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Pure Maths - 1

Differentiation

Exercise 1 Solution (Revision)

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Differentiation - Revision

Example 2. A curve has equation $y = \frac{12}{3-2x}$

(a) Find $\frac{dy}{dx}$. --- [2]

A point moves along this curve. As the point passes through A, the x-coordinate is increasing at a rate of 0.15 units per seconds and the y-coordinate is increasing at a rate of 0.4 units per seconds.

(b) Find the possible x-coordinates of A. $\boxed{SP-20/01/Q8}$ --- [4]

Solution (a) $y = \frac{12}{(3-2x)}$ or $y = (3-2x)^{-1}$

$\frac{dy}{dx} = 12(-1)(3-2x)^{-2} \times (-2)$

or $\frac{dy}{dx} = \frac{24}{(3-2x)^2}$ —— ①

(b) $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{0.4}{0.15} = 8/3$ —— ②

from ① and ②

$\Rightarrow \frac{24}{(3-2x)^2} = 8/3$

$\Rightarrow (3-2x)^2 = 9 \Rightarrow 3-2x = \pm 3$

$\Rightarrow x = 0$ or $x = 3$ ✓

Example 1. The following points,

A(0,1), B(1,6), C(1.5, 7.75), D(1.9, 8.79) and E(2,9)

lie on the curve $y = f(x)$. The table below shows the gradient of the chords AE and BE

Chord	AE	BE	CE	DE
Gradient of the Chord	4	3	--	--

$\boxed{SP-20/01/Q1}$

(a) Comple the table to show the gradient of CE and DE. --- [2]

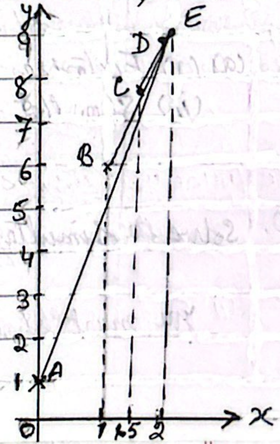
(b) State what the values in the table indicate about the value of $f'(2)$ --- [1]

Solution (a) Gradient of $CE = \frac{9-7.75}{2-1.5} = 2.5$ ✓

Gradient of $DE = \frac{9-8.79}{2-1.9} = 2.1$ ✓

(b) $x \to 2$, gradient at $E \to 2$

$\therefore f'(2) = 2$



Example 3: The function f is defined by $f(x) = \frac{1}{3x+2} + x^2$ for $x < -1$, Determine whether f is an increasing function, a decreasing function or neither. [M-20/12/Q 1] --[3]

Solution: $f(x) = (3x+2)^{-1} + x^2$ for $x < -1$

$f'(x) = -(3x+2)^{-2} \times 3 + 2x$

$= \frac{-3}{(3x+2)^2} + 2x < 0$

as $(3x+2)^2 > 0$ for $x < -1$ $\Rightarrow \frac{-3}{(3x+2)^2} < 0$

and $2x < 0$ as $x < -1$

So is there sum.

$\therefore$ $f(x)$ is decreasing function. ✓

Example 4: A curve has equation $y = x^2 - 2x - 3$. A point is moving along the curve in such a way that at P the y-coordinate is increasing at 4 units per second and the x-coordinate is increasing at 6 units per second. Find the x-coordinate of P. [M-20/12/Q 4] --[4]

Solution: $y = x^2 - 2x - 3$ —— ①

$\frac{dy}{dx} = 2x - 2$ —— ②

$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{4}{6}$ —③ [$\because$ Given $\frac{dy}{dt} = 4$; $\frac{dx}{dt} = 6$]

from ② and ③ $2x - 2 = \frac{4}{6} \Rightarrow x - 1 = 1/3$

$\Rightarrow x = 1 + \frac{1}{3} = \frac{4}{3}$ ✓

Example 5: The gradient of a curve at the point (x, y) is given by $\frac{dy}{dx} = 2(x+3)^{1/2} - x$. The curve has a stationary point at $(a, 14)$, where a is a positive constant.

(a) Find the value of a. --[3]

(b) Determine the nature of the stationary point. [M-20/12/Q 10] --[3]

Solution: $\left(\frac{dy}{dx}\right)_{x=a} = 2(a+3)^{1/2} - a = 0$

$\Rightarrow 4(a+3) = a^2$

$\Rightarrow a^2 - 4a - 12 = 0$

$(a-6)(a+2) = 0$

$a = 6$, $a = -2$ ✗ ($\because a > 0$)

$\frac{dy}{dx} = 2(x+3)^{1/2} - x$

$\frac{d^2y}{dx^2} = (x+3)^{-\frac{1}{2}} - 1$

$\left(\frac{d^2y}{dx^2}\right)_{x=6} = 9^{-\frac{1}{2}} - 1 = -\frac{2}{3} < 0$

$\therefore$ Max.

6. A curve is such that $\frac{dy}{dx} = \frac{6}{(3x-2)^3}$ and $A(1,-3)$ lies on the curve. A point is moving along the curve and at A, the y-coordinate of the point is increasing at 3 units per second.

Find the rate of increase at A of the x-coordinate of the point. ---[3]

[M-21/12/Q6(a)]

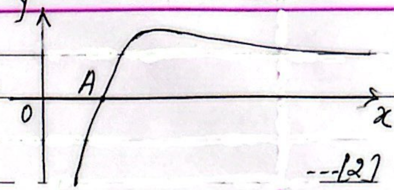
Solution: $\frac{dy}{dx} = \frac{6}{(3x-2)^3} \Rightarrow \frac{dy}{dx}$ at A, $\left(\frac{dy}{dx}\right)_{(1,-3)} = \frac{6}{(3\times1-2)^2} = 6$ ---①

and $\frac{dy}{dt} = 3$ (Given) ---②, $\left(\frac{dx}{dt}\right)_{at\,A} = ?$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \Rightarrow 3 = 6 \times \left(\frac{dx}{dt}\right) \Rightarrow \frac{dx}{dt} = \frac{3}{6} = \frac{1}{2}$ ✓

(from ① and ②)

7. The diagram shows the curve with equation $y = 9(x^{-\frac{1}{2}} - 4x^{-3/2})$. The curve crosses x-axis at the point A.



(a) Find the x-coordinate of A. --- [2]

(b) Find the equation of the tangent to the curve at A. -- [4]

(c) Find the x-coordinate of the maximum point of the curve. -- [2]

M-21/12/Q11

Solution(a): $y = 9(x^{-1/2} - 4x^{-3/2})$ --- ①

Curve intersects X-axis $\Rightarrow y=0$

$\Rightarrow 9(x^{-1/2} - 4x^{-3/2}) = 0$

$\Rightarrow 9x^{-\frac{1}{2}}[1 - 4x^{-1}] = 0$

$x^{-1/2} \neq 0,\ 1 - \frac{4}{x} = 0 \Rightarrow x = 4.$

$\therefore$ Curve intersects X-axis at A, $x=4$

(b) diff ① $\frac{dy}{dx} = 9[-\frac{1}{2}x^{-3/2} - 4\times(-\frac{3}{2})x^{-5/2}]$

$= 9 \cdot x^{-3/2}[-\frac{1}{2} + \frac{6}{x}]$ —— ②

at A, $\left(\frac{dy}{dx}\right)_{(4,0)} = 9 \times \frac{1}{4^{3/2}}\left[-\frac{1}{2} + \frac{6}{4}\right] = \frac{9}{8}$ ✓

$\therefore$ Eqn of the tangent to the curve at A(4,0)

$y - 0 = \frac{9}{8}(x-4) \Rightarrow y = \frac{9}{8}x - \frac{9}{2}$ ✓

(c) For any stationary point (maximum) $\frac{dy}{dx} = 0$

from ②

$9x^{-3/2}[-\frac{1}{2} + \frac{6}{x}] = 0$

$\Rightarrow \frac{6}{x} = \frac{1}{2}$

$\Rightarrow x = 12,$

8. It is given that a curve has equation $y = k(3x-k)^{-1} + 3x$, where k is a constant.

(a) Find, in terms of k, the values of x at which there is a stationary point. ---[4]

The function f has a stationary value at $x = a$ and is defined by $f(x) = 4(3x-4)^{-1} + 3x$ for $x \geq 3/2$

(b) Find the value of a and determine the nature of the stationary value. ---[3]

(c) The function g is defined by $g(x) = -(3x+1)^{-1} + 3x$ for $x \geq 0$

Determine, making your reasoning clear, whether g is an increasing function, a decreasing function or neither. ---[2]

M-22/12/Q11

Solution: (a) $y = k(3x-k)^{-1} + 3x$

$\frac{dy}{dx} = -k(3x-k)^{-2} \times 3 + 3$

for stationary point $\frac{dy}{dx} = 0$

$\Rightarrow \frac{-3k}{(3x-k)^2} + 3 = 0$

$\Rightarrow 3(3x-k)^2 = 3k$

$\Rightarrow 3x - k = \pm\sqrt{k}$

$\Rightarrow x = \frac{k \pm \sqrt{k}}{3}$ ✓

(b) $f(x) = 4(3x-4)^{-1} + 3x$ for $x \geq 3/2$

$f'(x) = -4(3x-4)^{-2} \times 3 + 3$ — (1)

for stationary point at $x = a$

$\frac{-4 \times 3}{(3a-4)^2} + 3 = 0$

$\Rightarrow (3a-4)^2 = 4$

$3a - 4 = \pm\sqrt{4}$

$3a = 6$ or 2

$a = 2, \cancel{2/3}$ $x \geq 3/2$

$a = 2$ ✓

Diff (1) $f''(x) = 24(3x-4)^{-3} \times 3$

$= \frac{72}{(3x-4)^3}$

$f''(2) = \frac{72}{2^3} = 9 > 0$

$\therefore$ at $x = 2$, Minimum.

(c) $g(x) = -(3x+1)^{-1} + 3x$ for $x \geq 0$

$g'(x) = +(3x+1)^{-2} \times 3 + 3$

$= \frac{3}{(3x+1)^2} + 3 > 0$

for all values of x. ✓

$\therefore$ $g(x)$ is increasing for $x \geq 0$

9. A curve has equation $y=\frac{1}{60}(3x+1)^2$ and a point is moving along the curve. Find the x-coordinate of the point on the curve at which the x- and y-coordinates are increasing at the same rate. --[4]

[M-23/12/Q3]

Solution: $y=\frac{1}{60}(3x+1)^2$ ---①

$\frac{dy}{dx}=\frac{1}{60}\times 2(3x+1)\times 3=\frac{1}{10}(3x+1)$ --②

Give $\frac{dy}{dt}=\frac{dx}{dt} \Rightarrow \frac{dy}{dt}\Big/\frac{dx}{dt}=1 \Rightarrow \frac{dy}{dx}=1 \Rightarrow \frac{1}{10}(3x+1)=1$ from ②

$\Rightarrow 3x+1=10 \Rightarrow \underline{x=3}$ ✓

10. At the point $(4,-1)$ on a curve, the gradient of the curve is $-\frac{3}{2}$. It is given that $\frac{dy}{dx}=x^{-1/2}+k$, where k is a constant.

(a) Show that $k=-2$ (b) Find an equation of the curve. [1]+[4]

(c) Find the coordinates of the stationary points. (d) Determine the nature of the stationary points. [3]+[2]

[M-23/12/Q10]

Solution: $\frac{dy}{dx}=x^{-\frac{1}{2}}+k$ ---①

(a) Gradient at $(4,1)=\left(\frac{dy}{dx}\right)_{x=4}=4^{-1/2}+k=-\frac{3}{2}$ (given)

$\Rightarrow \frac{1}{2}+k=-\frac{3}{2} \Rightarrow \underline{k=-2}$ ✓

(b) from ① $y=\int(x^{-1/2}-2)\,dx$ [as $k=-2$]

$=\frac{x^{1/2}}{1/2}-2x+c=2\sqrt{x}-2x+c$

$y=2\sqrt{x}-2x+c$ ---②

Passes through $(4,-1) \Rightarrow -1=2\sqrt{4}-2\times 4+c$

$\Rightarrow c=3$ ✓

∴ from ② Equation of the curve is:

$\underline{y=2\sqrt{x}-2x+3}$ ✓ ---③

(c) for stationary points $\frac{dy}{dx}=0$

from ① $\frac{dy}{dx}=x^{-1/2}-2=0$ (for $k=-2$)

$\Rightarrow x^{-1/2}=2 \Rightarrow x=\frac{1}{4}$

at from ③ $y=2\sqrt{\frac{1}{4}}-2\times\frac{1}{4}+3=\frac{7}{2}$

∴ Station point $\underline{\left(\frac{1}{4}, 3\frac{1}{2}\right)}$ ✓

(d) Diff ① $\frac{d^2y}{dx^2}=-\frac{1}{2}x^{-3/2}$

$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=\frac{1}{4}}=-\frac{1}{2}\left(\frac{1}{4}\right)^{-3/2}=-4<0$ ✓

∴ $\underline{\text{Maximum at}}\left(\frac{1}{4}, 3\frac{1}{2}\right)$

Example 11. The equation of a curve is $y=(3-2x)^3+24x$.

(a) Find the expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ --[4]

(b) Find the coordinates of each of the stationary points on the curve. --[3]

(c) Determine the nature of each stationary point. [S-20 | 11 | Q9] --[2]

Solution:

(a) Given $y=(3-2x)^3+24x$ —— ①

diff. $\frac{dy}{dx}=3(3-2x)^2\times(-2)+24$

$\frac{dy}{dx}=-6(3-2x)^2+24$ —— ②

diff. $\frac{d^2y}{dx^2}=-12(3-2x)\times(-2)$

$=24(3-2x)$ —— ③

(b) for stationary points $\frac{dy}{dx}=0$

for ② $\Rightarrow$ $-6(3-2x)^2+24=0$

$(3-2x)^2=4 \Rightarrow 3-2x=\pm 2$

$\Rightarrow$ $x=\frac{1}{2}$ or $x=5/2$

$\therefore$ Stationary point $(\frac{1}{2}, 20), (2\frac{1}{2}, 52)$.

(c) at $(\frac{1}{2}, 20)$ from ③

$\left(\frac{d^2y}{dx^2}\right)_{x=\frac{1}{2}}=24(3-2\times\frac{1}{2})=48>0$ ✓

$\therefore$ Minimum

and at $(2\frac{1}{2}, 52)$

$\left(\frac{d^2y}{dx^2}\right)_{x=5/2}=-48<0$ $\therefore$ Maximum ✓

Example 12. A weather balloon in the shape of a sphere is being inflated by a pump. The volume of the balloon is increasing at a constant rate of 600 cm³ per second. The balloon was empty at the start of pumping.

(a) Find the radius of the balloon after 30 seconds. --[2]

(b) Find the rate of increase of the radius after 30 seconds. --[3]

[S-20 | 12 | Q3]

Solution (a) Volume after 30 seconds $=600\times 30=18000$ cm³

$V=\frac{4}{3}\pi r^3=18{,}000$

$r^3=\frac{18{,}000\times 3}{4\pi}=4297$

$r=\sqrt[3]{4297}=16.257 = 16.3$ cm ✓

(b) $\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}$

$\left[V=\frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr}=4\pi r^2 \right.$

$\frac{dV}{dt}=600$

$\Rightarrow \frac{dr}{dt}=\frac{dV}{dt}\Big/\frac{dV}{dr}=\frac{600}{4\pi r^2}$

$\left(\frac{dr}{dt}\right)_{r=16.257}=\frac{600}{4\pi\cdot(16.257)^2}=0.181$ cm s⁻¹ ✓

Example 13: The equation of a curve is $y = 54x - (2x-7)^3$

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ ---[4]

(b) Find the coordinates of each of the stationary points of the curve. --[3]

(c) Determine the nature of each of the stationary points. $\boxed{S-20/12/Q10}$ --[2]

Solution: $y = 54x - (2x-7)^3$ —— ①

(a) $\frac{dy}{dx} = 54 - 3(2x-7)^2 \times 2$

$= 54 - 6(2x-7)^2$ —— ②

$\frac{d^2y}{dx^2} = -6 \times 2(2x-7) \times 2$

$= -24(2x-7)$ —— ③

(b) for stationary point $\frac{dy}{dx} = 0$

$\Rightarrow 54 - 6(2x-7)^2 = 0 \Rightarrow (2x-7)^2 = 9$

$\Rightarrow 2x-7 = \pm 3 \Rightarrow x = 5, x = 2$ ✓

from ① $(5, 243)$, $(2, 135)$ are the stationary points.

(c) at $(5, 243)$

$\left(\frac{d^2y}{dx^2}\right)_{x=5} = -24(10-7) = -72 < 0$

Max. ✓

and at $(2, 135)$

$\left(\frac{d^2y}{dx^2}\right)_{x=2} = -24(4-7) = 72 > 0$

Min ✓

Example 14. A point P is moving along a curve in such a way that the x-coordinate of P is increasing at a constant rate of 2 units per minute. The equation of the curve is $y = (5x-1)^{\frac{1}{2}}$.

(a) Find the rate at which the y-coordinate is increasing when $x = 1$. --[4]

(b) Find the value of x when the y-coordinate is increasing at 5/8 units per minute. $\boxed{S-20/13/Q6}$ --[3]

Solution: $y = (5x-1)^{\frac{1}{2}}$

(a) $\frac{dy}{dx} = \frac{1}{2}(5x-1)^{-\frac{1}{2}} \times 5$

$= \frac{5}{2\sqrt{5x-1}}$ —— ①

$\left(\frac{dy}{dx}\right)_{x=1} = \frac{5}{4}$ —— ②

Now $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$= \frac{5}{4} \times 2$

$= \frac{5}{2}$ ✓

$\left\{ \text{fm ② and } \frac{dx}{dt} = 2 \right.$

(b) Now given $\frac{dy}{dt} = \frac{5}{8}$

$\frac{dy}{dx} \times \frac{dx}{dt} = \frac{dy}{dt}$

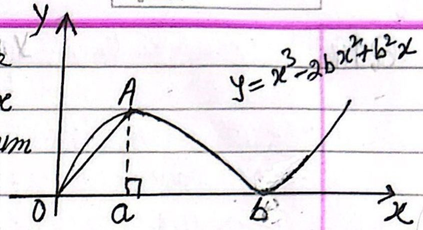
$\frac{5}{2\sqrt{5x-1}} \times 2 = \frac{5}{8}$ $\left\{ \text{for ① and } \frac{dx}{dt} = 2 \right.$

$\Rightarrow \sqrt{5x-1} = 8$

Sq $\rightarrow$ $5x - 1 = 64$

$x = 13$ ✓

Example 15 The diagram shows part of the curve with equation $y = x^3 - 2bx^2 + b^2x$ and the line OA, where A is the maximum point on the curve. The x-coordinate of A is a and the curve has a minimum point at $(0, b)$, where a and b are positive constants.



Show that $b = 3a$. $\boxed{S\text{-}20/13/Q\,11(a)}$ [4]

Solution: $y = x^3 - 2bx^2 + b^2x$

$$\frac{dy}{dx} = 3x^2 - 4bx + b^2$$

for stationary point $\frac{dy}{dx} = 0$

$$\Rightarrow 3x^2 - 4bx + b^2 = 0$$

$$(3x - b)(x - b) = 0 \Rightarrow x = \frac{b}{3} \text{ or } b$$

$$\therefore a = \frac{b}{3} \Rightarrow b = 3a$$

16. The equation of a curve is $y = 2\sqrt{3x+4} - x$

(a) Find the equation of the normal to the curve at the point (4,4), giving your answer in the form $y = mx + c$. ---[5]

(b) Find the coordinates of the stationary point. ---[3]

(c) Determine the nature of the stationary point. ---[2]

S-21/11/Q 11

Solution (a) $y = 2\sqrt{3x+4} - x$ —— ①

$\frac{dy}{dx} = 2 \times \frac{1}{2\sqrt{3x+4}} \times 3 - 1$

or $\frac{dy}{dx} = \frac{3}{\sqrt{3x+4}} - 1$ ---- ②

$\left(\frac{dy}{dx}\right)_{(4,4)} = \frac{3}{\sqrt{16}} - 1 = -\frac{1}{4}$

∴ Gredient of Normal at $(4,4) = 4$.

∴ Equation of the normal at (4,4),

$y - 4 = 4(x-4) \Rightarrow y = 4x - 12$ ✓

(b) For stationary point $\frac{dy}{dx} = 0$

from ② $\frac{3}{\sqrt{3x+4}} - 1 = 0$

$\Rightarrow 3 - \sqrt{3x+4} = 0$

$\Rightarrow \sqrt{3x+4} = 3 \Rightarrow 3x+4 = 9$

$\Rightarrow x = 5/3$ ✓

from ① $y = 2\sqrt{3 \times \frac{5}{3} + 4} - \frac{5}{3} = \frac{13}{3}$

∴ Station point $(5/3, 13/3)$ ✓

(c) diff ②

$\frac{d^2y}{dx^2} = 3 \times \left(-\frac{1}{2}\right) \times 3 (3x+4)^{-3/2}$

$= \frac{-9}{2(3x+4)^{3/2}}$

$\left(\frac{d^2y}{dx^2}\right)_{x=5/3} = -\frac{9}{2} \times \frac{1}{3^{3/2}} < 0$, Max

∴ Max at $(5/3, 13/3)$ ✓

17 The gradient of the curve is given by $\frac{dy}{dx} = 6(3x-5)^3 - kx^2$, where k is a constant. The curve has a stationary point at $(2, -3.5)$.

(a) Find the value of k. ---[2]

(b) Find the equation of the curve. ---[4]

(c) Find $\frac{d^2y}{dx^2}$ ---[2]

(d) Determine the nature of the stationary point at $(2, -3.5)$

[S-21 / 12 / Q11] ---[2]

Solution:

(a) $\frac{dy}{dx} = 6(3x-5)^3 - kx^2$ --- (1)

stationary point at $(2, -3.5)$

from (1) $\left(\frac{dy}{dx}\right)_{x=2} = 0$

$\Rightarrow 6(3\times2-5)^3 - k\times2^2 = 0$

$\Rightarrow 6 - 4k = 0 \Rightarrow k = 3/2$

(b) To find the equation of the curve

from (1) $y = \int [6(3x-5)^3 - kx^2]\,dx$

$y = \frac{6(3x-5)^4}{3\times4} - \frac{kx^3}{3} + C$

for $k = \frac{3}{2}$

$y = \frac{1}{2}(3x-5)^4 - \frac{3}{2}\cdot\frac{x^3}{3} + C$ --- (2)

Passes through a point $(2, -3.5)$

$\Rightarrow -3.5 = \frac{1}{2}(3\times2-5)^4 - \frac{1}{2}\times2^3 + C \Rightarrow C = 0$

from (2) Curve is: $y = \frac{1}{2}(3x-5)^4 - \frac{1}{2}x^3$ --- (3)

(c) Diff (1)

$\frac{d^2y}{dx^2} = 3\times6(3x-5)^2\times3 - 2kx$

$= 54(3x-5)^2 - 2\times\frac{3}{2}x \quad [k = \frac{3}{2}]$

$= 54(3x-5)^2 - 3x$ --- (4)

(d) from (4) at $(2, -3.5)$

$\left(\frac{d^2y}{dx^2}\right)_{x=2} = 54\times1^2 - 3\times2 = 48 > 0$; Minimum

18. The function f is defined by $f(x) = \frac{1}{3}(2x-1)^{3/2} - 2x$ for $\frac{1}{2} < x < a$.
It is given that f is a decreasing function.
Find the maximum possible value of the constant a. ---[4]

[S-21/13/Q2]

Solution: $f(x) = \frac{1}{3}(2x-1)^{3/2} - 2x \Rightarrow f'(x) = \frac{1}{3} \times \frac{3}{2} \times 2(2x-1)^{1/2} - 2$

$= (2x-1)^{1/2} - 2$

for f(x) to be decreasing → $f'(x) \leq 0 \Rightarrow (2x-1)^{1/2} - 2 \leq 0$

$\Rightarrow (2x-1)^{1/2} \leq 2$

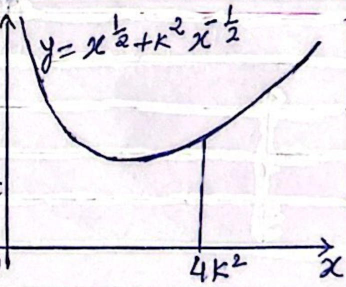
$\Rightarrow 2x - 1 \leq 4$

$\Rightarrow x \leq 5/2$

$\therefore$ Max value of $a = 5/2$

$a = 2\frac{1}{2}$ ✓

19. The diagram shows part of the curve with equation $y = x^{\frac{1}{2}} + k^2 x^{-\frac{1}{2}}$, where k is a positive constant.



(a) Find the coordinates of the minimum point of the curve, giving your answer in terms of k. ---[4]

The tangent at the point on the curve where $x = 4k^2$, intersects the y-axis at P.

(b) Find the y-coordinate of P in terms of k. ---[4]

(c) on next Page →

[S-21/13/Q11]

Solution:
(a) $y = x^{\frac{1}{2}} + k^2 x^{-\frac{1}{2}}$ ------ ①

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}k^2 x^{-3/2}$ --- ②

for stationary point $\frac{dy}{dx} = 0$

$\Rightarrow \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}k^2 x^{-3/2} = 0$

$\Rightarrow \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}k^2 x^{-3/2}$

$\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}} \times k^2 x^{-1}$

$\Rightarrow k^2 x^{-1} = 1 \Rightarrow x = k^2 \quad (x \neq 0)$

from ① at $x = k^2$; $y = (k^2)^{1/2} + k^2(k^2)^{-\frac{1}{2}}$

$= 2k$

$\therefore$ Minimum point $(k^2, 2k)$

("c" Part continued on next Page)

(b) when $x = 4k^2$

from ② $\frac{dy}{dx} = \frac{1}{2}(4k^2)^{-\frac{1}{2}} - \frac{1}{2}k^2(4k^2)^{-3/2}$

$= \left[\frac{1}{4k} - \frac{1}{16k}\right] = \frac{3}{16}k$ ✓

$y = (4k^2)^{1/2} + k^2(4k^2)^{-\frac{1}{2}} = \frac{5k}{2}$

Hence the equation of tangent at $(4k^2, \frac{5k}{2})$

$y - \frac{5k}{2} = \frac{3}{16}k[x - 4k^2]$

$y = \frac{5k}{2} + \frac{3}{16}k[x - 4k^2]$

for intersection with y-axis at P, Put $x = 0$

$y = \frac{5}{2}k + \frac{3}{16}k(-4k^2)$

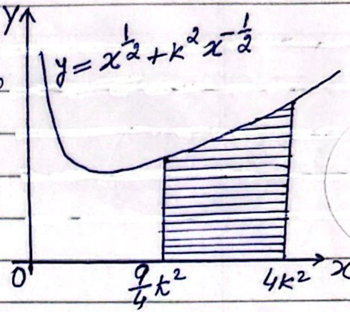
$y = \frac{5}{2}k - \frac{3}{4}k = \frac{7k}{4}$ ✓

(Continued →)

The shaded region is bounded by the curve, the x- axis and the lines $x=\frac{9}{4}k^2$ and $x=4k^2$.

19(c) The diagram shows part of the curve with equation $y=x^{1/2}+k^2x^{-1/2}$, where k is a positive constant. ---[3]

Find the area of the shaded region in terms of k.



[S-21/13/Q 11(C)]

Solution: Area $= \int y\,dx$

$$= \int_{\frac{9}{4}k^2}^{4k^2} \left(x^{1/2}+k^2x^{-\frac{1}{2}}\right)dx$$

$$= \left[\frac{2}{3}x^{3/2}+2k^2x^{1/2}\right]_{\frac{9}{4}k^2}^{4k^2}$$

$$= \left(\frac{2}{3}\cdot 8k^3+2k^2\times 2k\right)-\left(\frac{2}{3}\times\frac{27}{8}k^3+2k^2\times\frac{3}{2}k\right)$$

$$= \frac{49}{12}k^3$$

20 The equation of the curve is such that $\frac{d^2y}{dx^2} = 6x^2 - 4/x^3$. The curve has a stationary point at $(-1, 9/2)$.

(a) Determine the nature of the stationary point at $(-1, 9/2)$. --- [1]

(b) Find the equation of the curve. --- [5]

(c) Show the curve has no other stationary points. --- [3]

(d) A point A is moving along the curve and the y-coordinate of A is increasing at a rate of 5 units per second.
Find the rate of increase of the x-coordinate of A at the point where $x = 1$.

|5-22| 11| Q10| --- [3]

Solution:

$\frac{d^2y}{dx^2} = 6x^2 - 4/x^3$ --- (1)

(a) Stationary point at $(-1, 9/2)$

from (1) $\left(\frac{d^2y}{dx^2}\right)_{x=-1} = 6\times(-1)^2 - 4/(-1)^3$

$= 6 + 4 = 10 > 0$

$\therefore$ Minimum at $(-1, 9/2)$

(continued→)

(b) $\frac{dy}{dx} = \int(6x^2 - 4x^{-3})dx = 2x^3 + 2x^{-2} + c$

at $(-1, 9/2) \rightarrow \frac{dy}{dx} = -2 + 2 + c = 0$ as stationary point

$\Rightarrow c = 0$

$\therefore \frac{dy}{dx} = 2x^3 + 2x^{-2}$ --- (2)

Now $y = \int(2x^3 + 2x^{-2})dx = \frac{1}{2}x^4 - \frac{2}{x} + k$

Point $(-1, 9/2)$ lies on it $\Rightarrow 9/2 = \frac{1}{2} + 2 + k \Rightarrow k = 2$

Hence the eqn of curve: $y = \frac{1}{2}x^4 - \frac{2}{x} + 2$ --- (3)

21. The equation of a curve is $y = 3x+1-4(3x+1)^{1/2}$ for $x > -1/3$

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ --[3]

(b) Find the coordinates of the stationary point of the curve and determine its nature. --[4]

[S-22/12/Q9]

Solution (a) Curve: $y = 3x+1-4(3x+1)^{1/2}$ --①

diff ① $\frac{dy}{dx} = 3 - 4 \times \frac{1}{2}(3x+1)^{-1/2} \times 3$

$\frac{dy}{dx} = 3 - 6(3x+1)^{-1/2}$ --②

diff ② $\frac{d^2y}{dx^2} = -6 \times -\frac{1}{2}(3x+1)^{-3/2} \times 3$

$= \frac{9}{(3x+1)^{3/2}}$ --③

(b) for stationary point $\frac{dy}{dx} = 0$

from ② $3 - \frac{6}{\sqrt{3x+1}} = 0$

$\Rightarrow 3 = \frac{6}{\sqrt{3x+1}} \Rightarrow \sqrt{3x+1} = 2$

$3x+1 = 4 \Rightarrow x = 1$

from ① $y = 4 - 4 \times 2 = -4$

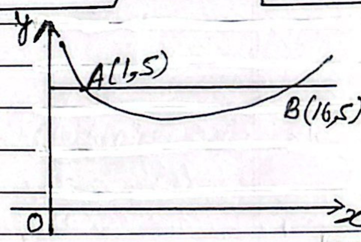
∴ Stationary point $(1, -4)$

Now to check the nature of stationary point $(1, -4)$

from ③ $\left(\frac{d^2y}{dx^2}\right)_{x=1} = \frac{9}{4^{3/2}} > 0$ so minimum

22. The diagram shows the curve with equation $y = x^{1/2} + 4x^{-1/2}$. The line $y = 5$ intersects the curve at the points $A(1,5)$ and $B(16,5)$.

Find the equation of the tangent to the curve at the point A. --[4]



[S-22/13/Q8]

Solution: Curve: $y = x^{1/2} + 4x^{-1/2}$ --①

differentiating ① w.r.t x; $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + 4 \times \left(-\frac{1}{2}\right) x^{-3/2}$

$= \frac{1}{2\sqrt{x}} - \frac{2}{x\sqrt{x}}$ --②

$\left(\frac{dy}{dx}\right)_{x=1} = \frac{1}{2\sqrt{1}} - \frac{2}{1\sqrt{1}} = \frac{1}{2} - 2 = -3/2$ is the gradient of the tangent at A.

∴ Equation of the tangent at $A(1,5)$

$y - 5 = -3/2(x-1)$

$2y - 10 = -3x + 3 \Rightarrow 3x + 2y = 13$

23. The function f is defined by $f(x) = (4x+2)^{-2}$ for $x > -\frac{1}{2}$

A point is moving along the curve $y = f(x)$ in such a way that, as it passes through a point A, its y-coordinate is decreasing at the rate k units per seconds, and its x-coordinate is increasing at the rate of k units per second. Find the coordinates of A. --- [6]

[S-22/13/Q10]

Solution: $f(x) = y = (4x+2)^{-2}$ --- (1)

$\frac{dy}{dx} = -2(4x+2)^{-3} \times 4$

$= \frac{-8}{(4x+2)^3}$ --- (2)

Now at A, y-coord. is decreasing

$\Rightarrow \frac{dy}{dt} = k$ --- (3)

and x-coord. is decreasing at A.

$\frac{dx}{dt} = -k$ --- (4)

$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = k/-k = -1$ -- (5)

from (2) and (5)

$\frac{-8}{(4x+2)^3} = -1 \Rightarrow (4x+2)^3 = 8$

$\Rightarrow 4x+2 = 2 \Rightarrow x = 0$ ✓

$\therefore$ from (1) $y = (4 \times 0 + 2)^{-2} = 1/4$ ✓

$\therefore$ coordinates of $A(0, 1/4)$ ✓

24. Water is poured into a tank at a constant rate of 500 cm^3 per second. The depth of water in the tank, t seconds after filling starts, is h cm. When the depth of water in the tank is h cm, the volume, V cm^3, of water in the tank is given by the formula $V = \frac{4}{3}(25+h)^3 - \frac{62500}{3}$.

(a) Find the rate at which h is increasing at the instant when $h = 10$ cm. ... [3]

(b) At another instant, the rate at which h is increasing is 0.075 cm per second. Find the value of V at this instant. --- [3]

S-23 | 11 | Q9

Solution:

$V = \frac{4}{3}(25+h)^3 - \frac{62500}{3}$ --- ①

$\left\{ \text{Given } \frac{dV}{dt} = 500 \text{ cm}^3/\text{s} \text{ --- ②} \right.$

(a) →

$\frac{dV}{dh} = \frac{4}{3} \times 3(25+h)^2 = 4(25+h)^2$ --- ①'

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ --- ③

$\left(\frac{dV}{dh}\right)_{h=10} = 4(25+10)^2 = 4900$ --- ④

from ②, ③ & ④

$500 = 4900 \times \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{500}{4900} = 0.102$ cm/s.

(b) Now given $\frac{dh}{dt} = .075$ cm/s --- ⑤

Now $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ --- ⑥

from ⑥

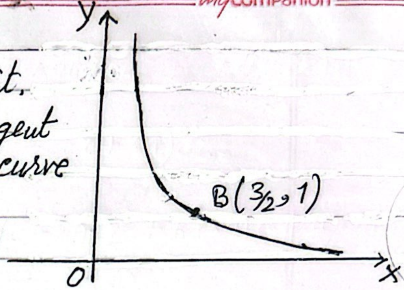
$\Rightarrow 500 = 4(25+h)^2 \times 0.075$ (from ①)

$\Rightarrow (25+h)^2 = \frac{500}{4 \times 0.075} \Rightarrow (25+h)^2 = \frac{5000}{3} \Rightarrow h = 15.8248$

∴ from ①

$V = \frac{4}{3}(25+15.8248)^3 - \frac{62500}{3} = 69900$

25. The diagram shows the curve:
$y = \frac{4}{(2x-1)^2}$ and a point $B(\frac{3}{2}, 1)$ on it.
A triangle is formed from the tangent to the curve at B, the normal to the curve at B and the x-axis.
Find the area of the triangle. ---[6]



Solution: Curve: $y = \frac{4}{(2x-1)^2}$ ---- (1)

S-23/11/Q10(b)

Diff. (1) $\frac{dy}{dx} = 4\times(-2)(2x-1)^{-3} = \frac{-8}{(2x-1)^3}$ --- (2)

Gradient of the tangent to the curve at $B(3/2, 1) = \frac{-8}{(2\times\frac{3}{2}-1)^2} = -2$

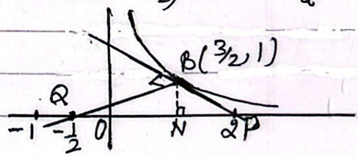
$\therefore$ Equation of tangent to the curve at B: $y-1 = -2(x-3/2)$ --- (3)

and the equation of Normal at B: $y-1 = \frac{1}{2}(x-3/2)$ -- (4)

Tangent (3) intersects x-axis (at $y=0$) is $-1 = -2(x-3/2) \Rightarrow x = 2$ at P

Normal (4) " " " " is $-1 = \frac{1}{2}(x-3/2) \Rightarrow x = -\frac{1}{2}$ at Q

Area of $\Delta BPQ = \frac{1}{2}\times PQ \times BN$
$= \frac{1}{2}\left(2-(-\frac{1}{2})\right)\times 1$
$= \frac{1}{2}\times\frac{5}{2}\times 1 = \frac{5}{4}$



26. The equation of a curve is such that $\frac{dy}{dx} = 6x^2 - 30x + 6a$, where a is a positive constant. The curve has a stationary point at $(a, -15)$.

(a) Find the value of a. ---[2]

(b) Determine the nature of this stationary point. ---[2]

(c) Find the equation of the curve. ---[3]

(d) Find the coordinates of any other stationary point on the curve [2]

S-23	11	Q 11

Solution (a) $\frac{dy}{dx} = 6x^2 - 30x + 6a$ --- ①

stationary point at $(a, -15) \Rightarrow \left(\frac{dy}{dx}\right)_{(a,-15)} = 0 \Rightarrow 6a^2 - 30a + 6a = 0$ from ①

$\Rightarrow 6a(a-4) = 0$

$\Rightarrow a = 4$ ✓ $(\because a > 0)$

(b) diff ① $\frac{d^2y}{dx^2} = 12x - 30 \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=} = 12\times4 - 30 = 18 > 0$ ✓

(Since the stationary point is at $(a, -15) \Rightarrow$ at $(4, -15)$ (as $a = 4$)

$\therefore$ The stationary point is a minimum at $(4, -15)$

(c) Equation of the curve from ① $y = \int (6x^2 - 30x + 6a)\,dx$

$\Rightarrow y = 6\times\frac{x^3}{3} - 30\times\frac{x^2}{2} + 6\times4x + c$ [$a = 4$]

$\Rightarrow y = 2x^3 - 15x^2 + 24x + c$ —— ②

Curve passes through $(4, -15)$ [$(a, -15)$, $a = 4$]

fr ② $\Rightarrow -15 = 2\times4^3 - 15\times4^2 + 24\times4 + c$

$\Rightarrow c = 1$

Hence fr ② Equation of the curve is: $y = 2x^3 - 15x^2 + 24x + 1$ --- ②

(d) $\frac{dy}{dx} = 6x^2 - 30x + 6\times4$ fr ①

$= 6x^2 - 30x + 24 = 6(x^2 - 5x + 4) = 6(x-1)(x-4)$

$\Rightarrow x = 1, \ x = 4$ (already done)

$\therefore$ hence another stationary point is at $x = 1$

fr ② $\left\{ y = 2\times1^3 - 15\times1^2 + 24\times1 + 1 = 12 \right.$

$\therefore$ another stationary point is at $(1, 12)$ ✓

27. The equation of a curve is $y = k\sqrt{4x+1} - x + 5$, where k is a positive constant.

(a) Find $\frac{dy}{dx}$. ...[2]

(b) Find the x-coordinate of the stationary point in terms of k. ...[2]

(c) Given $k = 10.5$, find the equation of the normal to the curve at the point where the tangent to curve makes an angle of $\tan^{-1}2$ with the positive x-axis. ...[4]

S-23/12/Q11

Solution: Curve: $y = k\sqrt{4x+1} - x + 5$ ---(1)

(a) $\frac{dy}{dx} = k \cdot \frac{1}{2}(4x+1)^{-\frac{1}{2}} \times 4 - 1$ $= \underline{2k(4x+1)^{-\frac{1}{2}} - 1}$ --- (2)

(b) for stationary point $\frac{dy}{dx} = 0 \Rightarrow \frac{2k}{\sqrt{4x+1}} - 1 = 0$ from (2)

$\Rightarrow \sqrt{4x+1} = 2k \Rightarrow 4x+1 = 4k^2$

$\Rightarrow x = \frac{4k^2-1}{4}$

(c) from (2) $\frac{dy}{dx} = 2k(4x+1)^{-\frac{1}{2}} - 1$ [for k = 10.4]

$= 2 \times 10.5 (4x+1)^{-\frac{1}{2}} - 1$

$\Rightarrow \frac{dy}{dx} = \frac{21}{\sqrt{4x+1}} - 1 = 2$

$\Rightarrow \sqrt{4x+1} = 7 \Rightarrow 4x+1 = 49$

$\Rightarrow x = 12.$

$\left\{ \begin{array}{l} \because \text{tangent makes an angle } \tan^{-1}2 = \theta \text{(let)} \\ \text{with the positive } x\text{-axis.} \\ \therefore \underline{\text{gradient of tangent}}\ \frac{dy}{dx} = \tan\theta = 2 \end{array} \right.$

from (1) $y = 10.5 \times \sqrt{4 \times 12 + 1} - 12 + 5 = 66.5$

$\therefore$ when grad of tangent $= 2$.

Point $(12, 66.5)$

$\Rightarrow$ Gradient of Normal $= -\frac{1}{2}$

$\therefore$ Equation the normal;

$y - 66.5 = -\frac{1}{2}(x - 12)$

$\Rightarrow y = -\frac{1}{2}x + 6 + 66.5$

or $y = -\frac{1}{2}x + 72.5$

28. The equation of a curve is $y = x + (x-1)^{-2} + 2$, the curve passes through a point $(0,3)$.

(a) The tangent to the curve at $(0,3)$ intersects the curve again at another point P. Show that the x-coordinate of P, satisfies the equation $(2x+1)(x-1)^2 - 1 = 0$ --[4]

(b) Verify that $x = 3/2$ satisfies this equation and hence find the y-coordinate of P. --[2]

S-23 / 13 / Q9 (b)(c)

Solution: Curve: $y = x + (x-1)^{-2} + 2$ --- ①

(a) $\dfrac{dy}{dx} = 1 - \dfrac{2}{(x-1)^3}$ --- ②

Gradient of the tangent at $(0,3) \rightarrow \left(\dfrac{dy}{dx}\right)_{x=0} = 3$ from ②

Equation of tangent at $(0,3)$;

$y - 3 = 3(x-0) \Rightarrow y = 3x+3$ --- ③

for the intersection of curve ① and tangent ③

$\Rightarrow 3x+3 = x + (x-1)^{-2} + 2 \Rightarrow 2x+1 = \dfrac{1}{(x-2)^2}$

$\Rightarrow \underline{(2x+1)(x-2)^2 - 1 = 0}$ ✓ --- ④

(b) Put $x = 3/2$ in ④ $\Rightarrow \left(2 \times \frac{3}{2} + 1\right)\left(\frac{3}{2} - 1\right)^2 - 1 = 0$

$\Rightarrow 4 \times \frac{1}{4} - 1 = 0$ True ✓

$\therefore$ from ① at $x = 3/2$, $y = \frac{3}{2} + \left(\frac{3}{2} - 1\right)^{-2} + 2$

$= \frac{3}{2} + 4 + 2 = 7\frac{1}{2}$ ✓

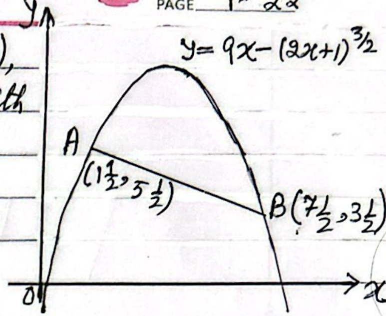
$\therefore$ when $x = 3/2$

$y = 7/2$ ✓

29 The diagram shows the points $A(1\frac{1}{2}, 5\frac{1}{2})$, and $B(7\frac{1}{2}, 3\frac{1}{2})$ lying on the curve with equation, $y = 9x - (2x+1)^{3/2}$

(a) Find the coordinates of the maximum point on the curve. ---[4]

(b) Verify that the line AB is the normal to the curve at A. ---[3]



S-23/13/Q10a,b

Solution: Curve: $y = 9x - (2x+1)^{3/2}$ --- ①

(a) $\frac{dy}{dx} = 9 - \frac{3}{2}(2x+1)^{1/2} \times 2$ -- ②

for max. $\frac{dy}{dx} = 0 \Rightarrow 9 - 3(2x+1)^{1/2} = 0 \Rightarrow 2x+1 = 9 \Rightarrow x = 4$

from ① $y = 9 \times 4 - (2 \times 4 + 1)^{3/2}$
$= 9$

∴ Max. point $(4, 9)$ ✓

(b) When $x = 1\frac{1}{2}$ (at A); $\frac{dy}{dx} = 9 - 3(2 \times \frac{3}{2} + 1)^{1/2}$ from ②

at A, Gradient of the tangent to curve $= 3$ --- ③

Gradient of AB, $\frac{5\frac{1}{2} - 3\frac{1}{2}}{1\frac{1}{2} - 7\frac{1}{2}} = \frac{2}{-6} = -\frac{1}{3}$ -- ④

fm ③ & ④ $3 \times -\frac{1}{3} = -1$ ✓

∴ AB is normal to the curve at A.

30. Air is being pumped into a balloon in the shape of a sphere so that its volume is increasing at a constant rate of $50\,cm^3s^{-1}$. Find the rate at which the radius of the balloon is increasing when the radius is 10 cm. [W-20/11/Q3] --[3]

Solution: Given $\frac{dV}{dt} = 50\ cm^3 s^{-1}$ —— ①   To find $\left(\frac{dr}{dt}\right)_{r=10} = ?$

Sphere, $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = \frac{4}{3}\pi \times 3r^2 = 4\pi r^2$ —— ②

Now $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

from ① & ② $\Rightarrow 50 = 4\pi r^2 \times \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{50}{4\pi r^2}$

$\therefore \left(\frac{dr}{dt}\right)_{r=10} = \frac{50}{4\pi \times 10^2} = \frac{1}{8\pi} = \underline{0.0398}$ ✓

---

31. The equation of the curve is $y = 2 + \sqrt{25 - x^2}$. Find the coordinates of the point on the curve at which the gradient is $4/3$. [W-20/11/Q6] --[5]

Solution:

$y = 2 + \sqrt{25 - x^2}$ —— ①

diff. $\frac{dy}{dx} = 0 + \frac{1}{2\sqrt{25-x^2}} \times \frac{d}{dx}(25 - x^2)$

$\frac{dy}{dx} = \frac{-2x}{2\sqrt{25-x^2}}$

$\because$ Using Chain Rule $\frac{d\sqrt{u}}{dx} = \frac{1}{2\sqrt{u}} \times \frac{du}{dx}$

$\therefore$ Gradient of Curve $= \frac{-x}{\sqrt{25-x^2}} = \frac{4}{3}$ (given) —— ②

$\Rightarrow 4\sqrt{25-x^2} = -3x \Rightarrow 16(25-x^2) = 9x^2$

$\Rightarrow 400 - 16x^2 = 9x^2$

$\Rightarrow 25x^2 = 400$

$\Rightarrow x^2 = 16$

$x = \pm 4$

$x = -4,\ \cancel{+4}$ from ② (*)

from ① $x = -4$

$\Rightarrow y = 2 + \sqrt{25-16} = 5$

$\therefore$ Required Point $\underline{(-4, 5)}$ ✓

32. The point (4,7) lies on the curve $y=f(x)$ and it is given that $f'(x)=6x^{-\frac{1}{2}}-4x^{-3/2}$
A point moves along the curve in such a way that the x-coordinate is increasing at a constant rate of 0.12 units per second.
Find the rate of increase of the y-coordinate when $x=4$.
[W-20/12/Q 7(a)]

Solution: Given $\frac{dx}{dt}=0.12$ —— ①

and $f'(x)=6x^{-1/2}-4x^{-3/2} \Rightarrow f'(4)=6\times4^{-1/2}-4\times4^{-3/2}$

or $\left(\frac{dy}{dx}\right)_{x=4}=\frac{6}{2}-\frac{4}{8}=\frac{5}{2}$ —— ②

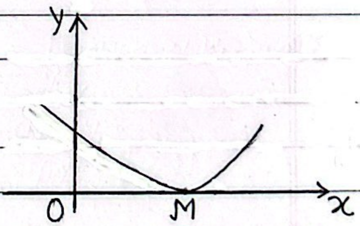
Now rate of increase of y (at $x=4$)

$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt} \Rightarrow \left(\frac{dy}{dt}\right)_{x=4}=\frac{5}{2}\times0.12$ from ① & ②

$=\underline{0.3}$ ✓

---

33. The diagram shows part of the curve, $y=\frac{2}{(3-2x)^2}-x$ and its minimum point M, which lies on the x-axis.



(a) Find the expressions for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ --- [4]
(b) Find the x-coordinate of M. ---- [2]
[W-20/12/Q 10(a)(b)]

Solution: $y=2(3-2x)^{-2}-x$ —— ①

diff ① $\frac{dy}{dx}=2\times(-2)(3-2x)^{-3}\times(-2)-1$

$=\frac{8}{(3-2x)^3}-1$ —— ② ✓

$\frac{d^2y}{dx^2}=8\times(-3)(3-2x)^{-4}\times(-2)$

$\Rightarrow \frac{d^2y}{dx^2}=\frac{48}{(3-2x)^4}$ —— ③

(b) for Minimum

$\frac{dy}{dx}=0$

from ① $\frac{8}{(3-2x)^3}-1=0$

$\Rightarrow (3-2x)^3=8$

$\Rightarrow 3-2x=2$

$\Rightarrow x=\frac{1}{2}$

$\therefore$ x-coordinate of M $=\frac{1}{2}$ ✓

34. The equation of a curve is $y = 2x+1+\frac{1}{2x+1}$ for $x > -\frac{1}{2}$

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ ...[3]

(b) Find the coordinates of the stationary point and the nature of the stationary point. ...[5]

W-20 / 13 / Q 8

Solution (a) $y = 2x+1+\frac{1}{(2x+1)}$ — ① (or $y = (2x+1)+(2x+1)^{-1}$)

$\frac{dy}{dx} = 2+(-1)(2x+1)^{-2}\times 2$

or $\frac{dy}{dx} = 2-2(2x+1)^{-2}$ — ② (or $\frac{dy}{dx} = 2-\frac{2}{(2x+1)^2}$)

$\therefore \frac{d^2y}{dx^2} = -2\cdot(-2)(2x+1)^{-3}\times 2$

or $\frac{d^2y}{dx^2} = \frac{8}{(2x+1)^3}$ — ③

(b) for stationary point $\frac{dy}{dx} = 0$

$\Rightarrow 2-\frac{2}{(2x+1)^2} = 0$ from ②

$\Rightarrow (2x+1)^2 = 1 \Rightarrow 2x+1 = 1$

$\Rightarrow x = 0$

from ① $y = 2$

$\therefore$ coordinates of stationary point $(0,2)$

Now from ③ $\left(\frac{d^2y}{dx^2}\right)_{(0,2)} = \frac{8}{1} > 0$

$\therefore$ Minimum at $(0,2)$ ✓

35. A curve has equation $y = \frac{1}{k}x^{1/2} + x^{-\frac{1}{2}} + \frac{1}{k^2}$ where $k > 0$ and k is a positive constant.
It is given that when $x = \frac{1}{4}$, the gradient of the curve is 3.
Find the value of k. --- [4]

[W-20 / 13 / Q 10(a)]

Solution: $y = \frac{1}{k}x^{1/2} + x^{-1/2} + \frac{1}{k^2}$ —— ①

diff. $\frac{dy}{dx} = \frac{1}{k}\cdot\frac{1}{2}x^{-\frac{1}{2}} + \left(-\frac{1}{2}\right)x^{-3/2} + 0$

$\Rightarrow \frac{1}{2k\cdot x^{1/2}} - \frac{1}{2x^{3/2}}$ [Given gradient of curve $= 3$ at $x = 1/4$]

$\left(\frac{dy}{dx}\right)_{x=\frac{1}{4}} = \frac{1}{2k\left(\frac{1}{4}\right)^{1/2}} - \frac{1}{2\left(\frac{1}{4}\right)^{3/2}} = 3$

$\Rightarrow \frac{1}{k} - \frac{1}{\frac{1}{4}} = 3$

$\Rightarrow \frac{1}{k} = 7 \Rightarrow k = \frac{1}{7}$ (or 0.143) ✓

36. Given a curve with equation: $y = \frac{1}{(3x-2)^{3/2}}$
The normal to the curve at (1,1), --- [4]
crosses the y-axis at the point A. Find the y-coordinate of A.

[W-21 / 11 / Q10(c)]

Solution: $y = (3x-2)^{-3/2}$ --- ①

$\frac{dy}{dx} = -\frac{3}{2}(3x-2)^{-5/2} \times 3$

$\left(\frac{dy}{dx}\right)_{(1,1)} = -9/2$.

Gradient of normal $= 2/9$

Equation of normal at (1,1)

$y - 1 = 2/9\,(x-1)$ -- ②

Normal ② intersect y-axis at $x = 0$

$y - 1 = -\frac{2}{9} \Rightarrow y = \frac{7}{9}$

$\therefore$ y coord. of A $= \frac{7}{9}$ ✓

37 The volume $V m^3$ of a large circular mound of iron ore of radius r m is modelled by the equation, $V = \frac{3}{2}(r-\frac{1}{2})^3 - 1$ for $r \geq 2$. Iron ore is added to the mound at a constant rate of $1.5 m^3$ per second.

(a) Find the rate at which the radius of the mound is increasing at the instant when the radius is 5.5 m. ---[3]

(b) Find the volume of the mound at the instant when the radius is increasing at 0.1 m per second. $\boxed{W-21/12/Q9}$ --[3]

Solution: $V = \frac{3}{2}(r-\frac{1}{2})^3 - 1$ —— ①

$\frac{dV}{dr} = \frac{3}{2} \times 3(r-\frac{1}{2})^2$

$\frac{dV}{dr} = \frac{9}{2}(r-\frac{1}{2})^2$ —— ②

$\frac{dV}{dt} = 1.5 \; m^3/s$ —— ③

Now $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$1.5 = \frac{9}{2}(r-\frac{1}{2})^2 \times \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{1.5}{\frac{9}{2}(r-\frac{1}{2})^2}$

$\left(\frac{dr}{dt}\right)_{r=5.5} = \frac{1.5}{\frac{9}{2}(5.5-\frac{1}{2})^2}$

$= \frac{2}{9} \times \frac{1.5}{(5)^2}$

$= \frac{2}{9} \times \frac{1.5}{25} = \frac{1}{75}$ ✓

$\left(\frac{dr}{dt}\right)_{r=5.5} = 0.0133 \; m/s.$

(b) Given $\frac{dr}{dt} = 0.1 \; m/s$

To find $V = ?$

$\frac{dV}{dr} = \frac{dV}{dt} / \frac{dr}{dt}$

$\frac{dV}{dr} = \frac{1.5}{0.1}$

$\frac{9}{2}(r-\frac{1}{2})^2 = 15$ from ②

$(r-\frac{1}{2})^2 = 15 \times \frac{2}{9} = \frac{10}{3}$

$(r-\frac{1}{2}) = \sqrt{\frac{10}{3}}$

$r = \frac{1}{2} + \sqrt{\frac{10}{3}}$ ✓

from ①

$V = \frac{3}{2}\left[\sqrt{\frac{10}{3}}\right]^3 - 1$

$= \frac{3}{2}[6.0858] - 1 = 8.128$

$V = 8.13 \; m^3$ ✓

38. The function f is defined by $f(x) = x^2 + \frac{k}{x} + 2$ for $x > 0$

(a) Given that the curve with equation $y = f(x)$ has a stationary point when $x = 2$ find k. ...[3]

(b) Determine the nature of the stationary point. ...[2]

(c) Given that this is the only stationary point of the curve, find the range of f.

$\boxed{W-21 / 12 / Q10}$ ...[2]

Solution: $f(x) = x^2 + \frac{k}{x} + 2 \quad x > 0$ --- (1)

(a) diff (1) $f'(x) = 2x - \frac{k}{x^2}$ --- (2)

given $x = 2$ is a stationary point $\Rightarrow f'(2) = 0$

from (2) $2 \times 2 - \frac{k}{2^2} = 0 \Rightarrow \frac{k}{4} = 4 \Rightarrow \underline{k = 16}$ ✓

(b) diff (2) $f''(x) = 2 + \frac{2k}{x^3}$

for $k = 16$, $f''(2) = 2 + \frac{2 \times 16}{2^3} > 0 \Rightarrow$ Minimum at $x = 2$ ✓

(c) from (1) $f(2) = 2^2 + \frac{16}{2} + 2$ (for $k = 16$)

$= 14$

as $x = 2$ is the only stationary point of $f(x)$.

Range of $f(x)$: $f(x) \geq f(2)$

$\Rightarrow f(x) \geq 14$ ✓

39. (a) Express $5y^2-30y+50$ in the form $5(y+a)^2+b$, where a and b are constants. ---[2]

(b) The function f is defined by $f(x)=x^5-10x^3+50x$ for $x \in R$. Determine whether f is an increasing function, a decreasing function or neither. ---[3]

[W-21|13|Q3]

Solution (a) $5y^2-30y+50$

$= 5[y^2-6y+10]$

$= 5[y^2-6y+3^2-9+10]$

$= 5[(y-3)^2+1]$

$= 5(y-3)^2+5$ ✓

(b) $f(x)=x^5-10x^3+50x \; ; x \in R$

$f'(x)=5x^4-20x^2+50$

$= 5(x^2-3)^2+5$ from part (a)

$5(x^2-3)^2+5>0$ for $x \in R$

$\therefore$ $f(x)$ is increasing function.

41. The normal to the curve $y=x^{-1/2}$ at the point $(1,1)$ intersects the y-axis at the point $(0,p)$. Find the value of p. [W-21|13|Q8(b)] -[4]

Solution: $y=x^{-1/2}$

$\frac{dy}{dx}=-\frac{1}{2}x^{-3/2}$

$\left(\frac{dy}{dx}\right)_{x=1}=-\frac{1}{2}$

$\therefore$ gradient of Normal $=2$

Equation of normal at $(1,1) \rightarrow y-1=2(x-1)$

intersects y-axis for $x=0 \Rightarrow y=-1$

at point $(0,-1) \equiv (0,p)$ (given)

$\Rightarrow p=-1$ ✓

41\. The equation of a curve is such that $\frac{dy}{dx} = 12(\frac{1}{2}x-1)^{-4}$. It is given that the curve passes through the point $P(6,4)$.
Find the equation of the tangent to the curve at P. ---[2]

[W-22 | 11 | Q2(a)]

Solution: Curve: $\frac{dy}{dx} = 12(\frac{1}{2}x-1)^{-4}$ --- ①

from ①
Gradient of the tangent to the curve at $P(6,4)$

$$m = 12(\tfrac{1}{2}\times 6-1)^{-4} = 12(2)^{-4} = 12/16 = 3/4$$

∴ Equation of the tangent at $P(6,4)$

$$y-4 = \tfrac{3}{4}(x-6) \qquad [y-y_1 = m(x-x_1)]$$

or $y = \frac{3}{4}x - \frac{18}{4} + 4$

$$y = \frac{3}{4}x - \frac{1}{2} \checkmark$$

42. A curve has equation $y = ax^{1/2} - 2x$, where $x > 0$ and a is a constant. The curve has a stationary point at the point P, which has x-coordinate 9. Find the y-coordinate of P.

[W-22 / 11 / Q3]

Solution: Curve: $y = ax^{1/2} - 2x$; $x > 0$ --- (1)

$\frac{dy}{dx} = \frac{1}{2}ax^{-1/2} - 2$

Given curve → Stationary point at $x = 9$

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=9} = \frac{1}{2}a\,9^{-1/2} - 2 = 0 \Rightarrow \frac{1}{2} \times \frac{a}{3} = 2 \Rightarrow a = 12$

Now → $y = 12x^{1/2} - 2x$ from (1) for $a = 12$

at $x = 9$, $y = 12 \times 9^{1/2} - 2 \times 9 = 12 \times 3 - 18 = 36 - 18 = 18$ ✓

43. The function f defined by $f(x) = 2 - \frac{3}{4x - p}$ for $x > p/4$, p is constant. Find $f'(x)$ and hence determine whether f is an increasing function, a decreasing function or neither. --- [3]

[W-22 / 11 / Q8(a)]

Solution: $f(x) = 2 - \frac{3}{4x - p}$ --- (1) for $x > p/4$

or $f(x) = 2 - 3(4x - p)^{-1} \Rightarrow f'(x) = -3(-1) \times (4x - p)^{-2} \times 4$

$= \frac{12}{(4x - p)^2} > 0$

Hence $f(x)$ is an increasing function.

44. The equation of a curve is such that $\frac{dy}{dx} = 3x^{1/2} - 3x^{-1/2}$. The curve passes through the point (3, 5).

(i) Find the x-coordinate of the stationary point. -- [2]

(ii) State the set of values of x for which y increases as x increases. ---[1]

[W-22 / 12 / Q8(b)(c)]

Solution: (i) For any stationary point $\frac{dy}{dx} = 0$

$\Rightarrow 3x^{1/2} - 3x^{-1/2} = 0$

$\Rightarrow 3x^{1/2} - \frac{3}{x^{1/2}} = 0$

$\Rightarrow 3(x - 1) = 0$

$\Rightarrow x = 1$ ✓

(ii) for increasing function $\frac{dy}{dx} > 0$

$\Rightarrow 3x^{1/2} - 3x^{-1/2} > 0$

$\Rightarrow 3x^{1/2} - \frac{3}{x^{1/2}} > 0$

$\Rightarrow 3(x - 1) > 0$

$\Rightarrow x > 1$ ✓

45. A point P is moving along the curve $y = 18 - \frac{3}{8}x^{5/2}$ in such a way that the x-coordinate of P is increasing at a constant rate of 2 units per second. ---[3]
Find the rate at which the y-coordinate of P is changing when $x = 4$.
[W-22/12/Q 11(C)]

Solution: $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$ --- ①

$y = 18 - \frac{3}{8}x^{5/2} \Rightarrow \frac{dy}{dx} = -\frac{3}{8} \times \frac{5}{2} x^{3/2} = \frac{-15}{16} x^{3/2}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=4} = \frac{-15}{16} \times 4^{3/2} = \frac{-15}{16} \times 8 = \frac{-15}{2}$ ✓ --- ②

hence from ① $\frac{dy}{dt} = -\frac{15 \times 2}{2} = -15$ ✓ (Given $\frac{dx}{dt} = 2$)

46. A large industrial water tank is such that, when the depth of the water in the tank is x metres, the volume $V m^3$ of water in the tank is given by $V = 243 - \frac{1}{3}(9-x)^3$. Water is being pumped into the tank at a constant rate of $3.6 m^3$ per hour. Find the rate of increase of the depth of the water when the depth is 4m, (answer in cm per minute)
[W-22/13/ Q4] ---[5]

Solution: $V = 243 - \frac{1}{3}(9-x)^3$ ---- ①

$\frac{dV}{dx} = -\frac{1}{3} \times 3(9-x)^2 \times (-1) = (9-x)^2$

$\Rightarrow \left(\frac{dV}{dx}\right)_{x=4} = (9-4)^2 = 5^2 = 25 \ m^3/h$ --- ②

Given $\frac{dV}{dt} = 3.6$ --- ③

Now $\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt} = \frac{1}{\frac{dV}{dx}} \times \frac{dV}{dt}$

$\left(\frac{dx}{dt}\right)_{x=4} = \frac{1}{25} \times 3.6 \ m/h$

$= \frac{3.6}{25} \times \frac{100}{60}$ cm/min

$= 0.24$ cm/min ✓

47. The curve $y = f(x)$ is such that $f'(x) = \dfrac{-3}{(x+2)^4}$.
The tangent at a point on the curve whose $x = a$ has gradient $-\dfrac{16}{27}$.
Find the possible values of a. ---[4]

W-22/13/Q 7(a)

Solution: $f'(x) = \dfrac{-3}{(x+2)^4}$ --- ①

Gradient of the tangent at $x = a$ is

$$f'(a) = \frac{-3}{(a+2)^4} = -\frac{16}{27} \text{ (Given)}$$

$$16(a+4)^4 = 81 \Rightarrow (a+2)^2 = \frac{9}{4}$$

$$\Rightarrow a+2 = \pm 3/2$$

$$\Rightarrow a = -\frac{1}{2} \text{ or } -\frac{7}{2}$$