$

$$= \pi \left[\frac{(y-1)^2}{2} \right]_1^5$$

$$= \pi \left[\left(\frac{4^2}{2} - 0 \right] = 8\pi \text{ Cubic Units} \right.$$

Example 29: The diagram shows the part of the curve $x = \frac{12}{y^2} - 2$. The shaded area is bounded by the curve, the y-axis and the lines $y=1$ and $y=2$. Find the volume, in terms of π, when this shaded region is rotated through 360° about the y-axis. ---[5]



[S-16/11/Q3]

Solution: Given $x = \frac{12}{y^2} - 2$ --- ①

Shaded region is rotated about y-axis,

$$V = \int_1^2 \pi x^2 \, dy$$

$$= \pi \int_1^2 \left(\frac{12}{y^2} - 2 \right)^2 dy$$

$$= \pi \int_1^2 \left(\frac{144}{y^4} + 4 - \frac{48}{y^2} \right) dy$$

$$= \pi \int_1^2 (144 y^{-4} + 4 - 48 y^{-2}) dy$$

$$= \pi \left[\frac{144 y^{-3}}{-3} + 4y - 48 \cdot \frac{y^{-1}}{-1} \right]_1^2$$

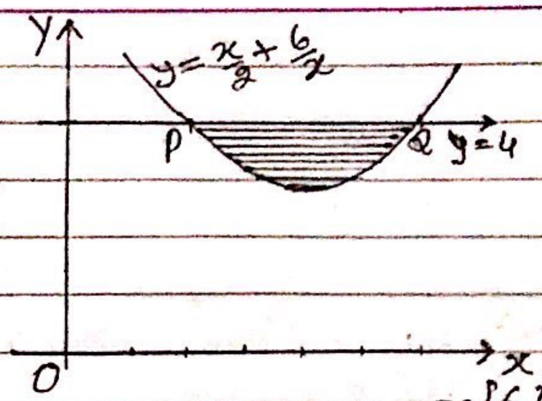
$$V = \pi \left[-\frac{48}{y^3} + 4y + \frac{48}{y} \right]_1^2$$

$$= \pi \left[\left(-\frac{48}{8} + 8 + 24 \right) - (-48 + 4 + 48) \right]$$

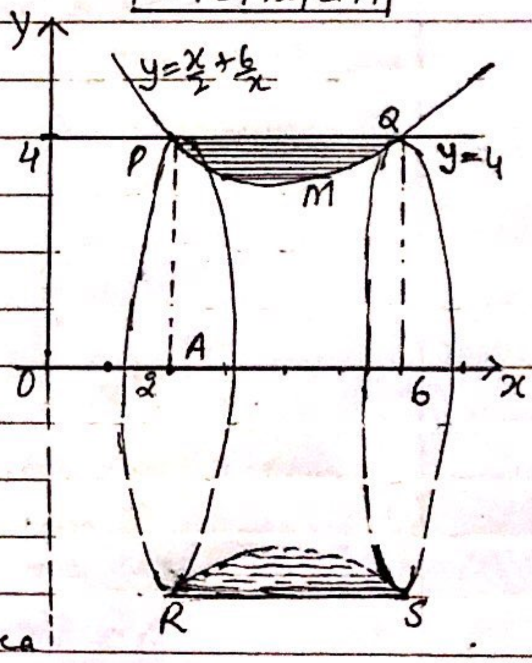
$$= \pi \left[-6 + 8 + 24 - 4 \right]$$

$$= 22\pi \text{ cubic units}$$

Example 30: The diagram shows part of the curve $y = \frac{x}{2} + \frac{6}{x}$. The line $y = 4$, intersects the curve at points P and Q. Find the volume obtained, when the shaded region is rotated through 360° about x-axis. Give your answer in terms of π. ---[6]



[S-18/12/Q11]



Solution: Curve: $y = \frac{x}{2} + \frac{6}{x}$ --- ①

line: $y = 4$ --- ②

To find the points of intersection of the line and curve from ① and ②

$\frac{x}{2} + \frac{6}{x} = 4$

$\Rightarrow x^2 + 12 = 8x$

$\Rightarrow x^2 - 8x + 12 = 0$

$(x-2)(x-6) = 0$

$x = 2;\ x = 6$

$P(2,4)$, $Q(6,4)$; $PQ = 6 - 2 = 4$ ✓

Volume obtained by rotating the shaded area

= Volume of cylinder PQRS − Volume by curve from $x = 2$ and 6 --- ③

Volume of cylinder $= \pi r^2 h$ $\left\{ \begin{array}{l} r = PA = 4 \\ h = PQ = 4 \end{array} \right.$

$= \pi \times 4^2 \times 4$

$= 64\pi$ --- ④

Volume obtained by rotating the curve PMQ between $x = 2$ & $x = 6$ about x-axis, $V = \int_2^6 \pi y^2 \, dx$

$= \pi \int_2^6 \left(\frac{x}{2} + \frac{6}{x}\right)^2 dx$

$= \pi \int_2^6 \left(\frac{x^2}{4} + 6 + \frac{36}{x^2}\right) dx$

$V = \pi \int_2^6 \left(\frac{1}{4}x^2 + 6 + 36x^{-2}\right) dx$

$V = \pi \left[\frac{1}{4} \cdot \frac{x^3}{3} + 6x - \frac{36}{x}\right]_2^6$

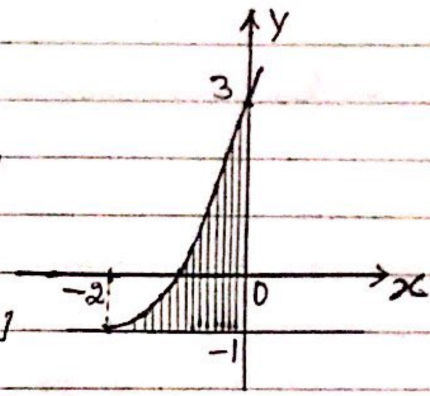
$= \pi \left[48 - \left(\frac{2}{3} - 6\right)\right] = \frac{160}{3}\pi$ --- ⑤

∴ Required Volume

$= 64\pi - \frac{160}{3}\pi$

$= \frac{32}{3}\pi = 10\frac{2}{3}\pi$ cubic units

<u>Example 31</u>: The diagram shows a shaded region bounded by the y-axis, the line $y=-1$ and the part of the curve $y=x^2+4x+3$ for $x \geq -2$.



(i) Express $y=x^2+4x+3$ in the form, $y=(x+a)^2+b$, where a and b are constants, hence, for $x \geq -2$, express x in terms of y. --[4]

(ii) Hence, find the volume obtained when the shaded region is rotated through 360° about y-axis. ---[6]

[W-19/11/Q11]

<u>Solution</u>: Curve: $y = x^2+4x+3$

(i)
$$= x^2+4x+2^2-4+3$$
$$= (x+2)^2-1 \quad ---(1)$$

from (1) $(x+2)^2 = 1+y$
$$x+2 = \pm\sqrt{1+y}$$
$$x = -2+\sqrt{1+y} \quad (\because x \geq -2) \quad ---(2)$$

(ii) Volume obtained by rotating the shaded region about y-axis

$$V = \int_{-1}^{3} \pi x^2 \, dy$$

from (2)
$$x = (-2+\sqrt{1+y})$$
$$\Rightarrow x^2 = 4+1+y-4(1+y)^{1/2}$$
$$= 5+y-4(1+y)^{\frac{1}{2}}$$

$$= \pi \int_{-1}^{3} \left(5+y-4(1+y)^{1/2}\right) dy$$

$$= \pi\left[5y+\frac{y^2}{2}-4\frac{(1+y)^{3/2}}{3/2}\right]_{-1}^{3}$$

$$= \pi\left[5y+\frac{y^2}{2}-\frac{8}{3}(y+1)^{3/2}\right]_{-1}^{3}$$

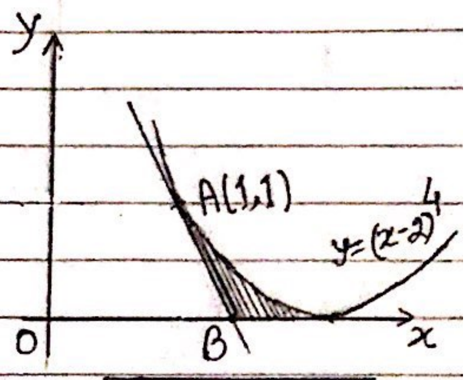
$$= \pi\left[\left(15+\frac{9}{2}-\frac{64}{3}\right)-\left(-5+\frac{1}{2}\right)\right]$$

$$= \frac{8\pi}{3} \quad (\text{or } 8.38) \text{ } \underline{\text{cubic units}}$$

Example 32: The diagram shows part of the curve $y=(x-2)^4$ and a point $A(1,1)$ on the curve. The tangent at A cuts the x-axis at B.

(i) Find the coordinates of B. ----[3]

(ii) Find the area of the shaded region.--[4]



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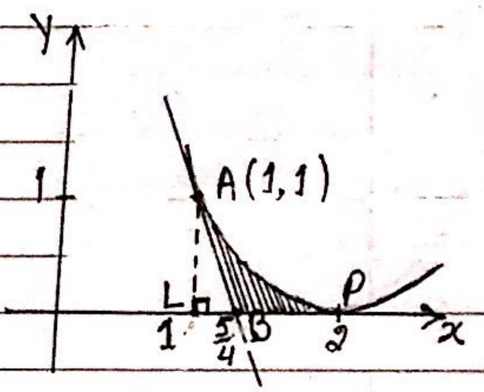
Solution: Curve: $y=(x-2)^4$ ---①

$\frac{dy}{dx} = 4(x-2)^3$

Gradient of the tangent to the cure at $A(1,1)$, $\left(\frac{dy}{dx}\right)_{(1,1)} = 4(1-2)^3 = -4$



$\therefore$ Equation of the tangent at $(1,1)$

$y-1=-4(x-1)$

$y=-4x+5$ ---②

Tangents ② cuts x-axis at $y=0$

from ② $0=-4x+5$

$\Rightarrow x=\frac{5}{4}$ $\therefore B\left(\frac{5}{4},0\right)$

Curve meets x-axis at $y=0$ from ①

$0=(x-2)^4 \Rightarrow x=2$

$P(2,0)$

Area of the shaded region = Area under the curve from $x=1$ to $x=2$ − area of ΔALB ———③

$(A=A_1-A_2)$

Area under the curve, above x-axis between $x=1$ and $x=2$

$A_1 = \int_1^2 y\,dx = \int_1^2 (x-2)^4 dx$

$= \left[\frac{(x-2)^5}{5}\right]_1^2 = (0)-\left(\frac{(-1)^5}{5}\right) = \frac{1}{5}$ ✓

area of $\Delta ALB = \frac{1}{2} AL \times LB$

$= \frac{1}{2}\times 1\times\left(\frac{5}{4}-1\right) = \frac{1}{2}\times1\times\frac{1}{4} = \frac{1}{8}$ ✓

$\therefore$ Required Shaded Area $= A_1 - A_2$

$= \frac{1}{5}-\frac{1}{8} = \frac{3}{40}$ sq. units ✓