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Pure Maths-1

Integration

Exercise 1. Solution (Revision)

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Integration - Revision

Example 1: A curve has equation $y=f(x)$. It is given that $f'(x)=\frac{1}{\sqrt{x+6}}+\frac{6}{x^2}$ and that $f(3)=1$, find $f(x)$ ---[5]

SP-20/01/Q4

Solution: $f'(x)=(x+6)^{-\frac{1}{2}}+6x^{-2}$

$f(x)=\int\{(x+6)^{-\frac{1}{2}}+6x^{-2}\}dx$

$f(x)=2(x+6)^{\frac{1}{2}}-\frac{6}{x}+c$ —①

given $f(3)=1$ fn ①

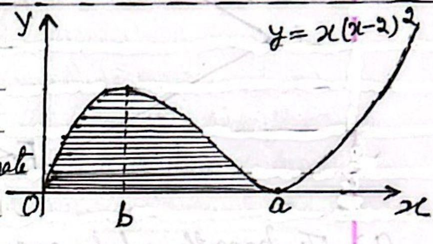
$1=2(3+6)^{\frac{1}{2}}-\frac{6}{3}+c$

$\Rightarrow 1=2\times3-2+c$

$\Rightarrow c=-3$

fn ① $f(x)=2(x+6)^{1/2}-\frac{6}{x}-3$ ✓

Example 2: The diagram shows the curve with equation $y=x(x-2)^2$. The minimum point on the curve has coordinate $(a,0)$ and the x-coordinate of the maximum is b, where a and b are constants.



(a) State the value of a. --[1]

(b) Calculate the value of b. --[4]

(c) Find the area of the shaded region. --[4]

(d) The gradient, $\frac{dy}{dx}$, of the curve has a minimum value m. Calculate the value of m. --[4]

SP-20/01/Q12

Solution: $y=x(x-2)^2$ —①

(a) $(a,0)$ lies on ① $\Rightarrow 0=a(a-2)^2 \Rightarrow a=2$

(b) $y=x^3-4x^2+4x$

$\frac{dy}{dx}=3x^2-8x+4$

$(x-2)(3x-2)=0$ for stationary point $\Rightarrow x=2, 2/3$

$\therefore b=2/3$ ✓ $[\because a=2]$

(c) Area $=\int_0^a y\,dx=\int_0^2(x^3-4x^2+4x)dx$

$=\left[\frac{x^4}{4}-\frac{4x^3}{3}+2x^2\right]_0^2$

Area $=4-\frac{32}{3}+8=\frac{4}{3}$ ✓

(d) Gradient $g=\frac{dy}{dx}=3x^2-8x+4$

$\frac{dg}{dx}=6x-8$

for gradient 'g' be minimum

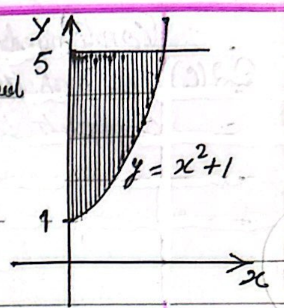
$\frac{dg}{dx}=0 \Rightarrow 6x-8=0$

$\Rightarrow x=4/3$

$\therefore$ Min. value of $\frac{dy}{dx}$; $m=4/3$ ✓

Example 3: The diagram shows part of the curve with equation $y = x^2+1$. The shaded region enclosed by the curve, the y-axis and the line $y=5$ is rotated through 360° about the y-axis. Find the volume obtained. ---[4]

[M-20/12/Q3]



Solution: $y = x^2+1 \Rightarrow x^2 = y-1$

$$V = \pi\int x^2\,dy = \pi\int_1^5 (y-1)\,dy$$

$$= \pi\left[\frac{y^2}{2} - y\right]_1^5$$

$$= \pi\left[\left(\frac{25}{2}-5\right) - \left(\frac{1}{2}-1\right)\right]$$

$$\therefore V = 8\pi$$

Example 4: The gradient of a curve at the point (x, y) is given by $\frac{dy}{dx} = 2(x+3)^{\frac{1}{2}} - x$. The curve has a stationary point at $(a, 14)$, where a is positive constant. Find the equation of the curve. [M-20/12/Q10] --[5]

Solution: $\frac{dy}{dx} = 2(x+3)^{\frac{1}{2}} - x$ —— ①

for stationary point $\left(\frac{dy}{dx}\right)_{(a,14)} = 0$

or

$$2(a+3)^{\frac{1}{2}} - a = 0$$

$$\Rightarrow 2(a+3)^{1/2} = a$$

$$4(a+3) = a^2$$

$$a^2 - 4a - 12 = 0$$

$$(a-6)(a+2) = 0$$

$$a = 6 \text{ or } \cancel{a = -2} \quad (\text{as } a > 0)$$

$$\therefore a = 6$$

Stationary point is $(6, 14)$ on the curve.

from ①

$$y = \int\left(2(x+3)^{1/2} - x\right)dx$$

$$y = \frac{2(x+3)^{3/2}}{3/2} - \frac{x^2}{2} + c$$

$$y = \frac{4}{3}(x+3)^{3/2} - \frac{x^2}{2} + c \quad \text{—— ②}$$

② passes through point $(6, 14)$

$$\Rightarrow 14 = \frac{4}{3}\times(6+3)^{3/2} - \frac{6^2}{2} + c$$

$$14 = \frac{4}{3}\times 27 - 18 + c$$

$$\Rightarrow c = -4$$

$\therefore$ from ② equ^n of the curve is

$$y = \frac{4}{3}(x+3)^{3/2} - \frac{x^2}{2} - 4$$

5. A curve is such that $\frac{dy}{dx} = \frac{6}{(3x-2)^3}$ and $A(1,-3)$ lies on the curve. Find the equation of the curve. --- [4]

[M-21/12/Q 6(b)]

Solution: $\frac{dy}{dx} = \frac{6}{(3x-2)^3}$

$\Rightarrow y = \int \frac{6}{(3x-2)^3} dx = \int 6(3x-2)^{-3} dx$

$= \frac{6(3x-2)^{-2}}{-2\times3} + c$

$\Rightarrow y = -(3x-2)^{-2} + c$ --- (1)

Point $A(1,-3)$ lies on the curve (1)

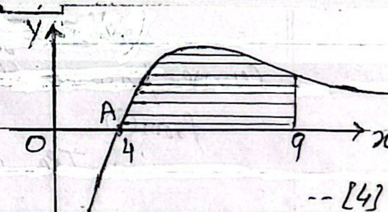
$\Rightarrow -3 = -(3\times1-2)^{-2} + c$

$\Rightarrow -3 = -1 + c$

$\Rightarrow c = -2$

$\therefore$ from (1) the equation of the curve: $y = -(3x-2)^{-2} - 2$ ✓

6. The diagram shows the curve with equation $y = 9(x^{-\frac{1}{2}} - 4x^{-3/2})$. The curve crosses the x-axis at the point A. Find the area of the region bounded by the curve, the x-axis and the line $x=9$ -- [4]



[M-21/12/Q11(d)]

$y = 9(x^{-1/2} - 4x^{-3/2})$

Solution: at the point A, $y=0$

$\Rightarrow y = 9(x^{-\frac{1}{2}} - 4x^{-3/2}) = 0$

$9x^{-1/2}[1 - 4x^{-1}] = 0$

$x^{-\frac{1}{2}} \neq 0$, $1 - \frac{4}{x} = 0 \Rightarrow x = 4$ ✓

at A, $x = 4$,

Area $= \int_4^9 9(x^{-\frac{1}{2}} - 4x^{-3/2}) dx$

$= 9\left[\frac{x^{1/2}}{\frac{1}{2}} - 4\times\frac{x^{-1/2}}{-\frac{1}{2}}\right]_4^9$

$= 9\left[2\sqrt{x} + \frac{8}{\sqrt{x}}\right]_4^9$

$= 9\left[\left(6+\frac{8}{3}\right) - (4+4)\right]$

$= 9\times\frac{2}{3} = 6$ ✓

7. A curve with equation $y = f(x)$ is such that $f'(x) = 2x^{-1/3} - x^{1/3}$. It is given that $f(8) = 5$. Find $f(x)$. ---[4]

[M-22 | 12 | Q 1]

Solution: $f'(x) = 2x^{-1/3} - x^{1/3} \Rightarrow f(x) = \int (2x^{-1/3} - x^{1/3})\,dx$

$$= 2\frac{x^{2/3}}{2/3} - \frac{x^{4/3}}{4/3} + c$$

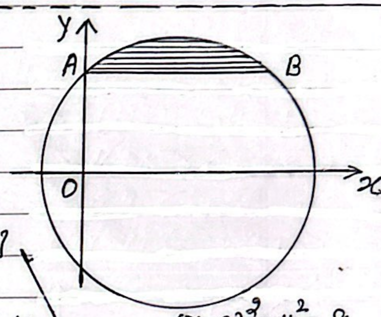
$$f(x) = 3x^{2/3} - \frac{3}{4}x^{4/3} + C \text{ ----①}$$

Now Given $f(8) = 5 \Rightarrow f(8) = 3 \times 8^{2/3} - \frac{3}{4} \times 8^{4/3} + C = 5$ (given)

$$\Rightarrow 12 - 12 + C = 5 \Rightarrow C = 5$$

$\therefore$ from ① $f(x) = 3x^{2/3} - \frac{3}{4}x^{4/3} + 5$

8.



The diagram shows the circle with equation, $(x-2)^2 + y^2 = 8$. The chord AB of the circle intersects the positive Y-axis at A and is parallel to X-axis.

(a) Find the coordinates of A and B. --- [3]

(b) Find the volume of revolution when the shaded segment, bounded by the circle and the chord AB, is rotated through 360° about the x-axis. --- [5]

$(x-2)^2 + y^2 = 8$

[M-22 | Q12 | Q 8]

Solution (a). Circle: $(x-2)^2 + y^2 = 8$ ---①

Circle ① intersects Y-axis at $x = 0$

$\Rightarrow (-2)^2 + y^2 = 8 \Rightarrow y^2 = 4 \Rightarrow y = 2 \; (y > 0)$

$\therefore A(0, 2)$.

Now AB ∥ X-axis; Eqn of AB: $y = 2$ -②

Solving ① & ② Put $y = 2$ in ①

$(x-2)^2 + 2^2 = 8 \Rightarrow (x-2)^2 = 4$

$x - 2 = \pm 2 \Rightarrow x = 0, 4$

$A(0, 2)$, $B(4, 2)$.

(b) Volume: when the arc AB is rotated about x-axis $= \int_0^4 \pi y^2\,dx \Rightarrow$

$= \int_0^4 [8 - (x-2)^2]\,dx$ {from ① $y^2 = 8 - (x-2)^2$}

$= \pi \left[8x - \frac{(x-2)^3}{3}\right]_0^4$

$= \pi \left[32 - \frac{16}{3}\right] = \frac{80\pi}{3}$ ——③

and the Volume of Cylinder obtained by rotating the chord AB, about x-axis

$= \pi \times OA^2 \times AB \quad [\pi r^2 h]$

$= \pi \times 2^2 \times 4 = 16\pi$ ——④

$\therefore$ Required Volume of revolution.

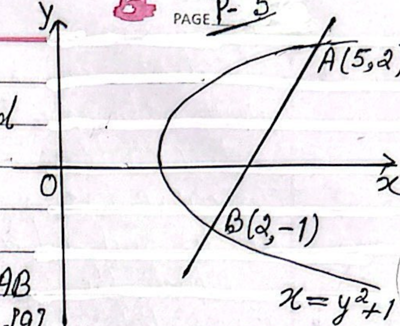
$= \frac{80}{3}\pi - 16\pi$ [from ③ & ④]

$= \frac{32}{3}\pi = 10\frac{2}{3}\pi$

9. The diagram shows the curve with equation $x = y^2 + 1$. The points $A(5,2)$ and $B(2,-1)$ lie on the curve.

(a) Find the equation of the line AB. ---[2]

(b) Find the volume of revolution when the region between the curve and the line AB is rotated through 360° about the y-axis ---[9]



M-23 | 12 | Q 11

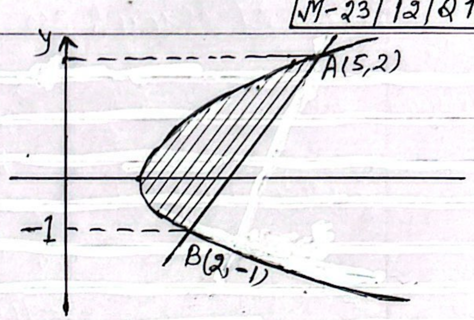
Solution (a) $A(5,2)$, $B(2,-1)$.

Gradient of $AB = \frac{2-(-1)}{5-2} = 1$

Equation of line AB,

$y - 2 = 1(x-5)$

$\Rightarrow y = x - 3$ ---- ①



(b)

Required V obtained by revolving the shaded area around y-axis (through 360°)

= Volume obtained by line AB − Volume obtained by the curve ---- ②

Volume obtained by revolving line AB $= \int \pi x^2\,dy = \pi \int (y+3)^2\,dy$ [from ① $x = y+3$]

$= \pi \frac{(y+3)^3}{3}$

∴ between, $y = -1$ to $2 = \frac{\pi}{3}\left[(y+3)^3\right]_{-1}^{2} = \frac{\pi}{3}\left[5^3 - 2^3\right] = \frac{117}{3}\pi = 39\pi$ ---- ③

Volume obtained by revolving the curve $= \int \pi x^2 = \pi \int (y^2+1)^2\,dy$

$= \pi \int (y^4 + 2y^2 + 1)\,dy = \left[\frac{y^5}{5} + \frac{2}{3}y^3 + y\right]$

Hence between $y = -1$ to 2

$= \pi\left[\frac{y^5}{5} + \frac{2}{3}y^3 + y\right]_{-1}^{2} = \pi\left[\left(\frac{32}{5} + \frac{16}{3} + 2\right) - \left(\frac{-1}{5} - \frac{2}{3} - 1\right)\right]$

$= \frac{78}{5}\pi$ ---- ④

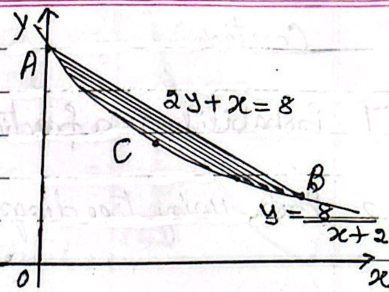
from ③ & ④ in ②

The required volume $= 39\pi - \frac{78}{5}\pi = \frac{(195-78)}{5}\pi = \frac{117}{5}\pi$

$V = \frac{117}{5}\pi = 73.513$

$V = 73.5$ cubic units

Example 10. The diagram shows part of the curve $y = \frac{8}{x+2}$ and the line $2y + x = 8$, intersecting at points A and B. The point C lies on the curve and the tangent to the curve at C is parallel to AB.



(a) Find, by calculation, the coordinates of A, B and C. ---[6]

(b) Find the volume generated when the shaded region, bounded by the curve and the line, is rotated through 360° about the x-axis. [S-20/11/Q11] ---[6]

Solution:

(a) curve; $y = \frac{8}{(x+2)}$ —— ①

line; $2y + x = 8$

or $y = \left(\frac{8-x}{2}\right)$ —— ②

Solving ① and ②

$$\frac{8}{(x+2)} = \frac{(8-x)}{2}$$

$\Rightarrow \quad x^2 - 6x = 0$

$x(x-6) = 0$

$x = 0$ or $x = 6$

from ① $\begin{cases} x=0 \\ y=4 \end{cases}$ and $\begin{cases} x=6 \\ y=1 \end{cases}$

$\therefore$ A(0,4) and B(6,1) ✓

Gradient of AB $= \frac{1-4}{6-0} = -\frac{1}{2}$ ✓

Gradient of the tangent at C, $\frac{dy}{dx} = -\frac{1}{2}$ —③

diff① $\frac{dy}{dx} = \frac{-8}{(x+2)^2} = -\frac{1}{2}$ fr③

$\Rightarrow (x+2)^2 = 16$

$x+2 = \pm 4$

fr① $\begin{cases} x = 2 \text{ or } \cancel{-6} \\ y = 2 \end{cases}$ $\therefore C(2,2)$

(b) Volume under the line

$= \pi \int y_{②}^2 dx = \pi \int_0^6 (4 - \frac{1}{2}x)^2 dx$

$= \pi \int_0^6 (16 + \frac{1}{4}x^2 - 4x) dx$

$= \pi \left[16x + \frac{x^3}{12} - 2x^2 \right]_0^6$

$= \pi [42 - 0] = 42\pi$ ← ④

Now area under the curve

$= \pi \int y_{①}^2 dx = \pi \int_0^6 \left(\frac{8}{x+2}\right)^2 dx$

$= \pi \left[\frac{-64}{x+2} \right]_0^6$

$= \pi [-8 - (-32)]$

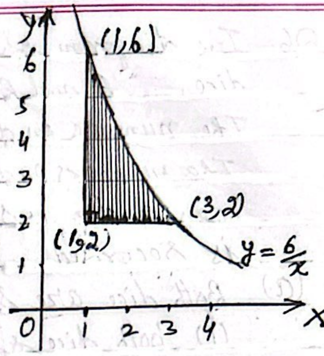
$= 24\pi$ —— ⑤

$\therefore$ Required shaded area

$= 42\pi - 24\pi$ {from ⑤ and ⑥}

$= 18\pi$ ✓

Example 11. The diagram shows part of the curve $y = \frac{6}{x}$. The points (1,6) and (3,2) lies on the curve. The shaded region is bounded by the curve and the lines $y = 2$ and $x = 1$.



(a) Find the volume generated when the shaded region is rotated through 360° about the y-axis. --[5]

(b) The tangent to the curve at a point X is parallel to line $y + 2x = 0$, show that X lies on the line $y = 2x$. $\boxed{5-20/12/Q8}$ -[3]

Solution: Volume by rotating the curve

(a) about the y-axis $= \pi \int x^2 dy$

$= \pi \int_2^6 \left(\frac{6}{y}\right)^2 dy \quad \left[y = \frac{6}{x}\right]$

$= \pi \int_2^6 36 \cdot y^{-2} dy$

$= \pi \left[-\frac{36}{y}\right]_2^6$

$= \pi [(-6) - (-18)]$

$= 12\pi$ —— ①

Volume of Cylinder by rotating the line $x = 1$ $\pi \int_2^6 x^2 dy$

$= \pi \int_2^6 1^2 dy = \pi [y]_2^6$

$= 4\pi$ —— ②

∴ Required Volume by rotating the shaded region about y-axis

fr ① & ② $= 12\pi - 4\pi$

$= 8\pi$ ✓

(b) l; $y + 2x = 0$

or $y = -2x$

gradient of line $= -2$ —— ③

gradient of tangent to the curve $= \frac{dy}{dx}$

Curve is $y = \frac{6}{x}$.

$\frac{dy}{dx} = \frac{-6}{x^2}$ —— ④

given: the tangent is parallel to line $y + 2x = 0$

∴ fr ③ & ④

$-\frac{6}{x^2} = -2$

$x^2 = 3$

$x = \sqrt{3}$ or $\cancel{(-\sqrt{3})}$

Curve is $y = \frac{6}{x} \Rightarrow \begin{cases} x = \sqrt{3} \\ y = \frac{6}{\sqrt{3}} = 2\sqrt{3} \end{cases}$

X $(\sqrt{3}, 2\sqrt{3})$ lies on $y = 2x$ ✓

Example 12. The equation of a curve is such that $\frac{dy}{dx} = 3x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$, It is given that the point (4,7) lies on the curve. Find the equation of the curve. [S-20/13/Q2] ---[4]

Solution: $\frac{dy}{dx} = 3x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$

$$\Rightarrow y = \int (3x^{1/2} - 3x^{-1/2})\, dx$$

$$= 3\frac{x^{3/2}}{3/2} - 3\frac{x^{1/2}}{\frac{1}{2}} + c$$

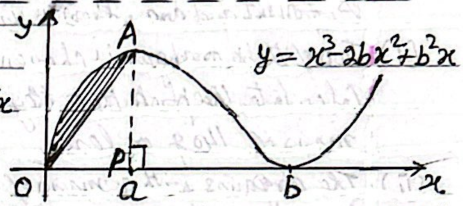
$$\text{or } y = 2x^{3/2} - 6x^{1/2} + c \quad (1)$$

Given (4,7) lies on the curve (1) $\Rightarrow 7 = 2\times(4)^{3/2} - 6\times(4)^{\frac{1}{2}} + c$

$$\Rightarrow 7 = 16 - 12 + c \Rightarrow c = 3$$

$\therefore$ from (1) Req. eqn of curve: $\underline{y = 2x^{3/2} - 6x^{\frac{1}{2}} + 3}$

Example 13. The diagram shows part of the curve with equation $y = x^3 - 2bx^2 + b^2x$ and the line OA, where A is the maximum point on the curve.



The x-coordinate of A is a and the curve has a minimum point at (b, 0), where a and b are positive constants.

(a) Show that $b = 3a$ --[4]

(b) Show that the area of the shaded region between the line and the curve is ka^4, where k is a fraction be found. [S-20/13/Q11] [7]

Solution: (a) has been done in differentiation revision; $b = 3a$.

(b) Shaded region area = area under the curve − area under the line — (1)

Area under the curve $= \int_0^a (x^3 - 2bx^2 + b^2x)\, dx$

for $b = 3a \rightarrow \int_0^a (x^3 - 6ax^2 + 9a^2x)\, dx$

$$A = \left[\frac{x^4}{4} - 6a\cdot\frac{x^3}{3} + 9a^2\times\frac{x^2}{2}\right]_0^a$$

$$= \left(\frac{a^4}{4} - 2a\cdot a^3 + \frac{9}{2}a^2\cdot a^2\right) - 0$$

$$= \frac{11}{4}a^4 \quad (2)$$

Now at $x = a$, $y = a^3 - 6a^3 + 9a^3$
$= 4a^3$
$A(a, 4a^3)$

Area under the line OA. [AP ⊥ x-axis]
$= \text{area of } \Delta OAP$
$= \frac{1}{2}\times a \times AP$
$= \frac{1}{2}\times a \times 4a^3 = 2a^4 \quad (3)$

from (2) and (3) in (1)

Area of the shaded region −
$$= \frac{11}{4}a^4 - 2a^4 = \underline{\frac{3}{4}a^4}$$

14 The equation of a curve is such that $\frac{dy}{dx} = \frac{3}{x^4} + 32x^3$. It is given that the curve passes through the point $(\frac{1}{2}, 4)$.
Find the equation of the curve. [S-21 | 11 | Q1] ---[4]

Solution: $\frac{dy}{dx} = 3x^{-4} + 32x^3$ --- ①

$\int dy = \int (3x^{-4} + 32x^3)\,dx$

$y = 3\frac{x^{-3}}{-3} + 32 \cdot \frac{x^4}{4} + c$

$y = -\frac{1}{x^3} + 8x^4 + c$ --- ②

curve passes through $(\frac{1}{2}, 4)$

from ② $4 = -\frac{1}{(\frac{1}{2})^3} + 8(\frac{1}{2})^4 + c$

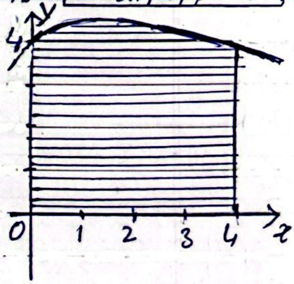
$\Rightarrow 4 = -8 + \frac{1}{2} + c \Rightarrow c = \frac{23}{2}$

Put $c = \frac{23}{2}$ in ②

Equation of the curve is

$y = -\frac{1}{x^3} + 8x^4 + \frac{23}{2}$ ✓

15. The equation of a curve is $y = 2\sqrt{3x+4} - x$.
Find the exact area of the region bounded by the curve, the x-axis and the lines $x=0$ and $x=4$. ---[4] [S-21 | 11 | Q11(d)]



Solution: Area $= \int_0^4 y\,dx = \int_0^4 \left(2(3x+4)^{1/2} - x\right) dx$

$= \left[\frac{2 \times \frac{2}{3}(3x+4)^{3/2}}{3} - \frac{1}{2}x^2 \right]_0^4$

$= \left[\left\{ \frac{4}{9}(16)^{3/2} - \frac{1}{2}(4)^2 \right\} - \left\{ \frac{4}{9}(4)^{3/2} - 0 \right\} \right]$

$= \left(\frac{4}{9} \times 64 - 8\right) - \left(\frac{32}{9}\right) = 16\frac{8}{9}$ ✓

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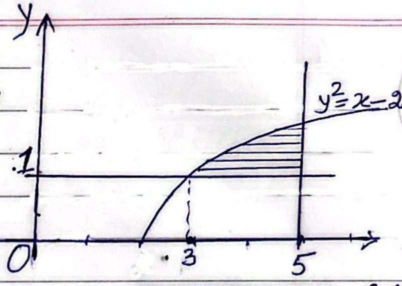
The diagram shows part of the curve with equation $y^2 = x-2$ and the lines $x=5$ and $y=1$. The shaded region enclosed by the curve and the lines is rotated through 360° about the x-axis. Find the volume obtained.



[S-21/12/Q9] --- [6]

Solution: $y^2 = x-2$ --- ①

Curve ① intersect the line $y=1$ at $x=3$

Volume obtained by rotating the curve around x-axis from $x=3$ to 5.

$$V_1 = \pi\int_3^5 y^2\,dx = \pi\int_3^5 (x-2)\,dx \text{ from ①}$$

$$= \pi\left[\frac{x^2}{2} - 2x\right]_3^5 = \left[\left(\frac{25}{2}-10\right)-\left(\frac{9}{2}-6\right)\right]\pi$$

$$= 4\pi \text{ --- ②}$$

Volume of cylinder $= \pi \times 1^2 \times (5-3) = 2\pi$ --- ③

from ② & ③

$\therefore$ Req. Volume obtained by rotating the shaded area $= 4\pi - 2\pi = 2\pi$ ✓

17 A curve with equation $y = f(x)$ is such that $f'(x) = 6x^2 - \frac{8}{x^2}$. It is given that the curve passes through the point $(2,7)$. Find $f(x)$.

[S-21/13/Q1] --- [3]

Solution: $f'(x) = 6x^2 - 8/x^2$

$$f(x) = \int (6x^2 - 8x^{-2})\,dx$$

$$= 6 \times \frac{x^3}{3} + \frac{8}{x} + c$$

$$f(x) = 2x^3 + \frac{8}{x} + c \text{ --- ①}$$

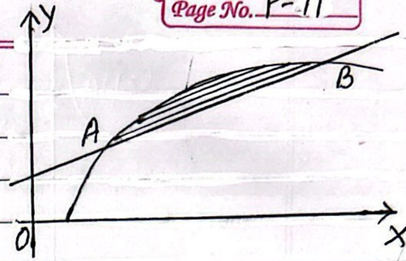
$f(x)$ passes through $(2,7)$ $\Rightarrow$

$$7 = 2 \times 2^3 + 8/2 + c \Rightarrow c = -13$$

from ①

$$f(x) = 2x^3 + \frac{8}{x} - 13$$ ✓

18. The diagram shows the curve with equation $y=(3x-2)^{1/2}$ and the line $y=\frac{1}{2}x+1$. The curve and the line intersect at points A and B.



(a) Find the coordinates of A and B. ---[4]

(b) Hence find the area of the region enclosed between the curve and the line. ---[5]

S-22 | 11 | Q7

Solution:

Curve: $y=(3x-2)^{1/2}$ --- (1)

Line: $y=\frac{1}{2}x+1$ --- (2)

(a) To find the coordinates of A and B.

from (2) in (1) $(3x-2)^{1/2}=\frac{1}{2}x+1$

$\Rightarrow \quad 3x-2=\left(\frac{1}{2}x+1\right)^2$

$\Rightarrow \quad 3x-2=\frac{1}{4}x^2+1+x$

$\Rightarrow \quad \frac{1}{4}x^2-2x+3=0 \Rightarrow x^2-8x+12=0$

$(x-6)(x-2)=0 \Rightarrow x=6 ; x=2$

from (2) $\quad y=4 ; y=2$

$A(2,2), \; B(6,4)$ ✓

(b) Shaded area = Area under the curve − area under the line from $x=2$ to $x=6$

$$\text{Area}=\int_2^6\left((3x-2)^{1/2}-\left(\frac{1}{2}x+1\right)\right)dx$$

$$=\left[\frac{2}{3}\frac{(3x-2)^{3/2}}{3}-\left(\frac{1}{4}x^2+x\right)\right]_2^6$$

$$=\left\{\frac{2}{9}\times 64-(9+6)\right\}-\left\{\frac{2}{9}\times 8-(1+2)\right\}$$

$$=\left(\frac{128}{9}-15\right)-\left(\frac{16}{9}-3\right)$$

$$=\frac{112}{9}-12=4/9$$ ✓

19. The equation of the curve is such that $\frac{dy}{dx} = 3(4x-7)^{1/2} - 4x^{-1/2}$. It is given that the curve passes through the point $(4, 5/2)$. Find the equation of the curve. ---[4]

[S-22/12/Q3]

Solution: $\frac{dy}{dx} = 3(4x-7)^{1/2} - 4x^{-1/2}$ ---(1)

$\Rightarrow y = \int \left(3(4x-7)^{1/2} - 4x^{-1/2}\right) dx$

$= \frac{3(4x-7)^{3/2}}{3/2 \times 4} - \frac{4x^{1/2}}{1/2} + c$

$y = \frac{1}{2}(4x-7)^{3/2} - 8x^{1/2} + c$ ---(2)

Point $(4, 5/2)$ lies on the curve,

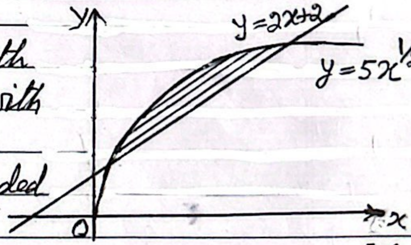
for (2) $\frac{5}{2} = \frac{1}{2}(9)^{3/2} - 8 \cdot 4^{1/2} + c$

$\Rightarrow \frac{5}{2} = \frac{27}{2} - 16 + c \Rightarrow c = 5$ ✓

$\therefore$ Equation of the curve from (2)

$y = \frac{1}{2}(4x-7)^{3/2} - 8x^{1/2} + 5$ ✓

20. The diagram shows the curve with equation $y = 5x^{1/2}$ and the line with equation $y = 2x+2$. Find the exact area of the shaded region which is bounded by the line and the curve. ---[5]



[S-22/12/Q6]

Solution: Curve: $y = 5x^{1/2}$ ---(1)

Line: $y = 2x+2$ ---(2)

To find the points of intersection from (1) and (2) $2x + 2 = 5x^{1/2}$

$\Rightarrow 2x - 5x^{1/2} + 2 = 0$

$2x - 4x^{1/2} - x^{1/2} + 2 = 0$

$(2\sqrt{x}(\sqrt{x}-2) - 1(\sqrt{x}-2)$

$(2\sqrt{x}-1)(\sqrt{x}-2) \Rightarrow \sqrt{x} = 2, \frac{1}{2} \Rightarrow x = 4, \frac{1}{4}$

$\therefore \text{Area} = \int_{1/4}^{4} \left[5x^{1/2} - (2x+2)\right] dx$

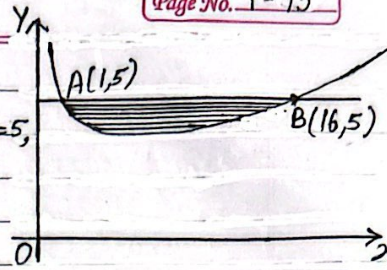
$= \int_{1/4}^{4} (5x^{1/2} - 2x - 2)\, dx$

$= \left[\frac{10}{3}x^{3/2} - x^2 - 2x\right]_{1/4}^{4}$

$= \left[\left(\frac{10}{3} \times 8 - 16 - 8\right) - \left(\frac{10}{3} \cdot \frac{1}{8} - \frac{1}{16} - \frac{1}{2}\right)\right]$

$= \frac{45}{16}$ or $2\frac{13}{16}$ ✓

21. The diagram shows the curve with equation $y = x^{1/2} + 4x^{-1/2}$. The line $y=5$, intersects the curve at the curve at the points $A(1,5)$ and $B(16,5)$.



(a) Find the equation of the tangent to the curve at the point A. ---[4]

(b) Calculate the area of the shaded region. ---[4]

[S-22/13/Q8]

Solution: Curve: $y = x^{1/2} + 4x^{-1/2}$ --- ①

Line: $y = 5$

(a) Diff. ① $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - 2x^{-3/2}$ --- ②

To find the tangent at $A(1,5)$ ✓

from ② $\left(\frac{dy}{dx}\right)_{x=1} = \frac{1}{2} - 2 = -\frac{3}{2}$ ✓

∴ Equation of the tangent at $A(1,5)$

$y - 5 = -3/2\,(x-1)$

$y = -3/2\,x + \frac{13}{2}$ ✓

(b) Area $= \int_a^b y\,dx$

= under the line − under the curve

$= \int_1^{16} [5 - (x^{1/2} + 4x^{-1/2})]\,dx$

$= \left[5x - \frac{2}{3}x^{3/2} - 8x^{1/2}\right]_1^{16}$

$= (80 - \frac{2}{3} \times 64 - 8 \times 4) - (5 - 2/3 - 8)$

$= 9$

∴ Area of the shaded region $= 9$ ✓

22. The function f is defined by $f(x) = (4x+2)^{-2}$ for $x > -\frac{1}{2}$

Find $\int_1^{\infty} f(x)\,dx$ --[4]

[S-22/13/Q10]

Solution: $f(x) = (4x+2)^{-2}$ for $x > -\frac{1}{2}$

$\therefore \int_1^{\infty} f(x)\,dx = \int_1^{\infty} (4x+2)^{-2}\,dx$

$= \left[\frac{(4x+2)^{-1}}{(-1) \times 4}\right]_1^{\infty}$

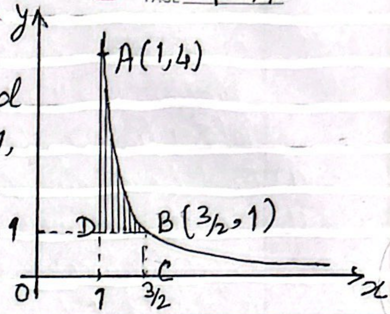
$= -\frac{1}{4}\left[\frac{1}{(4x+2)}\right]_1^{\infty}$

$= -\frac{1}{4}\left(0 - \frac{1}{6}\right)$

$= \frac{1}{24}$ ✓

23. The diagram shows part of the curve with equation $y = \frac{4}{(2x-1)^2}$ and part of the lines $x = 1$ and $y = 1$. The curve passes through the points $A(1, 4)$ and $B(\frac{3}{2}, 1)$.

Find the exact volume generated when the shaded region is rotated through 360° about the x-axis. [S-23/11/Q10(a)] --[5]



Solution: Volume generated by shaded area $= \int_1^{3/2} \pi y^2 \, dx$ − Volume of Cylinder $(r = 1, h = \frac{1}{2})$ —— (1)

Consider $\pi \int_1^{3/2} y^2 \, dx$

$$= \pi \int_1^{3/2} 16(2x-1)^{-4} \, dx = 16\pi \left[\frac{-1}{3 \times 2 \times (2x-1)^3} \right]_1^{3/2}$$

$$= -\frac{16\pi}{6} \left[\frac{1}{(3-1)^3} - \frac{1}{(2-1)^3} \right] = -\frac{16\pi}{6} \left[\frac{1}{8} - 1 \right]$$

$$= -\frac{16\pi}{6} \times -\frac{7}{8} = \frac{7}{3}\pi \quad \text{—— (2)}$$

Volume of the cylinder by revolving BD $\rightarrow \pi r^2 h = \pi \times 1^2 \times \frac{1}{2} = \frac{1}{2}\pi$ — (3)

$\therefore$ Required $V = \frac{7}{3}\pi - \frac{1}{2}\pi$

$$= \frac{11}{6}\pi \checkmark$$

24. The equation of a curve is such that $\frac{dy}{dx} = \frac{4}{(x-3)^3}$ for $x > 3$.
The curve passes through the point $(4,5)$.
Find the equation of the curve. ---[3]

S-23 / 12 / Q1

Solution: $\frac{dy}{dx} = \frac{4}{(x-3)^3}$ ------ ①

$\Rightarrow$ Equation of the curve, $y = \int \frac{4}{(x-3)^3} dx$

$\Rightarrow y = \frac{4(x-3)^{-3+1}}{-2} + c$

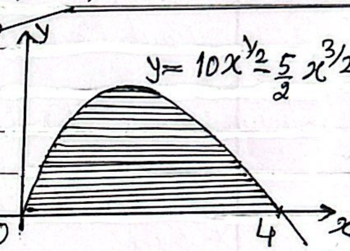
$\Rightarrow y = \frac{-2}{(x-3)^2} + c$ --- ②

Curve passes through $(4,5)$, from ② $5 = \frac{-2}{(4-3)^2} + c \Rightarrow c = 7$ -- ③

$\therefore$ The equation of the curve:

$y = \frac{-2}{(x-3)^2} + 7$ ✓ from ② & ③

25. The diagram shows the curve with equation $y = 10x^{1/2} - \frac{5}{2}x^{3/2}$ for $x > 0$.
The curve meets the x-axis at the points $(0,0)$ and $(4,0)$.
Find the area of the shaded region.



S-23 / 12 / Q5 ---[4]

Solution: The shaded area $= \int_a^b y\, dx$

$= \int_0^4 \left(10x^{1/2} - \frac{5}{2}x^{3/2}\right) dx$

$= \left[\frac{10x^{3/2}}{3/2} - \frac{5}{2} \cdot \frac{x^{5/2}}{5/2}\right]_0^4$

$= \left[\frac{20}{3}x^{3/2} - x^{5/2}\right]_0^4$

$= \left(\frac{20}{3} \times 4^{3/2} - 4^{5/2}\right) - (0-0)$

$= \frac{20}{3} \times 8 - 32$

$= \frac{64}{3}$ ✓ $\left(21\frac{1}{3}\right)$

26. A curve which passes through $(0,3)$, has equation $y=f(x)$.
It is given that $f'(x) = 1 - \frac{2}{(x-1)^3}$.
Find the equation of the curve. $\boxed{S\text{-}23/13/Q9(a)}$ --- [4]

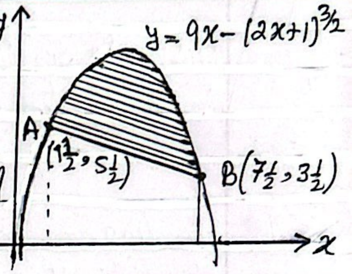
Solution: Given $f'(x) = 1 - \frac{2}{(x-1)^3} \Rightarrow y = \int\left(1 - \frac{2}{(x-1)^3}\right)dx$

$\Rightarrow y = x - 2\frac{(x-1)^{-2}}{-2} + C \Rightarrow y = x + (x-1)^{-2} + C$ --- (1)

Curve passes through $(0,3) \Rightarrow 3 = 0 + (0-1)^{-2} + C \Rightarrow C = 2$

$\therefore$ from (1) required equation of the curve: $y = x + (x-1)^{-2} + 2$ ✓

27. The diagram shows the points $A(1\frac{1}{2}, 5\frac{1}{2})$ and $B(7\frac{1}{2}, 3\frac{1}{2})$ lying on the curve with equation $y = 9x - (2x+1)^{3/2}$
Find the area of the shaded region. --- [5]



Solution: Curve: $y = 9x - (2x+1)^{3/2}$ $\boxed{S\text{-}23/13/Q10(c)}$

Area under the curve and x-axis (between $x = 1\frac{1}{2}$ & $x = 7\frac{1}{2}$) $= \int_{1.5}^{7.5}\left(9x - (2x+1)^{3/2}\right)dx$

$= \left[\frac{9}{2}x^2 - \frac{(2x+1)^{5/2}}{\frac{5}{2}\times 2}\right]_{1.5}^{7.5}$

$= \left[\frac{9}{2}(7.5)^2 - \frac{(2\times7.5+1)^{2.5}}{5}\right] - \left[\frac{9}{2}(1.5)^2 - \frac{1}{5}(2\times1.5+1)^{2.5}\right]$

$= 48.325 - 3.725 = 44.6$ --- (1)

Area under the segment AB and x-axis between $x = 3/2$ & $x = \frac{15}{2}$

$= \frac{1}{2}\left(5\frac{1}{2} + 3\frac{1}{2}\right)\times\left(7\frac{1}{2} - 1\frac{1}{2}\right) = \frac{1}{2}\times 9\times 6$

$= 27$ --- (2)

$\therefore$ Required shaded Area $= 44.6 - 27$

$= 17.6$ ✓ (from (1) & (2))

28. The equation of the curve is such that $\frac{dy}{dx} = \frac{1}{(x-3)^2} + x$.
It is given that the curve passes through the point $(2,7)$.
Find the equation of the curve. ---[4]

W-20/11/Q2

Solution: $\frac{dy}{dx} = \frac{1}{(x-3)^2} + x = (x-3)^{-2} + x$

$\therefore \quad y = \int\left((x-3)^{-2} + x\right)dx$

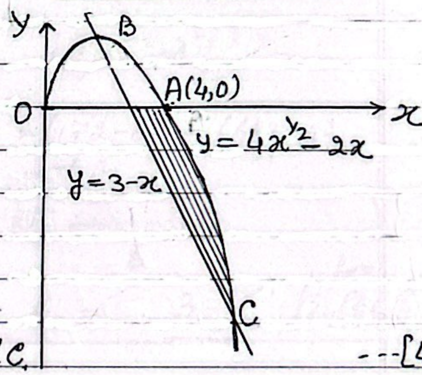
or $\quad y = \frac{(x-3)^{-1}}{-1} + \frac{x^2}{2} + C$

$y = \frac{-1}{(x-3)} + \frac{x^2}{2} + C$ ——— ①

Passes through $(2,7) \Rightarrow 7 = \frac{-1}{(2-3)} + \frac{2^2}{2} + C \Rightarrow C = 4$

$\therefore$ from ④ Equation of the curve is $y = \frac{-1}{(x-3)} + \frac{1}{2}x^2 + 4$ ✓

29. The diagram shows a curve with equation $y = 4x^{1/2} - 2x$ for $x \geq 0$, and a straight line with equation $y = 3 - x$. The curve crosses the x-axis at $A(4,0)$ and crosses the straight line at B and C.



(a) Find the x-coordinates of B and C. ---[4]
(b) Show that B is a stationary point on the curve. ---[2]
(c) Find the area of the shaded region. ---[6]

W-20/11/Q12

Solution: Line: $y = 3 - x$ ——— ①
Curve: $y = 4x^{1/2} - 2x$ ——— ②
from ① and ②
$3 - x = 4x^{1/2} - 2x$
$\Rightarrow \quad x - 4x^{1/2} + 3 = 0$ ——— ③

Let $x^{1/2} = z$ or $x = z^2$
$\therefore z^2 - 4z + 3 = 0$ from ③
$(z-1)(z-3) = 0$
$\Rightarrow z = 1,\ z = 3 \Rightarrow x = 1,\ x = 9$
$\therefore$ x-coord of $B = 1$ ✓
and x-coord of $C = 9$ ✓

(Continued →)

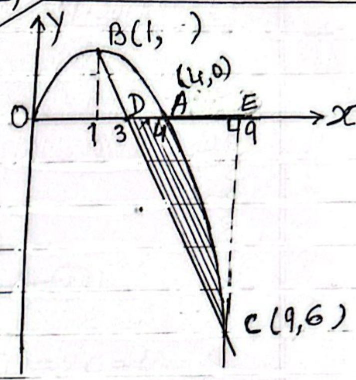
(Continued→)

29.(b) Curve: $y = 4x^{1/2} - 2x \Rightarrow \frac{dy}{dx} = 4 \times \frac{1}{2} x^{-1/2} - 2$

or $\frac{dy}{dx} = \frac{2}{\sqrt{x}} - 2$ —— ①

Gradient of the curve at B $(x=1)$, $\left(\frac{dy}{dx}\right)_{x=1} = \frac{2}{\sqrt{1}} - 2 = 0$

∴ B is a stationary point. ✓



(c) Draw AE perp to X-axis, E(9,0)

Let the line intersects the X axis at D(3,0)

Area of the shaded region

$=$ ar ΔCED − area between the curve and X-axis (from $x=4$ to $x=9$) —— ②

area of $\Delta CED = \frac{1}{2} \times DE \times CE$

$= \frac{1}{2} \times (9-3) \times 6 = 18$ —— ③

Area below the X-axis and the curve ($x=4$ to $x=9$)

$= \int_4^9 y\,dx = \int_4^9 (4x^{1/2} - 2x)\,dx$

$= \left[4\frac{x^{3/2}}{3/2} - x^2\right]_4^9 = \left[\frac{8}{3}x^{3/2} - x^2\right]_4^9$

$= (72 - 81) - \left(\frac{64}{3} - 16\right)$

$= -\frac{43}{3}$

Area $= \left|-\frac{43}{3}\right| = \frac{43}{3}$ —— ④

∴ Required shaded area

$= 18 - \frac{43}{3} = \frac{11}{3} = 3\frac{2}{3}$ ✓ (from ②, ③ & ④)

30. The point (4,7) lies on the curve $y = f(x)$ and it is given that $f'(x) = 6x^{-1/2} - 4x^{-3/2}$. Find the equation of the curve. ---[4]

[W-20/12/Q7(b)]

Solution:

$f'(x) = 6x^{-\frac{1}{2}} - 4x^{-3/2}$

$f(x) = \int (6x^{-1/2} - 4x^{-3/2})\,dx$

$f(x) = 6\frac{x^{1/2}}{1/2} - 4\frac{x^{-1/2}}{-1/2} + c$

$\Rightarrow f(x) = 12x^{1/2} + 8x^{-1/2} + c$ ··· ①

Curve $y = f(x)$ passes through point (4,7)

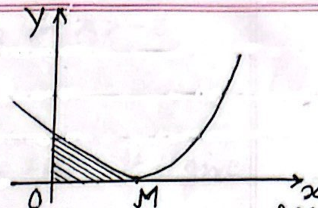
from ① $7 = 12 \times 4^{1/2} + 8 \times 4^{-\frac{1}{2}} + c$.

$\Rightarrow 7 = 24 + 4 + c \Rightarrow -21$

hence from ① Equation of the curve:

$f(x) = 12\sqrt{x} + \frac{8}{\sqrt{x}} - 21$ ✓

31. The diagram shows part of the curve, $y = \dfrac{2}{(3-2x)^2} - x$ and its minimum point M, which lies on the x-axis.



(a) Find an expression for $\int y\,dx$ ---[2]

(b) Find, by calculation, the x-coordinate of M. ---[2]

(c) Find the area of the shaded bounded by the curve and the coordinate axis. [W-20/12/Q10] ---[2]

Solution (a) $y = \dfrac{2}{(3-2x)^2} - x$ ——— ①

$\therefore \int y\,dx = \int \left(2(3-2x)^{-2} - x\right) dx$

$= \left(2 \times \dfrac{(3-2x)^{-1}}{(-1)\times(-2)} - \dfrac{x^2}{2}\right) = \dfrac{1}{(3-2x)} - \dfrac{1}{2}x^2 + c$ ——— ②

(b) for Min. $\dfrac{dy}{dx} = 0 \Rightarrow (3-2x)^3 - 8 = 0 \Rightarrow x = 1/2$ ✓

$\therefore$ x-coord of $M = \frac{1}{2}$, $M\left(\frac{1}{2}, 0\right)$

(c) The Shaded area $= \int_0^{\frac{1}{2}} y\,dx = \left[\dfrac{1}{(3-2x)} - \dfrac{1}{2}x^2\right]_0^{1/2}$ from ②

$= \left[\left(\frac{1}{2} - \frac{1}{8}\right) - \left(\frac{1}{3} - 0\right)\right]$

$= \frac{1}{2} - \frac{1}{8} - \frac{1}{3} = \frac{1}{24}$ ✓

32. The function f is defined by $f(x) = \frac{2}{(x+2)^2}$ for $x > -2$

(a) Find $\int_1^\infty f(x)\,dx$ --- [3]

(b) The equation of a curve is such that $\frac{dy}{dx} = f(x)$. It is given that the point $(-1,-1)$ lies on the curve. Find the equation of the curve. $\boxed{W-20/13/Q2}$ --- [2]

Solution (a) $f(x) = \frac{2}{(x+2)^2}$ for $x > -2$ —— ①

$$\therefore \int_1^\infty f(x)\,dx = \int_1^\infty \frac{2}{(x+2)^2}\,dx$$

$$= \int_1^\infty 2(x+2)^{-2}\,dx$$

$$= 2\left[\frac{-1}{x+2}\right]_1^\infty$$

$$= -2\left[0 - \frac{1}{3}\right] = \frac{2}{3} \checkmark$$

(b) $\frac{dy}{dx} = \frac{2}{(x+2)^2}$ Hence the equation of curve;

$$\Rightarrow y = \int 2(x+2)^{-2}\,dx$$

$$y = \frac{2x-1}{x+2} + C \quad \cdots\cdots ②$$

Given that the curve passes through the point $(-1,-1)$

from ② $-1 = \frac{-2}{1} + C \Rightarrow C = 1$

Hence from ② the equation of the curve is:

$$y = \frac{-2}{(x+2)} + 1 \checkmark$$

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33. It is given that $\int_{\frac{1}{4}k^2}^{k^2} \left(\frac{1}{k}x^{1/2} + x^{-\frac{1}{2}} + \frac{1}{k^2}\right) dx = \frac{13}{12}$; $k > 0$

Find the value of k. ... [5]

[W-20/13/Q10(b)]

<u>Solution:</u> $\int_{\frac{1}{4}k^2}^{k^2} \left(\frac{1}{k}x^{1/2} + x^{-\frac{1}{2}} + \frac{1}{k^2}\right) dx$ $\quad k > 0$

$$= \left[\frac{1}{k}\cdot\frac{2}{3}\cdot x^{3/2} + 2x^{1/2} + \frac{x}{k^2}\right]_{\frac{1}{4}k^2}^{k^2}$$

$$= \left[\left(\frac{2}{3k}\cdot k^3 + 2k + 1\right) - \left(\frac{2}{3k}\times\frac{1}{8}k^3 + 2\times\frac{1}{2}k + \frac{1}{4}\right)\right]$$

$$= \left[\frac{2}{3}k^2 + 2k + 1 - \frac{k^2}{12} - k - \frac{1}{4}\right]$$

$$= \frac{7}{12}k^2 + k + \frac{3}{4} = \frac{13}{12} \text{ (Given)}$$

$$\Rightarrow 7k^2 + 12k - 4 = 0$$

$$7k^2 + 14k - 2k - 4 = 0$$

$$7k(k+2) - 2(k+2) = 0$$

$$\Rightarrow (k+2)(7k-2) = 0$$

$$\Rightarrow k = \frac{2}{7} \text{ or } k = -2 \quad (\because k > 0)$$

$$\therefore k = \frac{2}{7} \text{ (or } k = 0.286)$$

34 A curve has equation $y = f(x)$; and $f'(x) = 2x^2 - 7 - 4/x^2$.

(a) Given that $f(1) = -\frac{1}{3}$, find $f(x)$. ---[4]

(b) Find the coordinates of the stationary points on the curve. ---[5]

(c) Find $f''(x)$ ---[1]

(d) Determine the nature of each of the stationary points. ---[2]

W-21/11/Q9

Solution: (a) $f'(x) = 2x^2 - 7 - 4x^{-2}$ --- ①

$\Rightarrow f(x) = \int (2x^2 - 7 - 4x^{-2})\,dx$

$f(x) = \frac{2}{3}x^3 - 7x + 4x^{-1} + c$ — ②

for $f(1) = -\frac{1}{3} \Rightarrow -\frac{1}{3} = \frac{2}{3} - 7\times 1 + 4\times 1^{-1} + c$

$\Rightarrow c = 2$ ✓

from ② eqⁿ of curve

$f(x) = \frac{2}{3}x^3 - 7x + 4x^{-1} + 2$ --- ③

(b) for Stationary points $\frac{dy}{dx} = 0$

from ① $2x^2 - 7 - \frac{4}{x^2} = 0$.

$\Rightarrow 2x^4 - 7x^2 - 4 = 0 \Rightarrow (2x^2+1)(x^2-4) = 0$

$\Rightarrow x = \pm 2$ or $x^2 = -\frac{1}{2}$ ✗

from ③ $x = 2$, $y = \frac{2}{3}\times 8 - 7\times 2 + \frac{4}{2} + 2 = -\frac{14}{3}$ ✓

and $x = -2$, $y = \frac{2}{3}\cdot(-8) - 7\cdot(-2) + \frac{4}{-2} + 2 = \frac{26}{3}$ ✓

$\therefore$ Stationary points are $(2, -\frac{14}{3})$, $(-2, \frac{26}{3})$

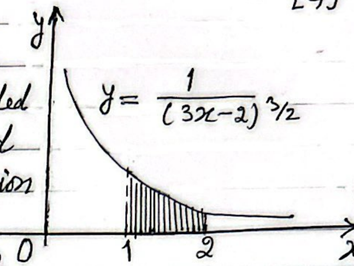
(c) $f''(x) = 4x + \frac{8}{x^3}$ — ④ ✓

(d) $f''(2) = 8 + 8/8 = 9 > 0$ Min at $(2, -\frac{14}{3})$

$f''(-2) = -8 - \frac{8}{8} = -9 < 0$ Max. at $(-2, \frac{26}{3})$ ✓

35. (a) Find $\int_1^{\infty} \frac{1}{(3x-2)^{3/2}}\,dx$ ---[4]

The diagram shows the curve with equation $y=\frac{1}{(3x-2)^{3/2}}$. The shaded region is bounded by the curve, the x-axis and the lines $x=1$ and $x=2$. The shaded region is roteted through 360° about the x-axis.



(b) Find the volume of revolution. ---[4]

W-21 | 11 | Q 10(a)(b)

Solution (a) $\int_1^{\infty} \frac{1}{(3x-2)^{3/2}}\,dx = \int_1^{\infty} (3x-2)^{-3/2}\,dx = \left[\frac{(3x-2)^{-1/2}}{-\frac{1}{2}\times 3}\right]_1^{\infty}$

$= -\frac{2}{3}\left[0-1\right] = \frac{2}{3}$ ✓

(b) Volume $= \int_1^2 \pi y^2\,dx = \pi\int_1^2 \left[(3x-2)^{-3/2}\right]^2 dx$

$= \pi\int_1^2 (3x-2)^{-3}\,dx$

$= \pi\left[\frac{(3x-2)^{-2}}{-2\times 3}\right]_1^2$

$= -\frac{\pi}{6}\left[\frac{1}{(3x-2)^2}\right]_1^2$

$= -\frac{\pi}{6}\left[\frac{1}{16}-1\right]$

$= -\frac{\pi}{6}\times -\frac{15}{16}$

$= \frac{5\pi}{32}$ ✓

36 A Curve is such that, $\frac{dy}{dx} = \frac{8}{(3x+2)^2}$. The curve passes through the point $(2, 5\frac{2}{3})$. Find the equation of the curve. ---[4]

[W-21/12/Q4]

Solution: $\frac{dy}{dx} = \frac{8}{(3x+2)^2}$

$y = \int 8\cdot(3x+2)^{-2}\,dx$

$y = \frac{8\cdot(3x+2)^{-1}}{-1\times 3} + c$

$y = -\frac{8}{3}\times\frac{1}{(3x+2)} + c$ ---(1)

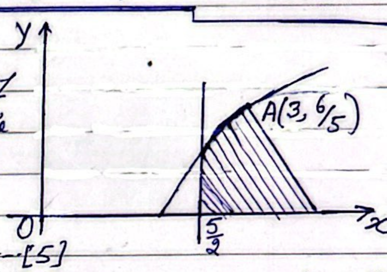
(1) Passes through the point $(2, 5\frac{2}{3})$

$\Rightarrow \frac{17}{3} = -\frac{8}{3}\times\frac{1}{8} + C \Rightarrow C = 6$

$\therefore$ from (1) Equation of the curve is:

$y = \frac{-8}{3(3x+2)} + 6$ ✓

37 The diagram shows the line $x = 5/2$, part of the curve $y = \frac{1}{2}x + \frac{7}{10} - \frac{1}{(x-2)^{1/3}}$ and the normal to the curve at $A(3, 6/5)$.



(a) Find the coordinates of the point where normal to the curve meets the x-axis ---[5]

(b) Find the area of the shaded region, giving your answer correct to 2dp.

[W-21/12/11] ---[6]

Solution: $y = \frac{1}{2}x + \frac{7}{10} - \frac{1}{(x-2)^{1/3}}$ ---(1)

(a) $\frac{dy}{dx} = \frac{1}{2} + \frac{1}{3(x-2)^{4/3}}$ ---(2)

$\left(\frac{dy}{dx}\right)_{x=3} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ ✓

$\therefore$ Gradient of Normal at $(3, \frac{6}{5}) = -\frac{6}{5}$

Equation of Normal,

$y - 6/5 = -\frac{6}{5}(x-3)$

$\Rightarrow 5y = -6x + 24$ ---(3)

(3) Normal intersect x-axis at $y=0$

at P; $0 = -6x + 24 = x = 4$, $P(4,0)$ ✓

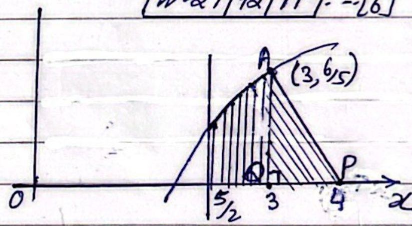
(b) Area of the shaded region; $AQ \perp x$-axis

= ar Δ PQP + area under the curve, $x = \frac{5}{2}$ to $x = 3$ ---(4)

Area of Δ AQP $= \frac{1}{2} \times AQ \times QP$

$= \frac{1}{2} \times \frac{6}{5} \times (4-3)$

$= \frac{3}{5} = 0.6$ ---(5)



Area under the curve;

$= \int_{5/2}^{3} \left(\frac{1}{2}x + \frac{7}{10} - (x-2)^{-1/3}\right) dx$

$= \left[\frac{1}{4}x^2 + \frac{7}{10}x - \frac{3(x-2)^{2/3}}{2}\right]_{5/2}^{3}$

$= \left(\frac{9}{4} + \frac{21}{10} - \frac{3}{2}\right) - \left(\frac{6.25}{4} + 1.75 - \frac{3\times(\frac{1}{2})^{2/3}}{2}\right)$

$= 0.4824$ ---(6)

$\therefore$ from (4), (5), (6) Req. Shaded area

$= 0.6 + 0.48$

$= 1.08$ ✓

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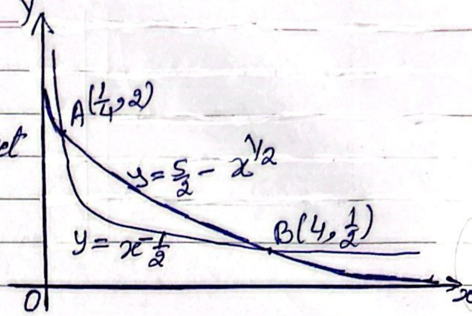
38. The diagram shows the curves with equations $y = x^{-\frac{1}{2}}$ and $y = 5/2 - x^{1/2}$. The curves intersect at the points $A(\frac{1}{4}, 2)$, $B(4, \frac{1}{2})$.
Find the area of the region between the two curves. ---- [6]



W-21/13/Q 8(a)

Solution: Area of the region between two curves;

$$\int_{\frac{1}{4}}^{4} \left(\frac{5}{2} - x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)dx = \left[\frac{5}{2}x - \frac{2}{3}x^{3/2} - 2x^{1/2}\right]_{\frac{1}{4}}^{4}$$

$$= \left(10 - \frac{16}{3} - 4\right) - \left(\frac{5}{8} - \frac{1}{12} - 1\right) = \frac{9}{8} \checkmark \text{ (or } 0.125)$$

39. A curve has equation $y = f(x)$ and $f'(x) = (\frac{1}{2}x + k)^{-2} - (1+k)^{-2}$, where k is a constant. The curve has a minimum point at $x = 2$

(a) Find $f''(x)$ in terms of k and x, hence find the set of values of k. ---[3]

It is now given that $k = -3$, and the minimum point is at $(2, 3\frac{1}{2})$.

(b) Find $f(x)$ ---[4]

(c) Find the other stationary point and determine its nature. --[4]

W-21/13/Q 10

Solution: $f'(x) = (\frac{1}{2}x + k)^{-2} - (1+k)^{-2}$ --- ①

(a) $f''(x) = -2 \times \frac{1}{2}(\frac{1}{2}x + k)^{-3}$

$= -(\frac{1}{2}x + k)^{-3}$ —— ②

$f''(2) = -(1+k)^{-3} > 0$ (Given $x = 2$ is min)

$\Rightarrow \frac{1}{(1+k)^3} < 0 \Rightarrow \frac{1}{1+k} < 0$

$\Rightarrow k + 1 < 0 \Rightarrow \underline{k < -1} \checkmark$

$\boxed{\text{for } k = -3}$

(b) $f(x) = \int \left((\frac{1}{2}x - 3)^{-2} - (-2)^{-2}\right) dx$

$f(x) = \frac{(\frac{1}{2}x - 3)^{-1}}{-1 \times \frac{1}{2}} - \frac{x}{4} + C$ —②

Minimum at $(2, 3\frac{1}{2})$ Put in ②

$3\frac{1}{2} = 1 - \frac{1}{2} + C \Rightarrow C = 3 \checkmark$

$\therefore$ from ②

$f(x) = \frac{-2}{(\frac{1}{2}x - 3)} - \frac{x}{4} + 3 \checkmark$ —③

(c) for stationary point (for $k = -3$) from ①

$\frac{dy}{dx} = 0 \Rightarrow (\frac{1}{2}x - 3)^{-2} - (-2)^{-2} = 0$

$\Rightarrow (\frac{1}{2}x - 3)^2 = 4 \Rightarrow \frac{1}{2}x - 3 = \pm 2$

$x = 10$ or ($x = 2$ already given) ×

from ③ $x = 10 \Rightarrow y = -\frac{1}{2}$

$\Rightarrow$ other stationary point $\underline{(10, -\frac{1}{2})} \checkmark$

$f''(x) = -(\frac{1}{2}x - 3)^{-3}$ [for $k = -3$]

$f''(10) = -(2)^{-3} = -\frac{1}{8} < 0$

$\therefore$ Max at $(10, -\frac{1}{2}) \checkmark$

40 The equation of a curve is such that $\frac{dy}{dx} = 12\left(\frac{1}{2}x-1\right)^{-4}$. It is given that the curve passes through the point $P(6,4)$. Find the equation of the curve. ---[2]

W-22 | 11 | Q2(b)

Solution: $\frac{dy}{dx} = 12\left(\frac{1}{2}x-1\right)^{-4} \Rightarrow y = \int 12\left(\frac{1}{2}x-1\right)^{-4}dx$

$\Rightarrow y = \frac{12\left(\frac{1}{2}x-1\right)^{-3}}{-3\times\frac{1}{2}} \Rightarrow y = -8\left(\frac{1}{2}x-1\right)^{-3}+c$ --- (1)

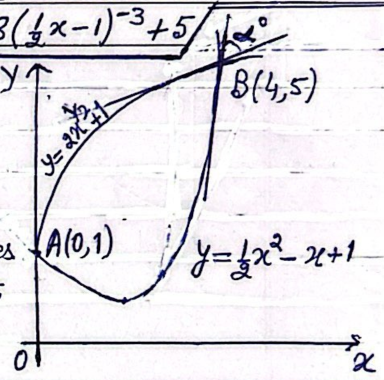
the curve passes through $P(6,4)$, from (1)

$\Rightarrow 4 = -8\left(\frac{1}{2}\times 6-1\right)^{-3}+c$

$\Rightarrow 4 = -8\times 2^{-3}+c \Rightarrow c = 5$

Hence from (1) Equation the curve: $y = -8\left(\frac{1}{2}x-1\right)^{-3}+5$

41 Curves with equations $y = 2x^{1/2}+1$ and $y = \frac{1}{2}x^2-x+1$ intersect at $A(0,1)$ and $B(4,5)$. --- [5]



(a) Find the area of the region between two curves.

The acute angle between the two tangent at B is denoted by $\alpha°$, and the scales on the axes are the same.

(b) Find α. ---[5]

W-22 | 11 | Q10

Solution: Area between the curves:

(a) $= \int_0^4 \left\{\left(2x^{1/2}+1\right)-\left(\frac{1}{2}x^2-x+1\right)\right\}dx$

$= \int_0^4 \left(2x^{1/2}-\frac{1}{2}x^2+x\right)dx$

$= \left[\frac{4x^{3/2}}{3} - \frac{x^3}{6} + \frac{x^2}{2}\right]_0^4$

$= \left[\left(\frac{4\times 4^{3/2}}{3} - \frac{4^3}{6} + \frac{4^2}{2}\right) - 0\right]$

$= \frac{32}{3} - \frac{32}{3} + 8$

$= 8$

(b) upper curve, $y = 2x^{1/2}+1$

$\frac{dy}{dx} = x^{-1/2}$

At $B(4,5)$ $\left(\frac{dy}{dx}\right)_{x=4} = 4^{-1/2} = \frac{1}{2} = m_1$ — (1)

for Lower curve, $y = \frac{1}{2}x^2-x+1$

$\frac{dy}{dx} = x-1$

at B, $m_2 = \left(\frac{dy}{dx}\right)_{x=4} = 4-1 = 3$ — (2)

$\therefore \alpha = \tan^{-1}3 - \tan^{-1}\frac{1}{2} = 71.75° - 26.57°$

$\alpha = 45°$

42. The equation of a curve is such that $\frac{dy}{dx} = 3x^{1/2} - 3x^{-1/2}$. The curve passes through the point (3,5). Find the equation of the curve. ---[4]

[W-22/12/Q8(a)]

Solution: $\frac{dy}{dx} = 3x^{1/2} - 3x^{-1/2} \Rightarrow y = \int (3x^{1/2} - 3x^{-1/2})dx = \frac{3x^{3/2}}{3/2} - \frac{3x^{1/2}}{1/2} + c$

$\Rightarrow y = 2x^{3/2} - 6x^{1/2} + c$ ---①

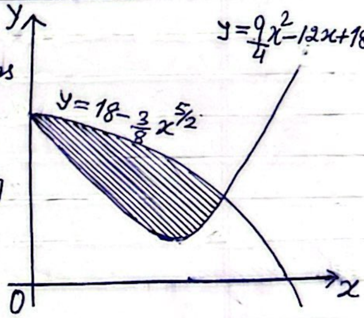
Point (3,5) lies on the curve ① $\Rightarrow 5 = 2 \times 3^{3/2} - 6 \times 3^{1/2} + c$

$\Rightarrow 5 = \cancel{6\sqrt{3}} - \cancel{6\sqrt{3}} + c \Rightarrow c = 5$

hence the equation of the curve is: $y = 2x^{3/2} - 6x^{1/2} + 5$ ✓

43. The diagram shows the curves with equations $y = \frac{9}{4}x^2 - 12x + 18$ and $y = 18 - \frac{3}{8}x^{5/2}$. The curves intersect at the points (0,18) and (4,6). Find the area of the shaded region. ---[5]

[W-22/12/Q11(b)]



Solution: Area of the shaded region $= \int_0^4 \left((18 - \frac{3}{8}x^{5/2}) - (\frac{9}{4}x^2 - 12x + 18) \right) dx$

$= \left[\left(\cancel{18x} - \frac{3 \times x^{7/2}}{8 \times 7/2} \right) - \left(\frac{9x^3}{4 \times 3} - \frac{12 \times x^2}{2} + \cancel{18x} \right) \right]_0^4$

$= \left[-\frac{3}{28}x^{7/2} - \frac{3}{4}x^3 + 6x^2 \right]_0^4$

$= \left(-\frac{3}{28} \cdot 4^{7/2} - \frac{3}{4} \times 4^3 + 6 \times 4^2 \right) - 0$

$= -\frac{3}{28} \times 128 - 48 + 96$

$= -\frac{96}{7} + 48 = \frac{240}{7}$ (or 34.3) ✓

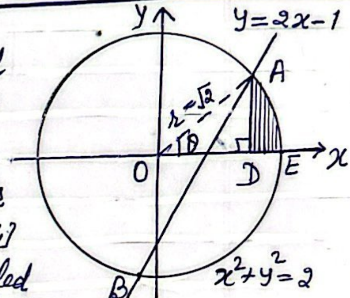
44. The curve $y = f(x)$ is such that $f'(x) = \dfrac{-3}{(x+2)^4}$
Find $f(x)$ given that the curve passes through the point $(-1,5)$ --- [3]

[W-22/13/Q7(b)]

Solution: $f(x) = \int \dfrac{-3}{(x+2)^4}\,dx = \int -3(x+2)^{-4}\,dx = -3 \times \dfrac{(x+2)^{-3}}{-3} + C$

$\Rightarrow f(x) = \dfrac{1}{(x+2)^3} + C$, given that it passes through $(-1,5)$

$\Rightarrow 5 = \dfrac{1}{1^3} + C \Rightarrow C = 4 \Rightarrow$ hence $f(x) = \dfrac{1}{(x+2)^3} + 4$ ✓



45 The diagram shows the circle $x^2+y^2=2$ and the straight line $y=2x-1$, intersecting at the points A and B. The point D on the x-axis is such that AD is perpendicular to the x-axis.

(a) Find the coordinates of A. --- [4]

(b) Find the volume of revolution when the shaded region is rotated through 360° about the x-axis. Give answer in the form $\dfrac{\pi}{a}(b\sqrt{c}-\pi)$, where a, b, c and d are integers. --- [4]

(c) Find an exact expression for the perimeter of the shaded region. --- [2]

[W-22/13/Q10]

Solution: Circle: $x^2+y^2=2$ --- ①

Line: $y = 2x-1$ --- ②

(a) from ① & ② $x^2+(2x-1)^2 = 2$

$\Rightarrow 5x^2-4x-1=0$

$(5x+1)(x-1)=0 \Rightarrow x = 1, -\frac{1}{5}$ ✗

$x=1$, $y=1$ from ② $\underline{A(1,1)}$ [∵ A lies in the Ist quad.]

(b) $V = \pi\int y^2\,dx$ Let the circle intersects x-axis at $E(\sqrt{2},0)$

$= \pi\int_1^{\sqrt{2}} (2-x^2)\,dx$ [from ① $y^2 = 2-x^2$

$= \pi\left[2x - \dfrac{x^3}{3}\right]_1^{\sqrt{2}}$

$= \pi\left[\left(2\sqrt{2} - \dfrac{(\sqrt{2})^3}{3}\right) - \left(2-\frac{1}{3}\right)\right]$

$= \dfrac{\pi}{3}(4\sqrt{2}-5)$ ✓

(c) Perimeter of shaded region

$= \text{arc } \widehat{AE} + AD + DE$ --- ③

$\tan\theta = \dfrac{AD}{OD} = \dfrac{1}{1} = 1$ [∵ A(1,1)

$\theta = \tan^{-1} = \dfrac{\pi}{4}$

length of arc $\widehat{AE} = r\theta = \sqrt{2}\times\dfrac{\pi}{4}$

$AD = 1$, $DE = OE - OD = (\sqrt{2}-1)$

Hence from ③

Perimeter $= \sqrt{2}\cdot\dfrac{\pi}{4} + 1 + (\sqrt{2}-1)$

$= \dfrac{\pi}{4}\sqrt{2} + \sqrt{2}$

$= \sqrt{2}\left(\dfrac{\pi}{4}+1\right)$ ✓