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F.P&S

Further Probability and Statistics

Probability Generating Function Notes and Revision

SP-20 / S-20 / W-20 / S-21 / W-21

Suresh Goel
(Former Director)
Alliance World School.
Noida. Delhi-NCR
INDIA.

(+91 9810 444 804)

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Probability Generating Function

§ Probability generating function:

For discrete random variable X_r, which has integer values x_r and associated probabilities $P(X=x_r)=p_r$, the prob. gen. fun. "PGF" denoted by $G_x(t)$ is defined as: $G_X(t)=p_1 t^{x_1}+p_2 x^{t_2}+p_3 x^{t_3}+\dots=\sum_r p_r t^{x_r}$

Here t is an arbitrary variable.

or $G_X(t)=E(t^X)$

{Note (i) $G_X(1)=p_1+p_2+\dots p_r+\dots=\sum_r p_r=1$ ✓

(ii) p_r = Prob of $x_r = P(X=r)$ = coeff of $\to t^r$.

§ Expectation and Variance:

Distribution	Expection	Variance
$B(n,p)$	np	$np\cdot(1-p)$
$P_0(\lambda)$	λ	λ
$Geo(p)$	$1/p$	$(1-p)/p^2$

$G_X'(1)=p_1x_1+p_2x_2+\dots$ → (See Page 9) $=\sum_r p_r x_r = E(X)$ ✓

$Var(X)=E(X^2)-[E(X)]^2$

$[G_X''(1)=E(X^2)-E(X)]$ → here $Var(X)=G_X''(1)+G_X'(1)-[G_X'(1)]^2$ ✓

§ (i) For Binomial distribution $B(n,p)\to G_X(t)=((1-p)+pt)^n$. [Proof on Page - 9]

(ii) For Poisson distribution $P_0(\lambda)\to G_X(t)=e^{\lambda(t-1)}$ [See Page - 3]

(iii) For Geometric distribution $Geo(p)\to G_X(t)=\dfrac{pt}{1-(1-p)t}$ [See Page - 3]

§ The sum of Independent random variables:

Given X and Y are independent random variables,

(i) $E(X+Y)=E(X)+E(Y)$

(ii) $Var(X+Y)=Var(X)+Var(Y)$

(iii) $G_{X+Y}(t)=G_X(t)\times G_Y(t)$

1. Find the prob. generating fu^n "PGF" for the following discrete random variable X.

x	0	1	2	3	4
$P(X=x)$	$1/2$	$1/5$	$3/20$	$1/10$	$1/20$

Solution: $G_X(t) = p_1 t^{x_1} + p_2 t^{x_2} + p_3 t^{x_3} + \cdots$

$$= \frac{1}{2}\cdot t^0 + \frac{1}{5}\times t^1 + \frac{3}{20}\times t^2 + \frac{1}{10}t^3 + \frac{1}{20}t^4$$

$$= \frac{1}{2} + \frac{1}{5}t + \frac{3}{20}t^2 + \frac{1}{10}t^3 + \frac{1}{20}t^4$$ ✓

2. A discrete random variable X has prob. gen. fu^n "PGF" such that:
$G_X(t) = \dfrac{k\alpha t}{1-\alpha t}$, where k and α are positive constants.

(a) By writing the PGF as a power series in t.
(i) find $P(X=1)$ in terms of α and k.
(ii) show that $P(X=r) = k\alpha^r$.
(b) By considering $G_X(1)$, find the value of α in terms of k.

Solution: $G_X(t) = k\alpha t(1-\alpha t)^{-1} = k\alpha t[1 + \alpha t + (\alpha t)^2 + (\alpha t)^3 + \cdots]$

(a) $= k\alpha t + k(\alpha t)^2 + k(\alpha t)^3 + k(\alpha t)^4 + \cdots$ ①

(i) $P(X=1) =$ Coeff of of t^1 in ① $= \underline{k\alpha}$ ✓

(ii) $P(X=r) =$ Coeff of t^r in $(G_X t)$ ① $= \underline{k\alpha^r}$ ✓

(b) $G_X(1) = \dfrac{k\alpha}{1-\alpha} = 1$ $\left[G_X(t) = \dfrac{k\alpha t}{1-\alpha t} \right]$; $\left(\because G_X(1) = \sum_r p_r = 1\right)$

$\Rightarrow k\alpha = 1-\alpha$

$\Rightarrow \alpha + k\alpha = 1$

$\Rightarrow \alpha\cdot(1+k) = 1$

$\Rightarrow \alpha = \dfrac{1}{(1+k)}$ ✓

3. The discrete random variable X is such that $X \sim Po(\lambda)$. Prove that the prob. gen. fn" 'PGF" is given by $G_X(t) = e^{\lambda(t-1)}$.

Solution: For $Po(\lambda)$; $P(X=r) = \frac{e^{-\lambda}\cdot\lambda^r}{r!}$

$$\begin{cases} P(r=0) = e^{-\lambda} \\ P(r=1) = e^{-\lambda}\cdot\lambda \\ P(r=2) = \frac{e^{-\lambda}\cdot\lambda^2}{2!} \\ --- \end{cases}$$

Hence

$$'PGF' \; G_X(t) = P(r=0)\cdot t^0 + p_{r=1}\cdot t^1 + p_{r=2}\cdot t^2 + --$$

$$= e^{-\lambda}\times 1 + e^{-\lambda}\cdot\lambda\cdot t + \frac{e^{-\lambda}\lambda^2}{2!}\cdot t^2 + \frac{e^{-\lambda}\cdot\lambda^3}{3}\cdot t^3$$

$$= e^{-\lambda}\left[1 + \lambda t + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + --\right]$$

(using Maclaurin theo.) $= e^{-\lambda}\cdot e^{\lambda t} = e^{\lambda(t-1)}$

$$\left[\text{using Maclaurin theorem: } e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + -- \right]$$

---

4. The discrete random variable $X \sim Geo(p)$
Prove that the 'PGF' is given by $G_X(t) = \frac{pt}{1-(1-p)t}$

Solution: $X \sim Geo(p) \rightarrow P(X=r) = q^{r-1}p$

X=	1	2	3	4	---
P(X=x)	p	qp	q^2p	q^3p	---

$$G_X(t) = p_1 t^{x_1} + p_2 t^{x_2} + p_3 t^{x_3} + --$$

$$= pt^1 + qp\cdot t^2 + q^2pt^3 + q^3p\cdot t^4 + --$$

$$= pt\left[1 + qt + q^2t^2 + q^3t^3 + --\right]$$

$$= pt \times \frac{1}{1-qt}$$

$$\left[\text{Infinite G.P. } S_\infty = a + ar + ar^2 -- = \frac{a}{1-r}\right]$$

$$= \frac{pt}{1-(1-p)t} \quad [q = 1-p]$$

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5. For Poission distribution $X \sim Po(\lambda)$, 'PGF' $= e^{\lambda(t-1)}$

(i) Show that $E(X) = \lambda$

(ii) $Var(X) = \lambda$.

<u>Solution</u> (i) $G_X(t) = e^{\lambda(t-1)}$ [for $X \sim Po(\lambda)$]

$G_X'(t) = \lambda e^{\lambda(t-1)}$ ---- (1)

for $t=1 \longrightarrow G_X'(1) = \lambda e^{\lambda(1-1)} = \lambda e^{\lambda \times 0} = \lambda$ ✓ ---- (2)

$\underline{E(X) = G_X'(t) = \lambda}$ ✓

(ii) Diff (1)

$G_X''(t) = \lambda^2 \cdot e^{\lambda(t-1)} \Rightarrow G_X''(1) = \lambda^2$ ✓ ---- (3)

$Var(X) = G_X''(1) + G_X'(1) - (G_X'(\lambda))^2$

$= \cancel{\lambda^2} + \lambda - \cancel{\lambda^2}$ (from (2) & (3))

$\underline{Var(X) = \lambda}$ ✓

6. For random variable modelled by geometric distribution, $X \sim Geo(p)$, use the definition of 'PGF' to verify the expectation is $1/p$ and the variance is $\frac{1-p}{p^2}$

<u>Solution:</u> For Geo. distribution: 'PGF' $G_X(t) = \frac{pt}{1-(1-p)t}$ ---- (1)

Diff (1) $G_X'(t) = \frac{p \times (1-(1-p)t) - pt(-(1-p))}{(1-(1-p)t)^2} = \frac{p}{(1-(1-p)t)^2}$ — (2)

from (2) $E(X) = G_X'(1) = \frac{p}{(1-(1-p))^2} = \frac{p}{p^2} = \underline{\frac{1}{p}}$ ✓ (3)

Diff (2) $G_X''(t) = \frac{2(1-p)p}{(1-(1-p)t)^3} \Rightarrow G_X''(1) = \frac{2(1-p)p}{(1-(1-p))^3} = \frac{2(1-p)}{p^2}$ ✓

Now $Var(X) = G_X''(1) + G_X'(1) - (G_X'(1))^2$

$= \frac{2\cdot(1-p)}{p^2} + \frac{p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{(1-p)}{p^2}$ ✓

7. Two discrete random variables X and Y have probability generating functions such that $G_X(t) = 0.1t^2(6+3t+t^2)$ and $G_Y(t) = \frac{0.05}{t^2}(8+12t^4)$. Find the PGF of the random variable X+Y and hence calculate the probability that X+Y is odd.

Solution: $G_{X+Y}(t) = G_X(t) \times G_Y(t)$

$$= 0.1t^2(6+3t+t^2) \times \frac{0.05}{t^2}(8+12t^4)$$

$$= 0.005[48+24t+8t^2+72t^4+36t^5+12t^6)$$

For P(X+Y is odd) the sum of the coeff. of t^r where r is odd.

$\therefore$ P(X+Y is odd) $= 0.005(24+36) = \underline{0.3}$ ✓

---

8. The discrete random variable X has 'PGF' $G_X(t)$ given by:

$$G_X(t) = 0.2t+0.5t^2+0.3t^3$$

The random variable Y is the sum of two independent observations of X.

(a) Find $G_Y(t)$, giving your answer as an expanded polynomial in t. --- [3]

(b) Use $G_Y(t)$ to find E(Y) and Var(Y). --- [5]

[S-20/43/Q4]

Solution: $G_X(t) = 0.2t+0.5t^2+0.3t^3$ --- ①

(a) as Y is the sum of two independent observations of X.

$G_Y(t) = G_X(t) \times G_X(t)$

$= (0.2t+0.5t^2+0.3t^3)^2$

$G_Y(t) = 0.04t^2+0.2t^3+0.37t^4+0.3t^5+0.09t^6$ — ①

diff ①

(b) $G_Y'(t) = 0.08t+0.6t^2+1.48t^3+1.5t^4+0.54t^5$ — ②

$E(Y) = G_Y'(1) = 0.08+0.6+1.48+1.5+0.54$

or $\underline{E(Y) = 4.2}$ ✓

diff ②

$G_Y''(t) = 0.08+1.2t+4.44t^2+6t^3+2.7t^4$

$G_Y''(1) = 0.08+1.2+4.44+6+2.7 = 14.42$ ✓

Hence $Var(Y) = G_Y''(1)+G_Y'(1)-(G_Y'(1))^2$

$= 14.42+4.2-(4.2)^2 = 0.98 \Rightarrow \underline{Var(Y) = 0.98}$ ✓

9. Aisha has a bag containing 3 red and 3 white balls. She selects a ball at random, notes its colour and returns it to the bag; the same process is repeated twice more. The number of red balls selected by Aisha is denoted by X.

(a) Find the probability generating function $G_X(t)$ of X. --- [2]

Basant also has a bag contain 3 red balls and 3 white balls. He selects three balls at random, without replacement, from his bag. The number of red balls selected by Bansaut is denoted by Y.

(b) Find the prob. generating function $G_Y(t)$ of Y. --- [3]

The random variable Z is the total of red balls selected by Aisha and Basant.

(c) Find the prob generating function of Z, expressing your answer as a polynomial. --- [3]

(d) Use the 'PGF' of Z to find E(Z) and Var(Z) --- [5]

[SP-20/04/Q6]

Solution (a) 3 red & 3 white, balls are drawn with replacement $P(\text{red}) = p = 3/6 = \frac{1}{2}$

$q = \frac{1}{2}$, $n = 3$

$B(3, \frac{1}{2})$

$P(r=0) = (\frac{1}{2})^3 = \frac{1}{8}$, $P(r=1) = {}^3C_1 (\frac{1}{2})^1 \cdot (\frac{1}{2})^2 = 3/8$

$P(r=2) = \frac{3}{8}$, $P(r=3) = \frac{1}{8}$

$\therefore G_X(t) = \frac{1}{8} \cdot t^0 + \frac{3}{8}t^1 + \frac{3}{8}t^2 + \frac{1}{8}t^3$

$G_X(t) = \frac{1}{8} + \frac{3}{8}t + \frac{3}{8}t^2 + \frac{1}{8}t^3$ --- (1)

(b) without replacement use combination

$P(\text{no red, 3 wh.}) = \frac{{}^3C_0 \times {}^3C_3}{{}^6C_3} = \frac{1}{20}$;

$P(\text{1 red, 2 wh.}) = \frac{{}^3C_1 \times {}^3C_2}{{}^6C_3} = \frac{9}{20}$

$P(\text{2 red \& 1 W}) = \frac{{}^3C_2 \times {}^3C_1}{{}^6C_3} = \frac{9}{20}$

$P(\text{3 red}) = \frac{{}^3C_3 \times {}^3C_0}{{}^6C_3} = \frac{1}{20}$

$G_Y(t) = \frac{1}{20} + \frac{9}{20}t + \frac{9}{20}t^2 + \frac{1}{20}t^3$ --- (2)

(c) $Z = X + Y$

$G_Z(t) = G_X(t) \times G_Y(t)$ from (1) & (2)

$= (\frac{1}{8} + \frac{3}{8}t + \frac{3}{8}t^2 + \frac{1}{8}t^3) \times (\frac{1}{20} + \frac{9}{20}t + \frac{9}{20}t^2 + \frac{1}{20}t^3)$

$= \frac{1}{160}(1 + 12t + 39t^2 + 56t^3 + 39t^4 + 12t^5 + t^6]$ --- (3)

(d) Diff (3)

$G_Z'(t) = \frac{1}{160}[12 + 78t + 168t^2 + 156t^3 + 60t^4 + 6t^5]$

$E(Z) = G_Z'(1) = \frac{1}{160} \times 480 = 3$ ✓ --- (4)

$\therefore E(Z) = 3$ ✓

Now Diff (4)

$G_Z''(t) = \frac{1}{160}[78 + 336t + 468t^2 + 240t^3 + 30t^4]$

$G_Z''(1) = \frac{1}{160} \times 1152 = 7.2$

$Var(Z) = G_Z''(1) + G_Z'(1) - [G_Z'(1)]^2$

$= 7.2 + 3 - 3^2$

$= 1.2$

$\therefore Var(Z) = 1.2$ ✓

10. A bag contains 4 red balls and 6 blue balls. Rassa selects two balls at random, without replacement, from the bag. The number of red balls selected by Rassa is denoted by X.

(a) Find the 'PGF', $G_x(t)$, of x. --[2]

Rassa also tosses two coins. One coin is biased so that the prob. of a head is 2/3. The other coin is biased so that the prob. of a head is p.

The prob. generating function of Y, the number of heads obtained by Rassa, is $G_Y(t)$. The coefficient of t in $G_Y(t)$ is 7/12.

(b) Find $G_Y(t)$. ---[3]

The random variable Z is the sum of the number of red balls selected and the number of heads obtained by Rassa.

(c) Find the 'PGF' of Z, expressing your answer as a polynomial. ---[3]

(d) Use 'PGF' of Z to find $E(Z)$ -- [2]

[S-20/41/Q6]

Solution (a) 4 Red & 6 Blue, two balls are drawn at random (with rep): $P(0\text{ red}) = P(BB) = \frac{6}{10}\times\frac{5}{9} = \frac{1}{3}$

$P(RB, BR) = \frac{6}{10}\times\frac{4}{9}\times 2 = \frac{8}{15}$; $P(RR) = \frac{4}{10}\times\frac{3}{9} = \frac{2}{15}$

$G_x(t) = p_0 t^0 + p_1 t^1 + p_2 t^2$

$\underline{G_x(t)} = \frac{1}{3} + \frac{8}{15}t + \frac{2}{15}t^2$ --- (1)

(b) $P(1H) = \frac{2}{3}(1-p) + \frac{1}{3}p = \frac{7}{12}$ [Coeff of t in $G_Y(t)$]

$\Rightarrow p = \frac{1}{4}$; $P(TT) = \frac{1}{3}\times\frac{3}{4} = \frac{1}{4} = P(0)$

'PGF' of $Y = \underline{G_Y(t)} = \frac{1}{4} + \frac{7}{12}t + \frac{1}{6}t^2$ -- (2)

$[\because P(HH) = \frac{2}{3}\times\frac{1}{4} = \frac{1}{6}]$

(c) $Z = X+Y$; 'PGF' $G_Z(t) = G_X(t)\times G_Y(t)$

$\Rightarrow G_Z(t) = \left(\frac{1}{3}+\frac{8}{15}t+\frac{2}{15}t^2\right)\left(\frac{1}{4}+\frac{7}{12}t+\frac{1}{6}t^2\right)$

$= \frac{1}{180}\left[15 + 59t + 72t^2 + 30t^3 + 4t^4\right]$ --(3)

(d) Diff. $G_Z(t) \rightarrow$ (3)

$G'_Z(t) = \frac{1}{180}[59 + 144t + 90t^2 + 16t^3]$ —(4)

$G'_Z(1) = \frac{1}{180}(59+144+90+16)$

$= \frac{309}{180} = 1.72$ —(5)

Hence $E(Z) = G_Z'(1)$

$= \underline{1.72}$ (from (5))

classmate

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11. Keira has two unbiased coins. She tosses both coins. The number of heads obtained by Keira is denoted by X.

(a) Find the prob. generating function (PGF) $G_X(t)$ of X. ---[1]

Hassan has three coins, two of which are biased so that the prob. of obtaining a head when the coin is tossed is $\frac{1}{3}$. The corresponding prob for the third coin is $\frac{1}{4}$. The number of heads obtained by Hassan when he tosses these three coins is denoted by Y.

(b) Find 'PGF' $G_Y(t)$ of Y. ---[3]

The random variable Z is the total number of heads obtained by Keira and Hassan.

(c) Find 'PGF' of Z, expressing your answer as a polynomial. ---[3]

(d) Use 'PGF' of Z to find E(Z) ---[2]

(e) Use 'PGF' of Z to find the most probable value of Z. ---[1]

W-20/41/Q5

Solution (a). $G_X(t) = \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2$ —① $\left\{\begin{array}{l} P(0H) = \frac{1}{2}\times\frac{1}{2} \\ P(1H) = 2\times\frac{1}{2}\times\frac{1}{2} \\ P(2H) = \frac{1}{2}\times\frac{1}{2} \end{array}\right.$

(b) $P(0H) = \frac{2}{3}\times\frac{2}{3}\times\frac{3}{4} = \frac{12}{36}$

$\left\{\begin{array}{l} \text{for Two Coins} \\ P(H) = \frac{1}{3},\ P(T) = \frac{2}{3} \\ \text{for the third coin} \\ P(H) = \frac{1}{4},\ P(T) = \frac{3}{4} \end{array}\right.$

$P(1H) = P(H_1T_1T_2) + P(T_1H_2T_3) + P(T_1T_2H_3)$

$= \frac{1}{3}\times\frac{2}{3}\times\frac{3}{4} + \frac{2}{3}\times\frac{1}{3}\times\frac{3}{4} + \frac{2}{3}\times\frac{2}{3}\times\frac{1}{4} = 16/36$

$P(2H) = \frac{7}{36}$, $P(3H) = \frac{1}{3}\times\frac{1}{3}\times\frac{1}{4} = \frac{1}{36}$

$G_Y(t) = \frac{12}{36} + \frac{16}{36}t + \frac{7}{36}t^2 + \frac{1}{36}t^3$ —②

(c) $G_Z(t) = G_X(t)\times G_Y(t)$

$= \left(\frac{1}{4}+\frac{1}{2}t+\frac{1}{4}t^2\right)\left(\frac{12}{36}+\frac{16}{36}t+\frac{7}{36}t^2+\frac{1}{36}t^3\right)$

$= \frac{1}{144}\left[12 + 40t + 51t^2 + 31t^3 + 9t^4 + t^5\right]$ —③

(d) Diff ③

$G_Z'(t) = \frac{1}{144}\left[40 + 102t + 93t^2 + 36t^3 + 5t^4\right]$ —④

$E(Z) = G_Z'(1) = \frac{1}{144}\left[40+102+93+36+5\right] = \frac{276}{144}$

$\therefore E(Z) = \frac{23}{12} = \underline{1.92}$ ✓

(e) In $G_Z(t)$

The term with largest coefficient, (probability) is $\frac{51}{144}t^2$

$\therefore$ Most probable value of $Z = \underline{2}$ ✓

§ 12. The random variable X has the binomial distribution $B(n, p)$.
(a) Write down an expression for $P(X=r)$ and hence show that the prob. generating function "PGF" of X is $(q+pt)^n$, where $q=1-p$ --[3]
(b) Use $G_X(t)$ to prove that $E(X)=np$ and $Var(X)=np(1-p)$ --[5]
W-20/43/Q5

Solution: (a) $X \sim B(n,p) \Rightarrow P(X=r) = {}^nC_r\, p^r q^{n-r}$; $q=1-p$

$$\therefore G_X(t) = \sum_{r=0}^{n} {}^nC_r \cdot p^r q^{n-r} \cdot t^r$$

$$= \sum_{r=0}^{n} {}^nC_r\,(pt)^r \cdot q^{n-r}$$

$$= (q+pt)^n \checkmark \quad \text{---} ①$$

(b) Diff ①

$$G_X'(t) = n(q+pt)^{n-1} \times p \quad \text{---} ②$$

$$E(X) = G_X'(1) = n(q+p)^{n-1} \cdot p = np \checkmark \text{---} ③$$
$(\because q+p=1)$

Diff ②

$$G_X''(t) = n(n-1)(q+pt)^{n-2} \cdot p \times p$$

$$\Rightarrow G_X''(1) = n(n-1)p^2 \quad \text{---} ④$$

Now $Var(X) = G_X''(1) + G_X'(1) - [G_X'(1)]^2$

$$= n(n-1)p^2 + np - (np)^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$\therefore Var(X) = np(1-p) \checkmark$$

13. Tanji has a bag containing 4 red and 2 blue balls. He selects 3 balls at random from the bag, without replacement. The number of red balls selected by Tanji is denoted by X.

(a) Find the prob. generating function (PGF) $G_X(t)$ of X. ---[2]

Tanji also has two coins, each baised so that the prob. of obtaining a head when it is thrown is $1/4$. He throws the two coins at the same time. The number of heads obtained is denoted by Y.

(b) Find the 'PGF' $G_Y(t)$ of Y. ---[2]

The random variable Z is the sum of the number of red balls selected by Tanji and the number of heads obtained.

(C) Find the 'PGF' of Z, expressing your answer as a polynomial. ---[3]

(d) Using 'PGF' of Z to find E(Z) and Var(Z). $\boxed{\text{S-21/41/Q6}}$ [5]

Solution (a) 4 red & 2 blue, drawn 3 balls.

$P(0R)=0$, (not Possible), $P(1R,2B)=\frac{{}^4C_1\times{}^2C_2}{{}^6C_3}=\frac{1}{5}$

$P(2R)=\frac{{}^4C_2\times{}^2C_1}{{}^6C_3}=\frac{3}{5}$; $P(3R)=\frac{{}^4C_3}{{}^6C_3}=\frac{1}{5}$

$G_X(t)=0+\frac{1}{5}t+\frac{3}{5}t^2+\frac{1}{5}t^3$ —— (1)

(b) $P(H)=\frac{1}{4}$, $P(T)=\frac{3}{4}$

$P(0H)=P(TT)=\frac{3}{4}\times\frac{3}{4}=\frac{9}{16}$; $P(1H)=2\times\frac{3}{4}\times\frac{1}{4}=\frac{6}{16}$

$P(2H)=\frac{1}{4}\times\frac{1}{4}=\frac{1}{16}$

$G_Y(t)=\frac{9}{16}+\frac{6}{16}t+\frac{1}{16}t^2$ —— (2)

(C) $Z=X+Y$

$G_Z(t)=G_X(t)\cdot G_Y(t)$

$=\left(\frac{1}{5}t+\frac{3}{5}t^2+\frac{1}{5}t^3\right)\left(\frac{9}{16}+\frac{6}{16}t+\frac{1}{16}t^2\right)$.

$G_Z(t)=\frac{9}{80}t+\frac{33}{80}t^2+\frac{28}{80}t^3+\frac{9}{80}t^4+\frac{1}{80}t^5$ --- (3)

(d) Diff (3)

$G_Z'(t)=\frac{1}{80}(9+66t+84t^2+36t^3+5t^4)$ —— (4)

$E(Z)=G_Z'(1)=\frac{1}{80}(9+66+84+36+5)$

$E(Z)=200/80=2.5$ —— (5)

Diff (4)

$G_Z''(t)=\frac{1}{80}(66+168t+108t^2+20t^3)$

$\therefore G''(1)=\frac{1}{80}(66+168+108+20)=\frac{362}{80}$

$=4.525$ —— (6)

$Var(Z)=G_Z''(1)+G_Z'(1)-[G_Z'(1)]^2$

$=4.525+2.5-(2.5)^2$

$=4.525+2.5-6.25$

$Var(Z)=0.775$

14 X is a discrete random variable which takes values 0, 2, 4, ---
The prob. generating function of X is given by $G_X(t) = \frac{1}{3-2t^2}$
(a) Find $E(X)$ and $Var(X)$ --- [5]
(b) Find $P(X=4)$ --- [3]

S-21/43/Q4

Solution: (a) $G_X(t) = \frac{1}{3-2t^2}$ $\left(\text{or} = (3-2t^2)^{-1}\right)$ --- ①

diff ①

$G_X'(t) = \frac{4t}{(3-2t^2)^2}$ --- ② $\Rightarrow G_X'(1) = 4$ --- ③

$\therefore \underline{E(X) = 4}$ ✓

diff. ②

$G_X''(t) = 4(3-2t^2)^{-2} + 32t^2(3-2t^2)^{-3} = \frac{12+24t^2}{(3-2t^2)^3}$ --- ④

$G''(1) = 36$ --- ⑤

$Var(X) = G_X''(1) + G_X'(1) - (G_X'(1))^2$

$= 36 + 4 - 4^2$

$\underline{Var(X) = 24}$ ✓

(b) from ① $G_X(t) = (3-2t^2)^{-1}$ $\left[(1-x)^{-1} = 1 + x + x^2 + \cdots\right]$

$= \frac{1}{3}\left[1 - \frac{2}{3}t^2\right]^{-1}$

$= \frac{1}{3}\left[1 + \frac{2}{3}t^2 + \frac{4}{9}t^4 + \cdots\right]$

$P(X=4) = \text{Coeff of } t^4 \text{ in } G_X(t) = \frac{4}{27}$

$\therefore \underline{P(X=4) = \frac{4}{27}}$ ✓

15. Nine balls labelled 1, 2, 3, 4, 5, 6, 7, 8 and 9, are placed in a bag. Kai selects three balls at random from the bag, without replacement. The random variable X is the number of balls selected by Kai that are labelled with a multiple of 3.

(a) Find the prob. generating function (PGF) $G_X(t)$ of X. --- [3]

The balls are replaced in the bag.

Jacob now selects two balls at random from the bag, without replacement. The random variable Y is the number of balls selected by Jacob that are labelled with an even number.

(b) Find the 'PGF' $G_Y(t)$ of Y. --- [2]

The random variable Z is the sum of the number of balls that are labelled with a multiple of 3 selected by Kai and the number of balls that are labelled with an even number selected by Jacob.

(c) Find the 'PGF' of Z, expressing your answer as a polynomial. -- [3]

(d) Use $G_Z(t)$ to find $E(Z)$ --- [2]

W-21 | 41 | Q5

Solution (a) 1, 2, 3, 4, 5, 6, 7, 8, 9 → Multiple of 3 → 3, 6, 9;

$P(\text{a multiple } 3) = 3/9$

$P(\text{None of } 3,6,9) = \frac{6}{9}\times\frac{5}{8}\times\frac{4}{7} = \frac{120}{504} = \frac{20}{84}$ ✓

$P(\text{One of } 3,6,9) = 3\times\frac{3}{9}\times\frac{6}{8}\times\frac{5}{7} = \frac{270}{508} = \frac{45}{84}$ ✓

$P(\text{Two of } 3,6,9) = 3\times\frac{3}{9}\times\frac{2}{8}\times\frac{6}{7} = \frac{108}{508} = \frac{18}{84}$

$P(\text{all three } 3,6,9) = \frac{3}{9}\times\frac{2}{8}\times\frac{1}{7} = \frac{6}{508} = \frac{1}{84}$

$\therefore\ G_X(t) = \frac{20}{84} + \frac{45}{84}t + \frac{18}{84}t^2 + \frac{1}{84}t^3$ —— (1)

(b) even no → 2, 4, 6, 8 → $P(\text{even}) = 4/9$, $P(\text{odd}) = \frac{5}{9}$

$P(\text{None even}) = \frac{5}{9}\times\frac{4}{8} = \frac{20}{72} = 5/18$

$P(\text{one even}) = 2\times\frac{4}{9}\times\frac{5}{8} = \frac{40}{72} = \frac{10}{18}$

$P(\text{both even}) = \frac{4}{9}\times\frac{3}{8} = \frac{12}{72} = \frac{3}{18}$

$\therefore\ G_Y(t) = \frac{5}{18} + \frac{10}{18}t + \frac{3}{18}t^2$ —— (2)

(c) $Z = X + Y$

$G_Z(t) = G_X(t)\times G_Y(t)$ from (1) & (2)

$= \frac{1}{84}(20 + 45t + 18t^2 + t^3)\times\frac{1}{18}(5 + 10t + 3t^2)$

$G_Z(t) = \frac{1}{1512}[100 + 425t + 600t^2 + 320t^3 + 64t^4 + 3t^5]$ -- (3)

(d) Diff (3)

$G_Z'(t) = \frac{1}{1512}[425 + 1200t + 960t^2 + 256t^3 + 15t^4]$

$E(Z) = G_Z'(1) = \frac{1}{1512}[425 + 1200 + 960 + 256 + 15]$

$E(Z) = \frac{17}{9}$ ✓

16. The random variable X is such that $P(X=r) = kr^2$ for $r = 1, 2, 3, 4$ where k is a constant.

(a) Find the value of k. ---[1]

(b) Find the prob. generating function $G_X(t)$ of X ---[2]

The random variable Y has 'PGF' $G_Y(t) = \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2$

The random variable Z is the sum of X and Y.

(c) Assuming X and Y are independent, find 'PGF' $G_Z(t)$ of Z, as a polynomial in t. ---[3]

(d) Given that $E(Z) = \frac{13}{3}$, Use $G_Z(t)$ to find $Var(Z)$ ---[3]

W-21/42/Q5

Solution (a) $\Sigma P(r) = (1+4+9+16)k = 1$

$\Rightarrow k = 1/30$ ✓

(b) $G_X(t) = \frac{1}{30}t + \frac{4}{30}t^2 + \frac{9}{30}t^3 + \frac{16}{30}t^4$ — (1)

(c) $G_Y(t) = \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2$ —— (2)

$Z = X + Y$

$\therefore G_Z(t) = G_X(t) \times G_Y(t)$

$= \left(\frac{1}{30}t + \frac{4}{30}t^2 + \frac{9}{30}t^3 + \frac{16}{30}t^4\right)$

$\times \left(\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2\right)$

$\therefore$

$G_Z(t) = \frac{1}{120}[t + 6t^2 + 18t^3 + 38t^4 + 41t^5 + 16t^6]$ -- (3)

diff (3)

$G_Z'(t) = \frac{1}{120}[1 + 12t + 54t^2 + 152t^3 + 205t^4 + 96t^5]$ — (4)

diff (4)

$G_Z''(t) = \frac{1}{120}[12 + 108t + 456t^2 + 820t^3 + 480t^4]$ — (5)

$G_Z'(1) = \frac{13}{3}$ Given — (5)

$G_Z''(1) = \frac{1}{120}[12 + 108 + 456 + 820 + 480] = \frac{1876}{120}$ — (6)

$Var(X) = G_Z''(1) + G_Z'(1) - [G_Z'(1)]^2$

$= \frac{1876}{120} + \frac{13}{3} - \left(\frac{13}{3}\right)^2 = \frac{107}{90}$ (or 1.19)

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