classmate
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Page

F. P&S

Further Probability and Statistics

χ^2 - test

Notes and Revision

SP-20	S-20	S-21	W-20	W-21

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FP&S

χ^2- test (Goodness of fit tests) (Contingency tables)

§ Observed frequency $O_K : \Sigma O_K = n$

Expected frequency $E_K = n \cdot p_K$ [p_K = Prob of a given class]

[Note: $E_K \geqslant 5$]

$$X^2 = \sum_k \frac{(O_K - E_K)^2}{E_K}$$

X^2 is a measure of the 'goodness of fit' of the model.
X^2 is also a statistic. The smaller the value of X^2, better the fit.

§ χ^2 distribution:

This distribution has a parameter ν (nu)
Degrees of freedom ν = number of class − number of constraints
and the prob. 'p' depending on the significance level (Normally 5%)

§ Goodness of fit tests where population parameters are known:

We try to fit the data to common distributions such as:
- Binomial - $B(n,p)$.
- Geometric - $Geo(p)$
- Poisson - $P_o(\lambda)$
- Uniform or Normal (μ, σ^2)

§ Goodness of fit-test:

"This section introduces a way of testing the hypothesis that observed frequencies 'O_K' are consistent with a given prob. distribution (expected frequencies 'E_K'), using χ^2-test"

1. A company has 200 machines producing parts of a specific process. The machine manufacturer states that the machines produce, on average, 2·5 defective parts a day, the company are investigating this claim and wish to see if their machine fit a poisson distribution with that mean. One day they collect data for all the machines, noting the number of defective parts made by each.

Number of defective parts	0	1	2	3	4	5	6 or more
Number of machines	28	49	50	44	16	8	5

(a) Stating your hpothesis carefully, perform a χ^2-test at 1% significance level.

(b) Under what assumptions would a poission distribution model be appropriate.

Solution (a) H_0: Defective parts can be modelled by Poisson's dist. $P_o(2.5)$; $\lambda = 2.5$

H_1: defective parts can not be modelled by $P_o(2.5)$ $\left\{ P(r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!} \right.$

1% Significance level.

	0	1	2	3	4	5	6 or more
Prob. p_k	0.08208	0.2052	0.2565	0.2138	0.1336	0.06680	0.04202
Obs O_k	28	49	50	44	16	8	5
Exp. E_k	16.42	41.04	51.30	42.75	26.72	13.36	8.402
$\frac{(O_k-E_k)^2}{E_k}$	8.172	1.543	0.03310	0.03640	4.301	2.151	1.379

Degree of freedom $V = 7-1 = 6$, at 1% significant level $(X > C)$

$p = 0.99 \Rightarrow$ critical value $= 16.81$ (for χ^2 distribution)

$\chi^2 = \sum_k \frac{(O_k - E_k)^2}{E_k} = 17.62$; $17.62 > 16.81$

Therefore reject H_0.

Hence the defective parts can be modelled by $P_o(2.5)$

(b) Defects should occur singularly and randomly.

2. Each of 200 identically biased dice is thrown repeatedly until an even number is obtained. The number of throws needed is recorded and the results are summarised in the following table.

Number of throws	1	2	3	4	5	6	≥ 7
Frequency	126	43	22	3	5	1	0

Carry out a goodness of fit test, at the 5% significance level, to test whether Geo(0.6) is a satisfactory model for the data. ---[7]

SP-20/04/Q2

Solution: Expected prob. Geo(p): $P(x) = q^{x-1} \cdot p$; Here Geo(0.6), $p=0.6$, $q=0.4$

$n = 200$, $P( ) = p$

H_0: Number throws/frequence can be modelled as Geo(0.6)

H_1: " Cannot be modelled as Geo(0.6)

at 5% Significant level.

	1	2	3	4	5	6	$7 \geq$
Prob. p_K	0.6	0.24	0.096	0.0384	0.01536	0.006144	0.004086
Obs.f. O_K	126	43	22	3	5	1	0
Exp.f. E_K	120	48	19.2	7.68	3.072	1.2288	0.8172
$X^2 = \frac{(O_K - E_K)^2}{E_K}$	0.3	0.5208	0.4083	2.852	Combining As $E_K \geq 5$		

New $O_K = 6$, $E_K = 5.12$ $\Big\}$ $X^2 = \frac{(6-5.12)^2}{5.12} = 0.15125$

$$X^2 = \Sigma X^2 = 4.23 \quad \text{---} \quad (1)$$

Degree of freedom $\nu = 5-1 = 4$; Significance level 5%, for $p > 0.95$

Critical value $= 9.49$ —— (2) (for X^2 distribution)

$4.23 < 9.43$. H_0 is accepted.

So distribution Geo(0.6) is a satisfactory fit.

3. Apples are sold in bags of 5. Based on her previous experience, Freya claims that the probability of any apple weighing more than 100 grams is 0.35, independently of other apples in the bag.

The apples in a random sample of 150 bags are checked and the number, x, in each bag weighing more than 100 grams is recorded. The results are shown in the following table:

x	0	1	2	3	4	5
Frequency	12	39	46	37	12	4

Carry out a goodness of fit test at 5% significance level and hence comment on Freya's claim. ---[7]

[W-20/41/Q3]

solution: H_0: The number of apples x in bag, weighing >100 gram follows Binomial distribution $B(n,p)$ $B(5, 0.35)$

H_1: x does not follow Binomial distribut.

$B(5, 0.35)$, $p = 0.35$, $q = 0.65$

No. of bags $N = 150$

$P(r) = {}^nC_r \cdot p^r q^{n-r} = {}^5C_r (.35)^r \cdot q^{(5-r)}$ $(r = 0,1,2,3,4,5)$

x	0	1	2	3	4	5
Obs. fre. O_K	12	39	46	37	(12 +	4) = 16
Bino Prob. P_K	0.11603	0.31236	0.33642	0.18115	0.04877	0.00525
Expected fre E_K	17.404	46.458	50.462	27.172	(7.316 +	0.7878 < 5
$\chi^2 = \frac{(O_K - E_K)^2}{E_K}$	$\frac{(12-17.404)^2}{17.404}$	$\frac{(39-46.458)^2}{46.858}$	→	→	= 8.104 (combine) $E_K \geq 5$; → $\frac{(16-8.104)}{8.104}$	
χ^2	✓	✓	✓	✓	✓	

$\Sigma \chi^2 = 14.65$

Degree of fredom $\nu = 5 - 1 = 4$; 5% Significance leve $p > 0.95$

Critical value $= 9.49$ (for χ^2 distribution)

Now $14.65 > 9.45 \Rightarrow$ Freya' claim is not supported.

or (Data does not fit the distribution)

4. A supermarket sells pears in packs of 8. Some of the pears in a pack may not be ripe and the supermarket manager claims that the number of unripe pears in a pack can be modelled by the distribution B(8, 0.15)

A random sample of 150 packs was selected and the number of unripe pears in each pack was recorded. The following table shows observed frequencies together with some of expected frequencies using the manager's binomial distribution.

Number of unripe pears per pack	0	1	2	3	4	5	≥ 6
Observed freq. O_K	35	48	43	15	6	3	0
Expected freq. E_K	40.874	p	35.641	12.579	2.775	0.392	q

(a) Find the values of p and q. -- [2]

(b) Carry out a goodness of fit test, at the 5% significance level, to test whether the manager's claim is justified. -- [6]

W-21/41/Q3

H_0: Number of unripe pears per pack follow binomial Dis.

Solution: $B(8, 0.15) \rightarrow n = 8, p = 0.15, q = 0.85, P(x) = {}^nC_x p^x q^{n-x}$

$P(x=1) = {}^8C_1 \cdot (0.15)^1 \cdot (0.85)^7 = 0.3847 \Rightarrow E(1) = 0.3847 \times 150 = 57.704$

$q = 1 - \sum_{K=0}^{5} E_K = 0.035$, $p = 57.704$

x	0	1	2	3	4	5	≥ 6
O_K	35	48	43	15	6	3	0
E_K	40.874	57.704	35.641	12.579	2.775	0.392	0.035
$W^2 = \frac{(O_K - E_K)^2}{E_K}$	0.8444	1.6319	1.5199	4.2803			

$E_K < 5$ Combine for $O_K = 24$ & $E_K = 15.7812$

Now $n = 4$, $E_K \geq 5$

$\sum W^2 = 8.24$

Degree of fredom $\nu = 4 - 1 = 3$ & for 5% Significant level $p = 0.95$

Using χ^2 test, critical value $= 7.815$; $8.24 > 7.815$

$\therefore$ Reject H_0.

Insufficient evidence to support manager's claim.

5. It is claimed that the heights of a particular age group of boys follow a normal distribution with mean 125 cm and standard deviation 12 cm. Observations for a randomly chosen group of 60 boys in this age group are summarised in the following table. Table also gives the expected frequencies.

Height x cm	$x < 100$	$100 \le x < 110$	$110 \le x < 120$	$120 \le x < 130$	$130 \le x < 140$	$x \ge 140$
Observed fre O_K	0	3	15	23	11	8
Expected fre E_K	1.12	5.22	13.97	19.38	13.97	6.34

(a) Show how the expected frequency for $130 \le x < 140$ is obtained. --[2]

(b) Carry out a goodness of fit test, at the 5% significance level, to determine whether the claim is supported by the data $\boxed{W\text{-}21/42/Q2}$ -- [6]

Solution (a) $P(130 \le x < 140) = P\left(\frac{130-125}{12} \le Z < \frac{140-125}{12}\right) = P(0.41167 < z < 1.25)$

$= 0.8944 - 0.6616 = 0.2328$

$n = 60 \rightarrow$ $\therefore E_K$ for $130 \le x \le 140) = n \cdot p_K = 60 \times 0.2328 = \underline{13.968}$ ✓

First Value $E_K = 1.12 < 5$; hence combining the first two values $\left\{\begin{array}{l} O_K = 0+3 = 3 \\ E_K = 1.12 + 5.22 \\ \quad = 6.34 \ge 5 \end{array}\right.$

Obs. fre O_K	$(0+3) = 3$	15	23	11	8
Expect fre E_K	$(1.12+5.22) = 6.34$	13.97	19.38	13.97	6.34
$W^2 = \frac{(O_K - E_K)^2}{E_K}$	1.7588	0.0761	0.6743	0.6309	0.4352

$\Sigma W^2 = 3.58$

H_0: Normal distribution fits the data - $N(125, 12^2)$.

Degree of freedom $\nu = (n-1) = 5-1 = 4$

At 5% significance level, for $p = 0.95$ } Critical value $= 9.488$

$3.58 < 9.488 \Rightarrow$ Accept H_0

Hence there is sufficient evidence that the given normal distribution fits the data.

6. A random sample of 200 observations of the continuous random variable X was taken and the values are summarised in the table:

Interval	$0 \le x < 0.5$	$0.5 \le x < 1$	$1 \le x < 1.5$	$1.5 \le x < 2$	$2 \le x < 2.5$	$2.5 \le x < 3$
Obs. fre O_K	5	23	40	41	46	45

It is required to test the goodness of fit of the distribution with prob. density function f given by:

$$f(x) = \begin{cases} \frac{1}{9}x(4-x) & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Most of the relevant expected frequencies, correct to 2 decimal places, are given in the table:

Interval	$0 \le x < 0.5$	$0.5 \le x < 1$	$1 \le x < 1.5$	$1.5 \le x < 2$	$2 \le x < 2.5$	$2.5 \le x < 3$
Expec. fre E_K	p	q	37.96	43.52	43.52	37.96

(a) Show that $p = 10.19$ and find the value of q. ---[3]

(b) Carry out a goodness of fit test, at the 5% significance level, to test whether f is a satisfactory model for the data. ---[4]

[W-20/42/Q3]

Solution (a) p_K for $0 \le x < 0.5 = \frac{1}{9}\int_0^{0.5} (4x - x^2)\,dx = \frac{1}{9}\left[2x^2 - \frac{x^3}{3}\right]_0^{0.5} = 0.050926$

$\therefore p = p_K \times 200 = 0.050926 \times 200 = \underline{10.19}$ ✓ ; $q = 200 - (\text{Sum of rest of } E_K)$

$(n = 200)$ or $\underline{q = 26.85}$ ✓

(b)

O_K	5	23	40	41	46	45
E_K	10.19	26.85	37.96	43.52	43.52	37.96

$\chi^2 = \frac{(O_K - E_K)^2}{E_K} = 2.6433 + 0.5520 + 0.1096 + 0.1459 + 0.1413 + 1.3056$

$= 4.89$ Test Statistic.

Now degree of fredom $\nu = 6 - 1 = 5$ and at 5% Significance level, critical value $= 11.07$.

$4.89 < 11.07$ (H_0 is acceptable)

$\therefore$ PDF is a satisfactory model for the data.

Contingency tables / Independent (two variables)

§ Expected value for the entry in i_{th} row and j_{th} column.

$$E_{ij} = \frac{\text{Total in } i_{th} \text{ row} \times \text{total in } j_{th} \text{ column}}{\text{Total frequency.}}$$

degree of freedom $\nu = (R-1)(C-1)$ $\begin{cases} R = \text{No of rows} \\ C = \text{No. of Columns.} \end{cases}$

7. Two randomly selected groups of students, with similar ranges of abilities, take the same examination in different roots. One group of 140 students takes the examination with background music playing. The other group of 210 students takes the examination in silence. Each student is awarded a grade for their performance in the examination and the members from each group gaining each grade are shown in the table:

	Grade awarded			
	A	B	C	Total
Background music	49	51	40	140
Silence	93	68	49	210
Total	142	119	89	350

Test at 10% significance level whether grade awarded are <u>independent</u> of whether background music is playing during the examination. ...[6]

[S-20/41/Q1]

Solution:

	A	B	C
E_K	$\frac{140\times142}{350} = 56.8$	$\frac{119\times140}{350} = 47.6$	35.6
E_K	$\frac{210\times142}{350} = 85.2$	71.4	53.4

$X^2 = ($

$$X^2 = \frac{(O_K - E_K)^2}{E_K} = \frac{(49-56.8)^2}{56.8} + \ -- \ + -- + \frac{(93-85.2)^2}{85.2} + \ -- + \ --$$

$$= 1.0711 + \ -- \ + -- + 0.7140 + \ - \ + \ --$$

Test Statistic $\rightarrow$ $\Sigma X^2 = 3.096$ ✓

degree of freedom $\nu = (3-1)(2-1) = 2\times1 = 2$

At 10% significance level $p = 0.9$ (90%)

Using X^2 table $\rightarrow$ Critical value $= 4.605$

Now $3.096 < 4.605$ (H_0 is accepted)

Hence the grades awarded are <u>independent</u> of the background.

8. Young children are learning to read using two different reading schemes, A and B. The standards achieved are measured against the national average standard achieved and classified as above average, average and below average. For two randomly chosen groups of young children, the number in each category are shown in the table:

	Standard achieved		
	Above average	Average	Below averge
Scheme A	31	35	22
Scheme B	19	50	43

Test at 5% significance level whether standard achieved is independent of the reading scheme used. [S-20/43/Q1] --[6]

Solution:

				Total
Scheme A O_K	31	35	22	88
Scheme B O_K	19	50	43	112
Total	50	85	65	200

Scheme A, E_K	$\frac{88\times 50}{200}=22$	$\frac{88\times 85}{200}=37.4$	28.6
Scheme B, E_K	$\frac{112\times 50}{200}=28$	$\frac{112\times 85}{200}=47.6$	36.6

$\chi^2 = \frac{(O_K-E_K)^2}{E_K}$: $\frac{(31-22)^2}{22}, \frac{(35-37.4)^2}{37.4}, \frac{(22-28.6)^2}{28.6}, \frac{(19-28)^2}{28}, \frac{(50-47.6)^2}{47.6}, \frac{(43-36.6)^2}{36.6}$

$\Sigma \chi^2 = 9.569$ ✓

Degree of freedom $\nu = (3-1)(2-1) = 2\times 1 = 2$

and at 5% significance level for $p=0.95$,

Using χ^2 table → Critical value $= 5.991$

Now $9.569 > 5.991$

Hence the standard achieve is not independent (dependent) on the reading scheme used.

9. A driving school employs four instructors to prepare people for their driving test. The allocation of people to instructors is random. For each of the instructors, the following table gives the number of people who passed and who failed their driving test last year.

	Instructor A	Inst B	Inst C	Inst D	Total
Pass	72	42	52	68	234
Fail	33	34	41	58	166
Total	105	76	93	126	400

Test at 10% significance level whether success in the driving test is independent of the instructor. [S-21/41/Q2] --- [7]

Solution:

	A	B	C	D
Pass	E_K $\frac{234\times105}{400}=61.425$	$\frac{76\times234}{400}=44.46$	54.405	73.71
Fail	E_K $\frac{105\times166}{400}=43.575$	$\frac{76\times166}{400}=31.54$	38.595	52.29
$X^2 = \frac{(O_K-E_K)^2}{E_K}$	$\frac{(72-61.425)^2}{61.425} = 1.8206$	0.1361	0.1063	$(68-73.71)^2/73.71 = 0.4423$

for Fail, 2.5664 ; 0.1919, 0.1499, 0.6235

$\Sigma X^2 = 6.037$ (Test statistic)

H_0: driving test success is independent of instructor.

$\left\{ \begin{array}{l} \text{Now 10\% significance level (for } p = 0.90) \\ \text{degree of fredom} = (4-1)(2-1) \Rightarrow \nu = 3 \end{array} \right.$

Using X^2 table; Critical value = 6.251

Now $6.037 < 6.251 \Rightarrow H_0$ is accepted.

Hence the driving test success is independent of instructor.

10 Chai packs china mugs into cardboard boxes. Chai's manager suspects that breakages occur at random times and that the number of breakages may follow a Poisson distribution. He takes a small sample of observations and finds that the number of breakages in one-hour period has a mean of 2.4 and standard deviation 1.5.

(a) Explain how this information tends to support the manager's suspicion. [2]

The manager now takes a larger sample and claims that the number of breakages in one-hour period follow a Poisson distribution. The number of breakages in a random sample of 180 one-hour periods are summarised in the following table.

Number of breakages	0	1	2	3	4	5	6	7 or more
Frequency (O_k)	21	33	46	31	23	16	10	0

The mean number of breakages calculated from this sample is 2.5.

(b) Use the data from this larger sample to carry out a goodness fit of test, at the 10% significance level, to test the claim. ---[8]

S-21/43/Q 5

Solution:

(a) Mean of the distribution $\lambda = 2.4$ and Variance $= \sigma^2 = (1.5)^2 = 2.25$ } mean approx. equal to variance, so the Poisson distribution might be suitable.

(b) $P(2.5)$ leads to the frequences; $n = 180$ } $P_k = \dfrac{e^{-\lambda}\lambda^k}{k!}$, $P_0 = e^{-2.5} = 0.08208$

$E_k \rightarrow$ 14.775, 36.938, 46.173, 38.477, 24.048, 12.024, 5.010, 2.554

$E_k(0) = 180 \times .08208 = 14.77$

Combine last two classes { $O_k = 10 + 0 = 10$, $E_k = 5.010 + 2.554 = 7.564$

$$\Sigma \chi^2 = \Sigma \frac{(O_k - E_k)^2}{E_k} = \frac{(21-14.775)^2}{14.775} + \cdots +$$

$$= 2.6227 + 0.4198 + 0.00065 + 1.4529 + 0.04570 + 1.3148 + 0.7845$$

$\Sigma \chi^2 = 6.64$ (Test Statistic)

H_0: Distribution fits the data:

degree of fredom $\nu = (6-1)(2-1) = 5$, At 10% Significance level ($p = 0.9$). } $\Rightarrow$ Using χ^2 table the critical value $= 9.236$

Now $6.64 < 9.236 \Rightarrow$ Accept H_0.

Hence data follow the Poisson distribution.