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F_P/S

Further Probability and Statistics

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Continuous random variables
Notes and Revision.

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Continuous random variables Notes

§ Probability density functions (PDFs):

The probability density function describes the probability of a continuous random variable X.

- For $f(x)$ to represent 'PDF', $f(x) \geq 0$; $\forall x$,
- The area under the 'PDF' must equal to 1. $\Rightarrow \int_{-\infty}^{\infty} f(x)dx = 1$
- $P(a \leq X \leq b) = \int_a^b f(x)dx$
- $P(X = a) = 0$ for any single value a.
- $P(a \leq X \leq b) = P(a < X < b) = P(a \leq X < b) = P(a < X \leq b)$.

Example 1: The random variable X has probability density function given by:

$$f(x) = \begin{cases} 4x^k & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

where k is a constant.

(i) Show that $k = 3$ ---[2]

(ii) Find the upper quartile of X. ---[2]

(iii) Find the interquartile range of X. ---[2]

(Question from S_2)

Solution (i) Total sum of prob. $\int_0^1 4x^k dx = 1 \Rightarrow \left[\frac{4x^{k+1}}{k+1}\right]_0^1 = 1$

$$\Rightarrow \frac{4}{(k+1)}[1-0] = 1 \Rightarrow k+1 = 4 \Rightarrow \underline{k = 3}$$

(ii) For upper quartile q_3: (75% Percentile) $\int_0^{q_3} 4x^3 dx = 0.75\ (75\%)$ $\left\{ \begin{array}{l} f(x) = 4x^3 \\ 0 \leq x \leq 1 \end{array} \right.$

$$\Rightarrow \left[x^4\right]_0^{q_3} = 0.75 \Rightarrow q_3^4 = 0.75 \Rightarrow q_3 = \sqrt[4]{0.75} = \underline{0.931}$$

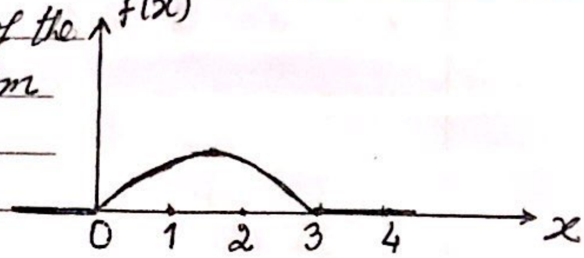
(iii) Now lower quartile q_1: (25% Percentile) $\int_0^{q_1} 4x^3 dx = 0.25 \Rightarrow \left[x^4\right]_0^{q_1} = 0.25$

$$\Rightarrow q_1^4 = 0.25 \Rightarrow q_1 = \sqrt[4]{0.25} = 0.707$$

$\therefore$ Inter quartile range $= q_3 - q_1 = 0.931 - 0.707$
$= \underline{0.224}$

Example 2: The diagram shows the graph of the prob. density function, f, of a random variable X, where,

$$f(x)=\begin{cases}\frac{2}{9}(3x-x^2): & 0\leq x\leq 3\\ 0 & \text{otherwise}\end{cases}$$



(a) State the value of $E(X)$ and find $Var(X)$ ---[4]

(b) State the value of $P(1.5\leq X\leq 4)$ ---[1]

(c) Given that $P(1\leq X\leq 2)=\frac{13}{27}$, find $P(X>2)$ ---[2]

(Question from S_2)

Solution (a) $E(X)=\underline{1.5}$ ✓

$$E(X^2)=\int_0^3 x^2\cdot f(x)\,dx=\int_0^3 x^2\cdot\frac{2}{9}(3x-x^2)\,dx=\frac{2}{9}\int_0^3(3x^3-x^4)\,dx$$

$$=\frac{2}{9}\left[\frac{3x^4}{4}-\frac{x^5}{5}\right]_0^3=\frac{2}{9}\left(\frac{243}{4}-\frac{243}{5}\right)=2.7\ ✓$$

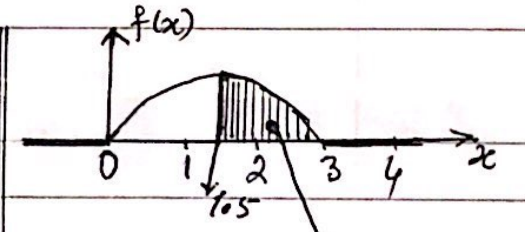
$$\therefore\ Var(X)=E(X^2)-(E(X))^2=2.7-(1.5)^2=\underline{0.45}\ ✓$$

(b) $$P(1.5\leq X\leq 4)=\int_{1.5}^3 f(x)\,dx+0$$

$$=\frac{2}{9}\int_{1.5}^3(3x-x^2)\,dx=\frac{2}{9}\left[\frac{3x^2}{2}-\frac{x^3}{3}\right]_{1.5}^3$$

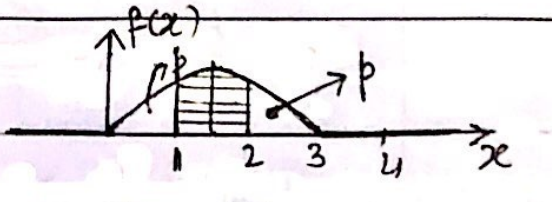
$$=\frac{2}{9}\left[\left(3\times\frac{3^2}{2}-9\right)-\left(\frac{3}{2}(1.5)^2-\frac{(1.5)^3}{3}\right)\right]$$

$$=\frac{2}{9}\left[\left(\frac{27}{2}-9\right)-(3.375-1.125)\right]=\frac{2}{9}\times 2.25=\underline{0.5}\ ✓$$



$P(1.5\leq X\leq 4)=0.5$

(c) $P(1\leq X\leq 2)=\frac{13}{27}$ (Given) ---(i)



Let $P(X>2)=p$

$$\Rightarrow\ p+\frac{13}{27}+p=1\Rightarrow 2p=1-\frac{13}{27}=\frac{14}{27}$$

$$\Rightarrow\ p=\underline{\frac{7}{27}}\ ✓\ (\text{or } 0.259)$$

$$P(X>2)=\frac{7}{27}\ ✓$$

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§ Cumulative distribution functions (CDFs) :

The cumulative distribution function, $F(x)$, for a continuous random variable, X:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx \quad [\text{Here } f(x) \text{ is PDF}]$$

§ Conversely: $\frac{d}{dx}F(x) = f(x)$ // $P(a \leq X \leq b) = F(b) - F(a)$

Example 3: The random variable X has probability density function f given by:
$$f(x) = \begin{cases} \frac{1}{30}\left(\frac{8}{x^2} + 3x^2 - 14\right) & 2 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$
Find the cumulative distribution function. ---[3]

[W-19/21/Q10]

Solution: Cumulative distribution function $F = \int_{-\infty}^{x} f(x)\,dx$

For, $x \in (-\infty, 2)$; $F(x) = \int_{-\infty}^{2} 0\,dx = 0$ ✓

For, $x \in (2, 4) = F(2) + \int_{2}^{x} \frac{1}{30}\left(\frac{8}{x^2} + 3x^2 - 14\right)dx = 0 + \frac{1}{30}\left[\frac{-8}{x} + x^3 - 14x\right]_2^x$

$= \frac{1}{30}\left(-\frac{8}{x} + x^3 - 14x + 24\right)$ ✓

For, $x \in (4, \infty)$, $F(x) = F(4) + \int_{4}^{\infty} 0\,dx = 1 + 0 = 1$

$\therefore$ Cumulative distribution function (CDF) $F(x) = \begin{cases} 0 & : x < 2 \\ \frac{1}{30}\left(-\frac{8}{x} + x^3 - 14x + 24\right) & \text{for } 2 \leq x \leq 4 \\ 1 & : x > 4 \end{cases}$

Example 4: For continuous random variable X has cumulative distribution function F given by:

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2}(1+x)^2 & -1 \leq x \leq 0 \\ 1 - \frac{1}{2}(1-x)^2 & 0 < x \leq 1 \\ 1 & x > 0 \end{cases}$$

(a) Find the probability density function of X. ---[2]

(b) Find $P(-\frac{1}{2} \leq X \leq \frac{1}{2})$. [W-21/41/Q2]

Solution: PDF; $f(x) = \frac{d}{dx}F(x)$

for $X < -1$, the Prob $= 0$ and for $x > 1$, Prob $= 0$

for $-1 \leq x \leq 0 \rightarrow \frac{d}{dx}(F(x)) = \frac{d}{dx}\left(\frac{1}{2}(1+x)^2\right) = 1 + x$ ✓

for $0 < x < 1 \rightarrow \frac{d}{dx}\left\{1 - \frac{1}{2}(1-x)^2\right\} = 1 - x$ ✓

$\therefore$ PDF; $f(x) = \begin{cases} 1+x & -1 \leq x \leq 0 \\ 1-x & 0 < x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

(b) $P(-\frac{1}{2} \leq X \leq \frac{1}{2}) = F(\frac{1}{2}) - F(-\frac{1}{2}) = \frac{7}{8} - \frac{1}{8}$
$= 3/4$

Example 5: A random variable X has a probability density function (PDF):

$$f(x)=\begin{cases} x^3 & 0\le x<1 \\ 1 & 1\le x<1.5 \\ -(x-2.5)^3 & 1.5\le x<2.5 \\ 0 & \text{otherwise.}\end{cases}$$

(a) Find the cumulative distribution $F(x)$.
(b) Calculate $P(X>1.2)$
(c) Find $P(0.5<X<2)$

Solution (a)

For, $x\in(0,1)$, $F(x)=\int_0^x x^3\,dx = \frac{x^4}{4}$ ✓

For, $x\in(1,1.5)$, $F(x)=F(1)+\int_1^x 1\,dx$

$= \frac{1}{4}+[x]_1^x = x-\frac{3}{4}$ ✓

For, $x\in(1.5,2.5) = F(1.5)+\int_{1.5}^x -(x-2.5)^3\,dx$

$= \frac{3}{4}+\left[-\frac{(x-2.5)^4}{4}\right]_{1.5}^x$

$= 1-\frac{(x-2.5)^4}{4}$ ✓

$$\therefore F(x)=\begin{cases} 0 & x<0 \\ \frac{x^4}{4} & 0\le x<1 \\ x-\frac{3}{4} & 1\le x<1.5 \\ 1-\frac{(x-2.5)^4}{4} & 1.5\le x<2.5 \\ 1 & x\ge 2.5\end{cases}$$

Solution:

(b) $P(X>1.2) = 1-P(X\le 1.2)$

$= 1-F(1.2)$

$= 1-(1.2-\frac{3}{4})$

$= \underline{0.55}$

(c) $P(0.5<X<2) = F(2)-F(0.5)$

$= 0.84375-0.015625$

$= 0.96875 = 0.969$ (3sf)

Example 6: A random variable X has a cumulative distribution function:

$$F(x)\begin{cases} 0 & x<5 \\ \frac{1}{50}(x-5)^2 & 5\le x<10 \\ \frac{1}{4}x-2 & 10\le x\le 12 \\ 1 & x>12\end{cases}$$

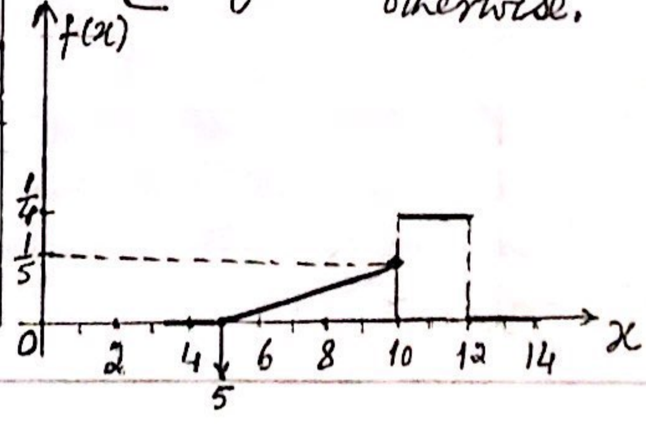
(a) Calculate $P(X>8)$.
(b) Sketch the graph of $f(x)$

Solution (a) $P(X>8) = 1-P(X\le 8)$ ✓

$= 1-\frac{1}{50}(8-5)^2 = \frac{41}{50}$

(b) PDF; $f(x)=\frac{d}{dx}F(x)$

$$f(x)=\begin{cases} \frac{1}{25}(x-5)\ ; & 5\le x\le 10 \\ \frac{1}{4} & 10\le x\le 12 \\ 0 & \text{otherwise.}\end{cases}$$



§ Functions of continuous random variables:

Consider another continuous random variable, Y, which is a function of a continuous random variable X.

Now to find the probability density function (PDF) of continuous random variable, Y, which is a function of a continuous random variable X, we follow

(i) Find the cumulative distribution function (CDF) of X, $F_X(x)$.
(ii) Find the inverse function of Y.
(iii) Set the corresponding limits of Y.
(iv) Find $F_X(g^{-1}(y)) \longrightarrow F_Y(y)$
(v) diff. $F_Y(y) \longrightarrow f_Y(y)$.

Example 7: The cumulative distribution function of a continuous random variable X is:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{16}x^2 & 0 \leq x \leq a \\ 1 & x > a \end{cases}$$

The random variable Y: $Y = X^2$

(a) Find the prob. density function of Y.
(b) Find $P(Y > 1)$

Solution (a) $\frac{1}{16}a^2 = 1 \Rightarrow a = 4, \cancel{-4} \; (a > 0)$

Change the limits:

X	Y
0	0
4	16

$F_Y(y) = P(X^2 \leq y) = F_X(y^{1/2}) = \frac{1}{16}y$

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{16}y & 0 \leq y \leq 16 \\ 1 & y > 16 \end{cases}$$

$f(y) = \frac{d}{dy}F(y) = \frac{d}{dy}\left(\frac{1}{16}y\right) = \frac{1}{16}$

$$\Rightarrow f(y) = \begin{cases} \frac{1}{16} & 0 \leq y \leq 16 \\ 1 & y > 16 \end{cases}$$

(PDF of Y)

(b) $P(Y > 1) = 1 - P(Y \leq 1) = 1 - F(1)$

$= 1 - \frac{1}{16} = \frac{15}{16}$ ✓

Example 8: The continuous random variable X has prob. density function f is given by: [S-21/43/Q6]

$$f(x)=\begin{cases}\frac{1}{8} & 0\le x<1\\ \frac{1}{28}(8-x) & 1\le x\le 8\\ 0 & \text{others.}\end{cases}$$

(a) Find the cumulative distribution function of X. ---[3]

(b) Find the value of the constant 'a' such that $P(X\le a)=5/7$ ---[3]

The random variable Y: $Y=\sqrt[3]{X}$

(c) Find the prob. density function of Y. ---[5]

Solution: For, $x\in(-\infty,0)\to F(x)=0$

(a) For, $x\in(0,1)\to F(x)=\int_0^1 \frac{1}{8}\,dx=\frac{1}{8}x$

For, $x\in(1,8)\to F(x)=F(1)+\int_1^x \frac{1}{28}(8-x)\,dx$

$$=\frac{1}{8}+\frac{1}{28}\left[8x-\frac{x^2}{2}\right]_1^x$$

$$=\frac{1}{8}+\frac{1}{28}\left[\left(8x-\frac{x^2}{2}\right)-\frac{15}{56}\right]$$

$$=\frac{1}{28}\left(8x-\frac{x^2}{2}\right)-\frac{1}{7}\ \checkmark$$

For $x\in(x>8)\to F(x)=0$

Hence Cumulative distribution fun. of X:

$$F(x)=\begin{cases}0 & x<0\\ \frac{1}{8}x & 0\le x<1\\ \frac{1}{28}\left(8x-\frac{1}{2}x^2\right)-\frac{1}{7} & 1\le x\le 8\\ 1 & x>8\end{cases}$$

(b) $F(a)=5/7$ $\quad(\because P(X\le a)=5/7)$

$$\Rightarrow \frac{1}{28}\left(8a-\frac{1}{2}a^2\right)-\frac{1}{7}=\frac{5}{7}$$

$$\Rightarrow \frac{1}{28}\left(8a-\frac{1}{2}a^2\right)=\frac{6}{7}$$

$$\Rightarrow a^2-16a+48=0$$

$$\Rightarrow \underline{a=4}\ \checkmark \text{ or } a=12^{\times}$$

(C) Now random variable $Y=\sqrt[3]{X}$

Change of limits:

X	Y
$0\le x<1$	$0\le y<1$
$1\le x<8$	$1\le y<2$

$F_Y(Y)=P(\sqrt[3]{X}\le Y)=F_X(Y^3)$

Hence 'CDF' of Y

$$F(Y)=\begin{cases}0 & y<0\\ \frac{1}{8}y^3 & 0\le y<1\\ \frac{1}{56}(16y^3-y^6)-\frac{1}{7}; & 1\le y<2\end{cases}$$

Now 'PDF' of $Y\to \frac{d}{dy}F(Y)=f(y)$

$$f(y)=\begin{cases}\frac{3}{8}y^2 & 0\le y<1\\ \frac{3}{28}(8y^2-y^5); & 1\le y<2\\ 0 & \text{otherwise.}\end{cases}$$

§ The expectation of a continuous random variable X and g(X):

(i) $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

(ii) $E[g(X)] = \int_{-\infty}^{\infty} f(x) \cdot g(x)\,dx.$

Example 9: The continuous random variable, X, has the probability density function:

$$f(x) = \begin{cases} \frac{1}{4}x & 0 \leq x < 2, \\ \frac{1}{4}(8-x) & 2 \leq x < 8 \\ 0 & \text{otherwise.} \end{cases}$$

The random variable Y is defined by $Y = x^2$.
Find the expected value of Y.

Solution: $E(Y) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_0^2 x^2 \left(\frac{x}{4}\right) dx + \int_2^8 x^2 \times \frac{1}{4}(8-x)\,dx$

$$= \frac{1}{4} \cdot \int_0^2 x^3\,dx + \frac{1}{4}\int_2^8 (8x^2 - x^3)\,dx$$

$$= \frac{1}{4}\left[\frac{x^4}{4}\right]_0^2 + \frac{1}{4}\left[\frac{8}{3}x^3 - \frac{x^4}{4}\right]_2^8$$

$$= \frac{1}{4}\left[4 + \left\{\left(\frac{4096}{3} - 1024\right) - \left(\frac{64}{3} - 4\right)\right\}\right]$$

$$= \frac{1}{4}[1344 - 1016] = \underline{82}$$

Example 10: The continuous random variable X has cumulative distribution function, F, given by:

$$F(x)=\begin{cases}0 & x<0,\\ \frac{1}{81}x^2 & 0\le x\le 9,\\ 1 & x>9\end{cases}$$

(a) Find $E(\sqrt{X})$ ---- [3]

(b) Find $Var(\sqrt{x})$ --- [2]

(c) The random variable Y is given by $Y^3=X$. Find the probability density function of Y. --- [3]

[S-21/41/Q3]

Solution: Given CDF, $F(X) \Rightarrow$ PDF, $f(x)=\frac{d}{dx}F(x)$

$\therefore$ PDF, $f(x)=\begin{cases}\frac{2}{81}x & 0\le x\le 9\\ 0 & \text{otherwise}\end{cases}$

(a) $E(\sqrt{x}) = \int_0^9 \sqrt{x}\cdot f(x)\,dx = \int_0^9 \sqrt{x}\cdot\frac{2}{81}x\,dx = \frac{2}{81}\int_0^9 x^{3/2}dx = \frac{2}{81}\left[\frac{2}{5}x^{5/2}\right]_0^9$

$= \frac{2}{81}\times(9)^{5/2} = \frac{2}{81}\times\frac{2}{5}\times 283$

$= 12/5 = 2.4$ ✓

(b) $Var(\sqrt{x}) = E((\sqrt{x})^2) - E(\sqrt{x})$ ---(i)

Consider $E((\sqrt{x})^2)=E(X)=\int_0^9 x\cdot f(x)dx = \int_0^9 x\cdot\frac{2}{81}x\,dx = \frac{2}{81}\int_0^9 x^2 dx$

$= \frac{2}{81}\left[\frac{x^3}{3}\right]_0^9 = \frac{2}{81}(273) = 6$ ---(ii)

from (i) $Var(\sqrt{x}) = 6 - (2.4)^2 = 6-5.76 = \underline{0.24}$ ✓

(c) Now $Y^3=X \Rightarrow F_Y(Y)=P(X^{1/3}\le Y)=F_X(Y^3)$

$\therefore$ 'CDF' of Y

$$F(Y)=\begin{cases}0 & y<0\\ \frac{1}{81}y^6, & 0\le y\le 9^{1/3}\\ 1, & y>9^{1/3}\end{cases}$$

Change of limit.

X	$Y=\sqrt[3]{X}$
$x<0$	$Y<0$
$0\le x\le 9$	$0\le y\le 9^{1/3}$

$\therefore$ Prob. density function of Y, $f(y)=\frac{d}{dy}F(y)=\begin{cases}\frac{2}{27}y^5 & 0\le y\le 9^{1/3}\\ 0 & \text{otherwise.}\end{cases}$

Example 11: The continuous random variable X has prob. density function f given by:

$$f(x) = \begin{cases} \frac{3}{16}(2-\sqrt{x}) & 0 \le x < 1 \\ \frac{3}{16\sqrt{x}} & 1 \le x \le 9 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $E(X)$ ---[3]

(b) The random value Y is such that $Y = \sqrt{X}$; find the prob. density function of Y. ...[5]

S-20/41/Q3

Solution: $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^1 \frac{3}{16} x(2-\sqrt{x})\,dx + \int_1^9 \frac{3}{16}\frac{x}{\sqrt{x}}\,dx$

$= \frac{3}{16}\int_0^1 (2x - x^{3/2})\,dx + \frac{3}{16}\int_1^9 x^{1/2}\,dx = \frac{3}{16}\left[x^2 - \frac{2}{5}x^{5/2}\right]_0^1 + \frac{3}{16}\left[\frac{2}{3}x^{3/2}\right]_1^9$

$= \frac{3}{16}\left[(1-\frac{2}{5}) - 0\right] + \frac{3}{16}\left[\frac{2}{3}\times 27 - \frac{2}{3}\right] = \frac{3}{16}\times\frac{3}{5} + \frac{3}{16}\times\frac{52}{3} = \underline{269/80}$ ✓

Now to find the 'CDF' of $f(x)$

For, $x \in (0,1) \rightarrow F(x) = \int_0^x \frac{3}{16}(2-\sqrt{x})\,dx = \frac{3}{16}\left[2x - \frac{2}{3}x^{3/2}\right]_0^x = \frac{3}{8}\left(x - \frac{1}{3}x^{3/2}\right)$ ✓

For, $x \in (1,9) \rightarrow F(x) = F(1) + \int_1^x \frac{3}{16} x^{-\frac{1}{2}}\,dx = \frac{1}{4} + \left[\frac{3}{16}(2\sqrt{x})\right]_1^x = \frac{1}{4} + \frac{3}{16}(2\sqrt{x} - 2)$

$= \frac{1}{4} + \frac{3}{8}\sqrt{x} - \frac{3}{8} = \underline{\left(\frac{3}{8}\sqrt{x} - \frac{1}{8}\right)}$ ✓

$$F(x) = \begin{cases} \frac{3}{8}\left(x - \frac{1}{3}x^{3/2}\right) & 0 \le x < 1 \\ \frac{3}{8}x^{1/2} - \frac{1}{8} & 1 \le x \le 9 \end{cases}$$ ✓

Now the random variable $Y = \sqrt{X} \rightarrow$ Change of Limit.

X	$Y = \sqrt{X}$
$0 \le x < 1$	$0 \le y < 1$
$1 \le x \le 9$	$1 \le y \le 3$

'CDF' of Y.

$G(Y) = P(y \le Y) = P(\sqrt{x} \le Y) = F_X(Y^2)$

$$G(Y) = \begin{cases} 0 & y < 0 \\ \frac{3}{8}\left(y^2 - \frac{1}{3}y^3\right) & 0 \le y < 1 \\ \frac{3}{8}y - \frac{1}{8} & 1 \le y \le 3 \end{cases}$$

Now 'PD' of Y $\quad \frac{d}{dy}G(Y) = g(y)$

Hence,

$$g(y) = \begin{cases} \frac{3}{8}(2y - y^2) & 0 \le y < 1 \\ \frac{3}{8} & 1 \le y \le 3 \\ 0 & \text{otherwise.} \end{cases}$$

Example 12: The continuous random variable, X, has cumulative distribution Function, F, given by,

$$F(x) = \begin{cases} 0 & x < 2 \\ \frac{1}{60}x^2 - \frac{1}{15} & 2 \leq x \leq 8 \\ 1 & x > 8 \end{cases}$$

(a) Find $P(3 \leq X \leq 6)$ ---- [1]

(b) Find $E(\sqrt{X})$ ---- [3]

(c) Find $Var(\sqrt{X})$ --- [2]

(d) The random variable Y is defined by $Y = X^3$.
Find the prob. density function of Y. ---- [3] $\boxed{W\text{-}20/42/Q4}$

Solution (a) $P(3 \leq X \leq 6) = F(6) - F(3) = \left(\frac{1}{60} \times 6^2 - \frac{1}{15}\right) - \left(\frac{1}{60} \times 3^2 - \frac{1}{15}\right)$
$= \frac{1}{60}(36 - 9) = \frac{1}{60} \times 27 = \frac{9}{20} = 0.45$

(b) $f(x) = \frac{d}{dx}F(x) \Rightarrow f(x) = \begin{cases} \frac{1}{30} x & 2 \leq x \leq 8 \\ 0 & \text{otherwise.} \end{cases}$

$E(\sqrt{x}) = \int_2^8 \sqrt{x} \cdot \frac{1}{30}x \, dx = \int_2^8 \frac{1}{30} x^{3/2} dx$
$= \frac{1}{30}\left[\frac{2}{5} \cdot x^{5/2}\right]_2^8 = \frac{1}{75} \times \left[8^{5/2} - 2^{5/2}\right]$
$= 2.3381$
$= 2.34$ ---(i)

(c) $E(\sqrt{x}^2) = \int_2^8 (\sqrt{x})^2 \cdot \frac{1}{30} x \, dx$
$= \frac{1}{30}\int_2^8 x^2 dx = \frac{1}{30}\left[\frac{x^3}{3}\right]_2^8 = \frac{1}{90}[512 - 8] = 5.6$ ---(ii)

$\therefore Var(\sqrt{X}) = E(\sqrt{X}^2) - E(\sqrt{X})$ from (i) & (ii)
$= 5.6 - (2.3381)^2 = \boxed{0.133}$

(d) For $Y = X^3$

for $G_Y(Y) = P(Y \leq Y) = P(X^3 \leq Y) = F_X(Y^{1/3})$

$\therefore G(Y) = \left\{ \frac{1}{60} y^{2/3} - \frac{1}{15} \right.$

for $Y = X^3$: Change of limits

X	$Y = X^3$
$2 \leq X \leq 8$	$8 \leq Y \leq 512$

For PDF of $Y \Rightarrow f(y) = \frac{d}{dy}F(y) = \frac{d}{dy}\left(\frac{1}{60}y^{2/3} - \frac{1}{15}\right) = \frac{1}{60} \times \frac{2}{3} y^{-1/3}$

$\therefore$ PDF of Y

$g(y) = \left\{ \frac{1}{90} y^{-\frac{1}{3}}, \quad 8 \leq y \leq 512 \right.$

§ The median and percentiles:

'p_{th}' percentile is a number 'N' such that $P(X \leq N)$
or $P(X \leq N) = \int_{-\infty}^{N} f(x)dx = F(N) = p\%$

Given "CDF" $\to F(X)$, then:

(i) For median M is 50_{th} percentile: $F(M) = 0.5$
(ii) For lower quartile Q_1; $F(Q_1) = 0.25$
(iii) For upper quartile Q_3; $F(Q_3) = 0.75$

Given 'PDF' $\to f(x)$
Then median - m
$\int_{-\infty}^{m} f(x)dx = 0.5$
and Q_1:
$\int_{-\infty}^{Q_1} f(x)dx = 0.25$

Example 13: A continuous random variable X has a cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{25}x^2 & 0 \leq x \leq 5 \\ 1 & x > 5 \end{cases}$$

(a) Find the median of X.
(b) Find the lower quartile and the upper quartile of X; and the interquartile range.

Solution (a) For median $F(X) = 0.5 \Rightarrow \frac{1}{25}x^2 = 0.5 \Rightarrow x^2 = 12.5$
$\Rightarrow x = 3.54$

$\therefore$ Median $M = 3.54$ ✓

(b) For lower quartile $F(Q_1) = 0.25 \Rightarrow \frac{1}{25} \cdot Q_1^2 = 0.25$
$\Rightarrow Q_1^2 = 6.25$
$\Rightarrow Q_1 = 2.5$ ✓

For upper quartile $F(Q_3) = 0.75$
$\Rightarrow \frac{1}{25}Q_3^2 = 0.75 \Rightarrow Q_3^2 = 25 \times 0.75 = 18.75$
$\Rightarrow Q_3 = 4.33$ ✓

$\therefore$ Interquartile range $= Q_3 - Q_1$
$= 4.33 - 2.5$
$= 1.83$ ✓

Example 14: The continuous random variable, X, has prob. density function, f, given by:

$$f(x)=\begin{cases}0 & x<0\\ \frac{6}{5}x & 0\le x\le 1\\ \frac{6}{5}x^{-4} & x>1\end{cases}$$

(a) Find $P(X>1)$ ---[1]

(b) Find the median value of X --- [2]

(c) Given that $E(X)=1$, Find the variance of X. --- [3]

(d) Find $E(\sqrt{X})$ $\boxed{SP\text{-}20/04/Q5}$ --- [2]

Solution (a) $P(X>1)=1-P(X\le 1)=1-\int_0^1 f(x)dx=1-\int_0^1 \frac{6}{5}x\,dx=1-\left[\frac{6}{5}\cdot\frac{x^2}{2}\right]_0^1$

$=1-\frac{6}{5}\times\frac{1}{2}=\frac{2}{5}$ ✓

(b) median m: $\int_0^m \frac{6}{5}x\,dx=0.5$ $(0\le \text{median}\le 1)$ $[\because P(X\le 1)=\frac{3}{5}$

$\Rightarrow \frac{6}{5}\left[\frac{x^2}{2}\right]_0^m=0.5 \Rightarrow \frac{6}{5}\times\frac{1}{2}m^2=0.5$

$\Rightarrow m^2=\frac{5}{6}$

$\Rightarrow m=\pm\sqrt{\frac{5}{6}}$

$m=\sqrt{\frac{5}{6}}$ ✓ ; $-\sqrt{\frac{5}{6}}$ ✗

(c) $E(X^2)=\int_{-\infty}^{\infty}x^2\cdot f(x)dx=\frac{6}{5}\int_0^1 x^2\cdot x\,dx+\frac{6}{5}\int_1^{\infty}x^2\cdot x^{-4}dx$

$=\frac{6}{5}\left[\int_0^1 x^3dx+\int_1^{\infty}x^{-2}dx\right]=\frac{6}{5}\left\{\left[\frac{x^4}{4}\right]_0^1+\left[-\frac{1}{x}\right]_1^{\infty}\right\}$

$=\frac{6}{5}\left[\frac{1}{4}+(0+1)\right]=\frac{3}{2}=1.5$ ✓ ; $E(X)=1$ Given.

$Var(X)=E(X^2)-(E(X))^2=1.5-(1)^2$

$=0.5$

(d) $E(\sqrt{X})=\int_{-\infty}^{\infty}\sqrt{x}\cdot f(x)\,dx=\int_0^1\sqrt{x}\cdot\frac{6}{5}x\,dx+\int_1^{\infty}\sqrt{x}\cdot\frac{6}{5}x^{-4}dx$

$=\frac{6}{5}\left[\int_0^1 x^{3/2}dx+\int_1^{\infty}x^{-7/2}dx\right]$

$=\frac{6}{5}\left\{\left[\frac{2}{5}x^{5/2}\right]_0^1+\left[-\frac{2}{5}x^{-\frac{5}{2}}\right]_1^{\infty}\right\}=\frac{6}{5}\left[\frac{2}{5}+\left(0-\left(-\frac{2}{5}\right)\right)\right]$

$=\frac{6}{5}\times\frac{4}{5}=\frac{24}{25}$ (or 0.96)

Example 15: The continuous random variable, X, has prob. density function f given by:

$$f(x) = \begin{cases} \frac{1}{5}x & 0 \leq x < 2 \\ \frac{2}{15}(5-x) & 2 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the cumulative distribution function of X -- [3]

(b) Find the median value of X. -- [2]

(c) Find $E(X^2)$ --- [2]

(d) Find $P[1 \leq X \leq 3]$ --- [2]

[S-20 | 43 | Q3]

Solution (a) For, $x \in (-\infty, 0)$, $F(x) = \int_{-\infty}^{0} 0\,dx = 0$

For $x \in (0,2)$, $F(x) = \int_0^x \frac{1}{5}x\,dx = \left[\frac{x^2}{10}\right]_0^x = \frac{x^2}{10}$ ✓

For $x \in (2,5)$; $F(x) = F(2) + \int_2^x \frac{2}{15}(5-x)\,dx = \frac{2}{5} + \left[\frac{2}{15}\left(5x - \frac{x^2}{2}\right)\right]_2^x$

$$= \frac{2}{5} + \frac{2}{15}\left(5x - \frac{x^2}{2} - 8\right) = \frac{1}{15}(10x - x^2) - \frac{2}{3}, \checkmark$$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{10} & 0 \leq x < 2 \\ \frac{1}{15}(10x - x^2) - \frac{2}{3} & 2 \leq x \leq 5 \\ 1 & x > 5 \end{cases}$$

$\left\{ x \in (5,\infty) \atop F(x) = F(5) + \int_5^\infty 0\,dx = 1 \right.$

(b) For median 'm'; $F(m) = \frac{1}{2}$

$\Rightarrow \frac{1}{15}(10m - m^2) - \frac{2}{3} = \frac{1}{2}$

$\Rightarrow 2m^2 - 20m + 35 = 0$

$m = \frac{20 \pm 2\sqrt{30}}{4} = 5 - \frac{1}{2}\sqrt{30}$, $\cancel{\left(5 + \frac{1}{2}\sqrt{30}\right)} > 5$

$m = 5 - \frac{\sqrt{30}}{2}$ or $\underline{2.26}$ ✓

(c) $E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_0^2 x^2 \times \frac{x}{5}\,dx + \int_2^5 x^2 \times \frac{2}{15}(5-x)\,dx$

$$= \left[\frac{x^4}{20}\right]_0^2 + \left[\frac{2}{15}\left(\frac{5}{3}x^3 - \frac{x^4}{4}\right)\right]_2^5 = \frac{16}{20} + \frac{2}{15}\left[\frac{625}{12} - \frac{28}{3}\right]$$

$= 6.5$ ✓

(d) $P(1 \leq X \leq 3) = F(3) - F(1)$

$$= \frac{11}{15} - \frac{1}{10} = \underline{19/30} \checkmark$$

Example 16: The continuous random variable X has cumulative distribution function F given by:

$$F(x)=\begin{cases}0 & x<-1\\ \frac{1}{2}(1+x)^2 & -1\leq x\leq 0\\ 1-\frac{1}{2}(1-x)^2 & 0<x\leq 1\\ 1 & x>1\end{cases}$$

(a) Find the prob. density function of X --- [2]
(b) Find $P(-\frac{1}{2}\leq X\leq \frac{1}{2})$ ----- [2]
(c) Find $E(X^2)$ --- [2]
(d) Find $Var(X^2)$ --- [2]

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Solution: $\frac{d}{dx}F(x)=f(x)$

(a) $\frac{d}{dx}\frac{1}{2}(1+x)^2=(1+x)$

$\frac{d}{dx}\{1-\frac{1}{2}(1-x)^2=(1-x)$

$\Rightarrow$ PDF of X: $f(x)=\begin{cases}1+x & -1\leq x\leq 0\\ 1-x & 0<x\leq 1\\ 0 & \text{otherwise}\end{cases}$

(b) $P(-\frac{1}{2}\leq X\leq \frac{1}{2}) = F(\frac{1}{2})-F(-\frac{1}{2}) = \{1-\frac{1}{2}(1-\frac{1}{2})^2\}-\{\frac{1}{2}(1+(-\frac{1}{2}))^2\}$

$= 7/8 - 1/8 = \frac{3}{4}$ ✓

(c) $E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_{-1}^{0} x^2 \times (1+x)\,dx + \int_0^1 x^2\cdot(1-x)\,dx$

$= \int_{-1}^{0}(x^2+x^3)\,dx + \int_0^1 (x^2-x^3)\,dx$

$= \left[\frac{x^3}{3}+\frac{x^4}{4}\right]_{-1}^{0} + \left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1 = \frac{1}{12}+\frac{1}{12}$

$= 1/6$ --- (i)

(d) $Var(X^2) = E((X^2)^2)-[E(X^2)]^2$ ------- (ii)

Consider $E((X^2)^2) = \int_{-\infty}^{\infty} x^4\cdot f(x)\,dx = \int_{-1}^{0} x^4(1+x)\,dx + \int_0^1 x^4(1-x)\,dx$

$= \int_{-1}^{0}(x^4+x^5)\,dx + \int_0^1 (x^4-x^5)\,dx$

$= \left[\frac{x^5}{5}+\frac{x^6}{6}\right]_{-1}^{0} + \left[\frac{x^5}{5}-\frac{x^6}{6}\right]_0^1 = \frac{1}{30}+\frac{1}{30}$

$= \frac{1}{15}$ ---- (iii)

from (i) and (iii) in (ii) $\rightarrow$

$Var(X^2) = \frac{1}{15}-\left(\frac{1}{6}\right)^2 = \frac{1}{15}-\frac{1}{36} = \frac{7}{180}$ ✓

Example 17: The continuous random variable X has prob. density function f is given by:

where a is a constant.

$$f(x)=\begin{cases} a+\frac{1}{5}x & 0\le x<1 \\ 2a-\frac{1}{5}x & 1\le x\le 2 \\ 0 & \text{otherwise}\end{cases}$$

(a) Find the value of a. ---[3]

(b) Find $E(X^2)$ --[2]

(c) Find the cumulative distribution function of X ---[3]

W-21/42/Q3

Solution (a) $\int_{-\infty}^{\infty} f(x)=1 \Rightarrow \int_0^1 (a+\frac{1}{5}x)\,dx+\int_1^2 (2a-\frac{1}{5}x)\,dx = 1$

$$\Rightarrow \left[ax+\frac{x^2}{10}\right]_0^1+\left[2ax-\frac{x^2}{10}\right]_1^2=1\Rightarrow \left(a+\frac{1}{10}\right)+\left(2a-\frac{3}{10}\right)=1$$

$$\Rightarrow 3a-\frac{1}{5}=1 \Rightarrow a=\frac{2}{5} \checkmark$$

(b) $E(X^2)=\int_{-\infty}^{\infty} x^2\cdot f(x)\,dx$

$$\Rightarrow f(x)=\begin{cases} \frac{2}{5}+\frac{1}{5}x & 0\le x<1 \\ \frac{4}{5}-\frac{1}{5}x & 1\le x\le 2 \\ 0 & \text{otherwise}\end{cases}$$

$$=\int_0^1 x^2\left(\frac{2}{5}+\frac{1}{5}x\right)dx+\int_1^2 x^2\cdot\left(\frac{4}{5}-\frac{x}{5}\right)dx$$

$$=\int_0^1 \frac{1}{5}(2x^2+x^3)\,dx+\int_1^2 \frac{1}{5}(4x^2-x^3)\,dx$$

$$=\frac{1}{5}\left[\frac{2}{3}x^3+\frac{1}{4}x^4\right]_0^1+\frac{1}{5}\left[\frac{4}{3}x^3-\frac{1}{4}x^4\right]_1^2=\frac{1}{5}\left(\frac{11}{12}\right)+\frac{1}{5}\left(\frac{67}{12}\right)=\frac{78}{60}=\frac{13}{10} \checkmark$$

For 'CDF' of X $\longrightarrow$ $F(x)=\int_{-\infty}^{\infty} f(x)\,dx$

(c) For, $x\in(-\infty,0)$, $F(X)=\int_{-\infty}^{0} 0\,dx=0 \checkmark$

For, $x\in(0,1)$; $F(X)=F(0)+\int_0^x \left(\frac{2}{5}+\frac{1}{5}x\right)dx=0+\left(\frac{2}{5}x+\frac{1}{5}\cdot\frac{x^2}{2}\right) \checkmark$

For, $x\in(1,2)$, $F(X)=F(1)+\int_1^x\left(\frac{4}{5}-\frac{1}{5}x\right)dx=\frac{1}{2}+\left[\frac{4}{5}x-\frac{1}{5}\cdot\frac{x^2}{2}\right]_1^x$

$$=\frac{1}{2}+\left(\frac{4}{5}x-\frac{1}{5}\frac{x^2}{2}-\frac{7}{10}\right)$$

$$=\frac{1}{5}\left(4x-\frac{x^2}{2}-1\right) \checkmark$$

For, $x\in(2,\infty)=F(2)+\int_2^{\infty} 0\,dx=1+0=1 \checkmark$

Hence the cumulative distribution function of X

$$F(X)=\begin{cases} 0 & x<0 \\ \frac{1}{5}\left(2x+\frac{x^2}{2}\right) & 0\le x<1 \\ \frac{1}{5}\left(4x-\frac{1}{2}x^2-1\right) & 1\le x\le 2 \\ 1 & x>2\end{cases}$$