Date 25.12.23.
Page

P_1

Pure Maths_1

Series_ A.P and G.P

Exercise 1. Solution (Revision)

SP-20	M-20	M-22	S-20	S-22	W-20
W-22	M-21	M-23	S-21	S-23	W-21

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Series (AP and GP) - Revision

Example 1: An arithmetic progression has first term 7. The n_{th} term is 84 and the $(3n)_{th}$ term is 245. Find the value of n.
SP-20/01/Q3 ---[4]

Solution: First term $a = 7$.

n_{th} term: $a+(n-1)d = 84$

or $7+(n-1)d = 84$

$(n-1)d = 77$ — ①

and $(3n)_{th}$ term: $7+(3n-1)d = 245$

or $(3n-1)d = 238$ — ②

from ① ÷ ② ⇒ $\dfrac{n-1}{3n-1} = \dfrac{\cancel{77}\ 11}{\cancel{238}\ 34} = \dfrac{11}{34}$

$\Rightarrow 33n - 11 = 34n - 34 \Rightarrow n = 23$ ✓

Example 2: A woman's basic salary for her first year with a particular company is \$30 000 and at the end of the year she also gets a bonus of \$600.

(a) For her first year, express her bonus as a percentage of her basic salary. --[1]

At the end of each complete year, the woman's basic salary will increase by 3% and her bonus will increase by \$100.

(b) Express the bonus she will be paid at the end of her 24_{th} year as a percentage of the basic salary paid during that year. ---[5]
M-20/12/Q8

Solution (a) $\dfrac{600}{30{,}000} \times 100 = 2\%$ ✓

(b) Bonus $= 600 + 23 \times 100 = 2900$ ✓

Salary $= 30\,000 \times (1.03)^{23} = 59207.60$ ✓

$\therefore \dfrac{2900}{59200} \times 100 \quad 4.9\%$ ✓

3. The first term of a progression is $\cos\theta$, where $0<\theta<\frac{1}{2}\pi$
(a) For the case where the progression is geometric, the sum to infinity is $\frac{1}{\cos\theta}$.
(i) Show that the second term is $\cos\theta \cdot \sin^2\theta$ ---[3]
(ii) Find the sum of the first 12 terms, when $\theta=\frac{1}{3}\pi$, giving your answer correct to 4 significant figures. ---[2]
(b) For the case when the progression is arithmetic, the first two terms are again $\cos\theta$ and $\cos\theta \cdot \sin^2\theta$ respectively. Find the 85_{th} term when $\theta=\frac{1}{3}\pi$. --[4]

M-21/12/Q9

Solution: First term $a=\cos\theta$; Sum of infinte geometric series:
(i)
$$S_\infty=\frac{a}{1-r}=\frac{1}{\cos\theta}$$
$$\Rightarrow \frac{\cos\theta}{1-r}=\frac{1}{\cos\theta} \Rightarrow 1-r=\cos^2\theta$$
$$\Rightarrow r=1-\cos^2\theta=\sin^2\theta$$
$\therefore$ Second term of G.P. $= ar = \cos\theta \cdot \sin^2\theta$

(ii) $$S_{12}=\frac{a[1-r^{12}]}{(1-r)}=\frac{\frac{1}{2}\left[1-\left(\frac{3}{4}\right)^{12}\right]}{1-3/4}$$
$\because$ for $\theta=\frac{\pi}{3}$: $a=\cos\frac{\pi}{3}=\frac{1}{2}$ and $r=\sin^2\frac{\pi}{3}=\frac{3}{4}$.
$$=2\left[1-\left(\frac{3}{4}\right)^{12}\right]=1.9366$$
$$=1.937 \text{ (4s.f)}$$

(b) Now $\theta=\frac{\pi}{3}$, $a=\cos\frac{\pi}{3}=\frac{1}{2}$ for A.P
and second term $=\cos\theta\sin^2\theta=\cos\frac{\pi}{3}\cdot\sin^2\frac{\pi}{3}=\frac{1}{2}\times\frac{3}{4}=\frac{3}{8}$
$$\Rightarrow a+d=\frac{3}{8}$$
$$\frac{1}{2}+d=3/8 \Rightarrow d=\frac{3}{8}-\frac{1}{2}=-\frac{1}{8}$$

$\therefore$ 85_{th} term $=a+(n-1)d=\frac{1}{2}+84\times\left(-\frac{1}{8}\right)$
$$=\frac{1}{2}-\frac{21}{2}=-10$$
$\therefore$ 85_{th} term of A.P $=\underline{-10}$

4 The first term of a G.P and the first term of an A.P are both equal to a. The third term of G.P is equal to the second term of the A.P. The fifth term of the G.P is equal to the sixth term of the A.P. Given that all the terms are positive and not all equal, find the sum of first twenty terms of the A.P in terms of a. ---[6]

M-22 / 12 / Q4

Solution: Let AP: $a, a+d, a+2d, a+3d, a+4d, a+5d, ---$

GP: $a, ar, ar^2, ar^3, ar^4, ----$

Given: Third term of G.P = Second term of A.P $\Rightarrow ar^2 = a+d$ ---(i)

Fifth term of G.P = Sixth term of A.P $\Rightarrow ar^4 = a+5d$ --(ii)

multiply (ii) by $a \Rightarrow a^2r^4 = a(a+5d)$ -- (3)

and squaring (i) $\Rightarrow a^2r^4 = (a+d)^2$ -- (4)

from (3) & (4) $(a+d)^2 = a(a+5d)$

$\Rightarrow \cancel{a^2} + d^2 + 2ad = \cancel{a^2} + 5ad$

$\Rightarrow 5ad - 2ad - d^2 = 0$

$\Rightarrow 3ad - d^2 = 0$

$\Rightarrow d(3a-d) = 0 \Rightarrow d = 3a$ ✓ or $d = 0$ ✗

$\therefore$ Sum of 20 terms of A.P

$\left[S_n = \frac{n}{2}(2a+(n-1)d) \right]$
$\left[n = 20, \quad d = 3a \right]$

$S_{20} = \frac{20}{2}[2a + (20-1) \times 3a]$

$= 10(2a + 57a)$

$S_{20} = 590a$ ✓

5. The circumference round the trunk of a large tree is measured and found to be 5.00 m. After one year the circumference is measured again and found to be 5.02 m.

(a) Given that the circumference at yearly intervals form an A.P., find the circumference 20 years after the first measurement. --- [2]

(b) Given instead that the circumference at yearly intervals form a G.P., find the circumference 20 years after the first measurement. --- [3]

M-23 / 12 / Q4

Solution (a) first term = 5.00

Common difference = 5.02 − 5.00 = 0.02

Number of terms $n = 21$

for A.P ∴ Required value $= a + (n-1)d$

$= 5.00 + (21-1) \times 0.02$

$= 5.40\,m$ ✓

(b) first term $a = 5.00$

Common ratio $r = \frac{5.02}{5} = 1.004$

number of terms $n = 21$ of G.P

∴ Required Value $= a \cdot r^{n-1} = 5 \times (1.004)^{20}$

$= 5.41557$

$= 5.42\,m$ ✓

Example 6 The sum of the first nine terms of an arithmetic progression is 117. The sum of the next four terms is 91. Find the first term and the common difference of the progression. [S-20/11/Q 1] ---[4]

Solution: $S_n = \frac{n}{2}[2a+(n-1)d]$

$S_9 = \frac{9}{2}[2a+8d] = 117 \Rightarrow 2a+8d = 26$

$\Rightarrow a+4d = 13$ — ①

Sum of next 4 terms $= 91$

$\Rightarrow$ Sum of total 13 terms $= 117+91$

$S_{13} = 208 \Rightarrow \frac{13}{2}[2a+12d] = 208$

$\Rightarrow 2a+12d = 32$

$\Rightarrow a+6d = 16$ — ②

Solving ① and ② $\Rightarrow$ $a = 7$ and $d = 1.5$

7. Each year the selling price of diamond necklace increases by 5% of the price year before. The selling price of the necklace in year 2000 was \$ 36000.

(a) Write down an expression for the selling price of the necklace n years later and hence find the selling price in 2008. ---[3]

(b) The company that makes the necklace only sells one each year. Find the total amount of money obtained in the ten-year period starting in the year 2000. ---[2]

[S-20/11/Q3]

Solution: Price in 2000 = \$ 36000, Rate of increase = 5% = 0.05

(a) G.P → Common ratio = After one year will be $(1+0.5) = 1.05$ times

Selling price after 'n' years $= \$\,36000 \times (1.05)^n$ [Note: It is not the n_{th} term of G.P. but $(n+1)_{th}$ term.

Selling price after 8 years $= 36000 \times (1.05)^8$

$= \$\,53188$

$= \$\,53200$ (3 sf)

(b) $S_{10} = 36000 \times \frac{(1.05^{10}-1)}{(1.05-1)} = \$\,453000$ $\left\{ S_n = \frac{a[r^n-1]}{(r-1)} \right.$

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Example 8. The n_{th} term of an arithmetic progression is $\frac{1}{2}(3n-15)$
Find the value of n for which the sum of the first n terms is 84.
[S-20/12/Q4] ---[5]

Solution: n_{th} term $= \frac{1}{2}(3n-15)$

for $n=1$, first term $= \frac{1}{2}(3\times1-15) = -6 \Rightarrow a=-6$ ✓

for $n=2$, second term $= \frac{1}{2}(3\times2-15) = -4.5$

2nd term $a+d = -4.5$

$-6+d = -4.5$

$d = 1.5$ ✓ $= 3/2$ ✓

Now $S_n = \frac{n}{2}[2a+(n-1)d] = 84$

$n[2\times(-6)+(n-1)\times 3/2] = 168$

$n[-12 + \frac{3}{2}n - \frac{3}{2}] = 168$

$\Rightarrow 3n^2 - 27n - 336 = 0$

$\Rightarrow n^2 - 9n - 112 = 0$

$(n-16)(n+7) = 0$

$n = 16$ ✓ (or $n = -7$ ✗)

$\therefore n = 16$ ✓

Example 9: The first term of a progression is $\sin^2\theta$, where $0<\theta<\frac{\pi}{2}$,
The second term of the progression is $\sin^2\theta\cos^2\theta$.

(a) Given that the progression is geometric, find sum to infinity. ---[3]

It is now given instead that the progression is arithmetic.

(b) (i) Find the common difference of the progression in terms of $\sin\theta$. --[3]

(ii) Find the sum of the first 16 terms when $\theta=\frac{1}{3}\pi$. --[3]

[S-20/13/Q8]

(a) $a=\sin^2\theta$; G.P, second term $ar=\sin^2\theta\cos^2\theta$

$\Rightarrow \sin^2\theta \times r=\sin^2\theta\cos^2\theta$ $\left\{\begin{array}{l}0<\theta<\frac{\pi}{2}\\ r<1\end{array}\right.$

$\Rightarrow$ common ratio $r=\cos^2\theta$.

$\therefore S_\infty=\frac{a}{1-r}=\frac{\sin^2\theta}{1-\cos^2\theta}$

$=\frac{\sin^2\theta}{\sin^2\theta}=1$ ✓

(b) $a=\sin^2\theta$; Second term of A.P $=a+d=\sin^2\theta\cos^2\theta$

(i) or $\sin^2\theta+d=\sin^2\theta\cos^2\theta$

$\Rightarrow d=\sin^2\theta\cos^2\theta-\sin^2\theta=-\sin^2\theta(1-\cos^2\theta)$

$\Rightarrow d=-\sin^4\theta$ ✓

(ii) $a=\sin^2\theta=\sin^2\frac{\pi}{3}=\left(\frac{\sqrt{3}}{2}\right)^2=\frac{3}{4}$ ✓ $\left(\theta=\frac{\pi}{3}\right)$

$d=-\sin^4\theta=-\left(\frac{\sqrt{3}}{2}\right)^4=-\frac{9}{16}$ ✓

$n=16$ ✓

$S_n=\frac{n}{2}\left[2a+(n-1)d\right]$

$S_{16}=\frac{16}{2}\left[2\times\frac{3}{4}+(16-1)\times\left(-\frac{9}{16}\right)\right]$

$=8\left[\frac{3}{2}-\frac{135}{16}\right]=-\frac{111}{2}$

$\therefore S_{16}=-55\frac{1}{2}$ ✓

10. The sum of the first 20 terms of an A.P. is 405 and the sum of first 40 terms is 1410.
Find the 60_{th} term of the progression. ...[5]
S-21/11/Q2

Solution: $S_{20} = \frac{20}{2}(2a+(20-1)d)$ $\left(\because S_n = \frac{n}{2}[2a+(n-1)d] \text{ for A.P}\right)$
$\Rightarrow 10(2a+19d) = 405$ given
$\Rightarrow 20a + 190d = 405$ —— ①
$S_{40} = \frac{40}{2}(2a+(40-1)d) \Rightarrow$
$= 20(2a+39)d = 1410$ (given) $\Rightarrow 4a+78d = 141$ —— ②
Solving ① & ② $a=6$, $d=1.5$
60_{th} term $= 6+(60-1)\times 1.5$ $[n_{th}\text{ term} = a+(n-1)d]$
$= 6 + 59\times 1.5 = 94.5$ ✓

11. The fifth, sixth and seventh terms of a geometric progression are $8k$, -12 and $2k$ respectively. Given k is negative, find the sum to infinity of the progression. ...[4]
S-21/11/Q5

Solution: $8k$, -12 and $2k$ are in G.P
$\Rightarrow \frac{-12}{8k} = \frac{2k}{-12} \Rightarrow (-12)^2 = 8k\times 2k$
$\Rightarrow 16k^2 = 144 \Rightarrow k^2 = 9 \Rightarrow k = -3, 3$ (Given $k<0$)
Now $r = \frac{-12}{8k} = \frac{-12}{8\times(-3)} = \frac{1}{2}$; 6_{th} term $ar^5 = -12 \Rightarrow a = \frac{-12}{(\frac{1}{2})^5} = -384$
Hence sum to infinity of G.P $= \frac{a}{1-r} = \frac{-384}{1-\frac{1}{2}} = -768$ ✓

12. The first, second and third terms of an A.P are a, $\frac{3}{2}a$ and b, where a and b are positive constants. The first, second and third terms of a G.P are a, 18 and $b+3$ respectively.

(a) Find the values of a and b. --- [5]

(b) Find the sum of the first 20 terms of the A.P. --- [3]

[S-21/12/Q 8]

Solution (a) a, $\frac{3}{2}a$ and b are in A.P. [$\because$ a, b, c are in A.P $\Rightarrow a+c=2b$]

$\Rightarrow a+b = 2\times\frac{3}{2}a \Rightarrow b=2a$ — ①

and $a, 18, (b+3)$ are in G.P [$\because$ a, b, c are in G.P $= b^2=ac$]

$\Rightarrow 18^2 = a(b+3)$ — ②

$324 = a(2a+3)$ [from ① $b=2a$]

$\Longrightarrow 2a^2+3a-324=0$

$2a^2+27a-24a-324=0$

$a(2a+27)-12(2a+27)=0$

$(a-12)(2a+27)=0$

$a=12$ or $a=-\frac{27}{2}$ ($\because a>0$)

from ① $b=24$

(b) common diff of AP $=\frac{3}{2}a-a$

$a=12$; $d=\frac{a}{2}=\frac{12}{2}=6$

Sum of first 20th terms, $n=20$

$S_{20}=\frac{20}{2}[2\times12+(20-1)6]$ $\left[S_n=\frac{n}{2}[2a+(n-1)d]\right]$

$=10(24+114)$

$=1380$

13. A Geometric progression is such that the second term is equal to 24% of the sum to infinity. Find the possible values of the common ratio. ---[3]

[S-21/13/Q9(a)]

Solution: Let G.P is $a, ar, ar^2, --- \infty$ $\left[S_\infty = \frac{a}{1-r}\right]$

Given $ar = \frac{24}{100} \times \frac{a}{1-r} \Rightarrow 100r^2 - 100r + 24 = 0$

$(20r-8)(5r-3) = 0$

$r = 2/5, \; r = 3/5$ ✓

An A.P. 'P' has the first term a and common difference d. Another A.P. 'Q' has first term $2(a+1)$ and common difference $(d+1)$, it is given:

$\frac{5th \text{ term of } P}{12th \text{ term of } Q} = \frac{1}{3}$ and $\frac{\text{Sum of 5 terms of } P}{\text{Sum of 5 terms of } Q} = 2/3$ ---[6]

Find the value of a and the value of d.

[S-21/13/Q9(b)]

Solution: 5th term of $P = a + 4d$

12th term of $Q = 2(a+1) + 11(d+1)$

$\Rightarrow \frac{a+4d}{2(a+1)+11(d+1)} = \frac{1}{3}$

$\Rightarrow a + d = 13$ — (1)

Sum of 5 terms of $P = \frac{5}{2}(2a+4d)$

and Sum of 5 terms of $Q = \frac{5}{2}[2 \times 2(a+1) + 4(d+1)]$

$\Rightarrow \frac{\frac{5}{2}(2a+4d)}{\frac{5}{2}[4(a+1)+4(d+1)]} = \frac{2}{3}$

$-a + 2d = 8$ — (2)

Solving (1) and (2)

$d = 7, \; a = 6$ ✓

14. The thirteen term of an A.P is 12 and the sum of the first 30 terms is −15. Find the sum of the first 50 term of the progression.
[S-22 | 11 | Q2] ---[5]

Solution: Let A.P : $a, a+d, a+2d, ---$

13_{th} term of A.P $= a+12 = 12$ --- ① Given.

$S_{30} = \frac{30}{2}[2a+(30-1)d] = -15 \Rightarrow 15(2a+29d) = -15$

$\Rightarrow 2a+29d = -1$ --- ②

Solving ① and ② $\rightarrow a = 72,\ d = -5$

Now $S_n = [2a+(n-1)d] \Rightarrow S_{50} = \frac{50}{2}[2\times72+(50-1)\times(-5)]$

$= 25\times[144-245]$

$\therefore S_{50} = 25\times(-101) = \underline{-2525}$ ✓

15. The second and third terms of a G.P are 10 and 8 respectively. Find the sum to infinity. --[4]
[S-22 | 12 | Q2]

Solution: Given a G.P: $a, ar, ar^2, ---$

Second term $ar = 10$ --- ①

Third term $ar^2 = 8$ --- ②

② ÷ ① $\Rightarrow r = \frac{8}{10} = 4/5$

from ① $a\times\frac{4}{5} = 10 \Rightarrow a = \frac{25}{2}$

$S_\infty = \frac{a}{1-r} = \frac{\frac{25}{2}}{1-4/5}$

$= \frac{25}{2}\times\frac{5}{1} = 125/2$ or $62\frac{1}{2}$ ✓

16. The first, second and the third terms of an A.P are $k, 6k, k+6$.

(a) Find the value of k. ---[2]

(b) Find the sum of the first 30 terms of the progression. ---[3
[S-22 | 12 | Q4]

Solution (a) The first the terms of an A.P are $k, 6k, k+6$

$\therefore$ Common difference:

$6k - k = k+6-6k$

$5k = -5k+6$

$10k = 6$

$k = 3/5$ ✓

(b) for $k = 3/5$, first 3 terms are

$3/5,\ 6\times3/5$ and $3/5+6$

$3/5,\ \ /5$ and $33/5$

$\therefore$ first term $a = 3/5$ ✓

common diff. $d = 18/5 - 3/5 = 3$ ✓

$\therefore S_{30} = \frac{30}{2}[2\times\frac{3}{5}+(30-1)\times3]$

$= 15\cdot[6/5+29\times3]$

$= \underline{1323}$ ✓

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17. An A.P has first term 4 and common difference d. The sum of the first n terms of the progression is 5863.

(a) Show that $(n-1)d = \frac{11726}{n} - 8$ --[1]

(b) Given that the n_{th} term is 139, find the values of n and d, giving the value of n as a fraction. ---[4]

S-22 / 13 / Q3

Solution: An A.P: $a=4$ and common difference $=d$, $S_n = 5863$

(a) now $S_n = \frac{n}{2}[2a+(n-1)d] = 5863$

$\Rightarrow \frac{n}{2}[2\times4+(n-1)d] = 5863$

$\Rightarrow 8+(n-1)d = 5863\times\frac{2}{n}$

$\Rightarrow (n-1)d = \frac{11726}{n} - 8$ ✓ --- (1)

(b) n_{th} term $a+(n-1)d = 139$

$\Rightarrow 4+(n-1)d = 139$

$\Rightarrow (n-1)d = 135$ --- (2)

from (1) & (2) $\frac{11726}{n} - 8 = 135$

$\Rightarrow \frac{11726}{n} = 143 \Rightarrow n = 82$ ✓

from (2) $(82-1)d = 135$

$\Rightarrow d = \frac{135}{81} = 5/3$ ✓

18. The first three terms of an arithmetic progression are, $\frac{p^2}{6}$, $2p-6$ and p.

(a) Given that the common difference of the progression is not zero, find the value of p. ---[3]

(b) Using this value, find the sum to infinity of the geometric progression with first two terms $\frac{p^2}{6}$ and $2p-6$. ---[2]

S-23/11/Q6

Solution: Given $\frac{p^2}{6}$, $2p-6$ and p are in A.P

a, b, c are in AP $\Rightarrow b-a = c-b \Rightarrow 2b = a+c$

(a) $2(2p-6) = \frac{p^2}{6} + p \Rightarrow \frac{p^2}{6} - 3p + 12 = 0$

$\Rightarrow p^2 - 18p + 72 = 0$

$(p-6)(p-1)=0$

$p = 12$ or $p = 6$ ---(1)

Common difference $d = p-(2p-6)$

$d = 6-p$

$\therefore$ for $p=6 \Rightarrow d = 6-6 = 0$ ✗

for $p = 12 \Rightarrow d = 6-12 = -6 \neq 0$

$\therefore p = \underline{12}$ ✓

(b) first two term of G.P are

$\frac{p^2}{6}$, $2p-6$ (for $p=12$)

or $\frac{12^2}{6}$, $2\times 12 - 6 \Rightarrow 24, 18,$

$\Rightarrow$ common ratio of G.P. $r = \frac{18}{24} = \frac{3}{4}$ ✓

First term $a = 24$ ✓

$S_\infty = \frac{a}{1-r} = \frac{24}{1-\frac{3}{4}} = \frac{24}{\frac{1}{4}} = \underline{96}$ ✓

19. The second term of a G.P is 16 and the sum to infinity is 100.
(a) Find the two possible values of the first term. ---[4]
(b) Show that the n_{th} term of one of the two possible G.P is equal to 4^{n-2} multiplied by the n_{th} term of the other G.P. ---[4]

S-23/12/Q9

Solution (a) Let the first term of G.P $= a$

given second term $= 16$; $S_\infty = 100$,

or $ar = 16$

$\Rightarrow r = \frac{16}{a}$ ---- (1)

$\therefore S_\infty = \frac{a}{1-r} = 100$ (Given)

$\Rightarrow \frac{a}{1-\frac{16}{a}} = 100$

$\Rightarrow a^2 = 100(a-16)$

$\Rightarrow a^2 - 100a + 1600 = 0$

$(a-20)(a-80) = 0 \Rightarrow a = 20, a = 80$

(b) from (1) $a_1 = 20$, $r = \frac{16}{20} = \frac{4}{5}$

and for $a_2 = 80$, $r_2 = \frac{16}{80} = \frac{1}{5}$

n_{th} term of first G.P $= a_1 r_1^{n-1}$

$U_n = 20\left(\frac{4}{5}\right)^{n-1}$ --- (2)

and the n_{th} term of second G.P

$V_n = 80\cdot\left(\frac{1}{5}\right)^{n-1}$ --- (3)

$\Rightarrow$ Now $U_n = 20\times\left(\frac{4}{5}\right)^{n-1} = 20\cdot\left(\frac{1}{5}\right)^{n-1}\times 4^{n-1}$

$= 20\times 4\cdot\left(\frac{1}{5}\right)^{n-1}\times 4^{n-2}$

$U_n = 80\cdot\left(\frac{1}{5}\right)^{n-1}\times 4^{n-2}$

$\Rightarrow U_n = V_n \times 4^{n-2}$ from (3)

20. A progression has first term a and the second term $\frac{a^2}{a+2}$, where a is a positive constant.
(a) For the case where the progression is G.P and sum to infinity is 264, Find the value of a. --[5]
(b) For the case where the progression is A.P and $a = 6$, determine the least value of n required for the sum of the first n terms is less than -480.

S-23/13/Q8

Solution (a) First term $= a$

for G.P $\rightarrow$ Second term $ar = \frac{a^2}{a+2}$

$\Rightarrow r = \frac{a}{a+2}$

$S_\infty = \frac{a}{1-r} = 264$ given

$\Rightarrow \frac{a}{1-\frac{a}{a+2}} = 264 \Rightarrow \frac{a(a+2)}{a+2-2} = 262$

$\Rightarrow a^2 + 2a = 528 \Rightarrow a^2 + 2a - 528 = 0$

$\Rightarrow (a-22)(a+24) = 0$

$\Rightarrow a = 22$, $a = -24$ $(a > 0)$

(b) For A.P,

$a = 6$, first two term are $a, \frac{a^2}{a+2}$

First two terms $\rightarrow 6, \frac{36}{6+2} \Rightarrow 6, \frac{9}{2} \Rightarrow d = \frac{9}{2} - 6$

$\Rightarrow d = -\frac{3}{2}$

Given $S_n < -480$

$\Rightarrow \frac{n}{2}[2a + (n-1)d] < -480$

$\Rightarrow n\left[6\times 2 + (n-1)\times\left(-\frac{3}{2}\right)\right] < -960$

$\Rightarrow -3n^2 + 27n + 1920 < 0$

$\Rightarrow n^2 - 9n - 640 > 0$

$n < -21.2$ or $n > 30.2$

$\therefore$ least value of $n = 31$

$n^2 - 9n - 640 = 0$

Critical values

$n = \frac{9 \pm \sqrt{2641}}{2}$

$n = 30.2, -21.2$

Q1. A geometric progression has first term a, common ratio r and sum to infinity S. A second geometric progression has first term a, common ratio R and sum to infinity $2S$.

(a) Show that $r = 2R - 1$ --[3]

It is now given that the third term of the first progression is equal to the second term of the second progression.

(b) Express S in terms of a. --[4]

W-20/11/Q8

Solution (a) $S = \dfrac{a}{1-r}$ —— ①

and $2S = \dfrac{a}{1-R}$ —— ② $\Rightarrow$ from ① & ② $2 \times \dfrac{a}{1-r} = \dfrac{a}{1-R}$

$\Rightarrow 2(1-R) = 1-r$

$\Rightarrow r = 2R - 2 + 1$

$\Rightarrow r = 2R - 1$ ✓ —— ③

(b) Third term of the first progression $= ar^2$ $(\because a, ar, ar^2, --)$
Second " " " second " $= aR$ $(\because a, aR, aR^2, --)$

Given $aR = ar^2 \Rightarrow R = r^2$ —— ④

from ③ $R = \frac{1}{2}(1+r)$ —— ⑤

from ④ & ⑤

$\frac{1}{2}(1+r) = r^2$

$\Rightarrow 2r^2 - r - 1 = 0$

$(2r+1)(r-1) = 0$

$\Rightarrow r = -\frac{1}{2}$ or $r = 1^{\times}$ $\left(\text{for } S_\infty \ |r| < 1 \right)$

$\therefore$ from ① $S = \dfrac{a}{1-(-\frac{1}{2})}$ $(\because r = -\frac{1}{2})$

or $S = \dfrac{2a}{3}$ ✓

22. The first, second and third terms of a geometric progression are $2p+6$, $-2p$ and $p+2$ respectively, where p is positive.
Find the sum to infinity of the progression. ...[5]

W-20/12/Q2

Solution: $2p+6, -2p$ and $p+2$ are in G.P $\left\{ \because a, b, c \text{ are in G.P} \Rightarrow \frac{b}{a} = \frac{c}{b} \Rightarrow b^2 = ac \right.$

$\Rightarrow (-2p)^2 = (2p+6)(p+2)$

$\Rightarrow 4p^2 = 2p^2 + 10p + 12$

$\Rightarrow 2p^2 - 10p - 12 = 0 \Rightarrow p^2 - 5p - 6 = 0 \Rightarrow (p-6)(p+1) = 0$ Given.

$\Rightarrow p = 6$; $p = -1^{\times}$ $(\because p > 0)$

Hence first term $= 2p+6 = 2\times6+6 = 18$ (for $p=6$)

and common ratio $r = \frac{-2p}{2p+6} = \frac{-2\times6}{2\times6+6} = -\frac{12}{18} = -\frac{2}{3}$

$\therefore S_\infty = \frac{a}{1-r} = \frac{18}{1-(-2/3)} = 18 \times \frac{3}{5} = \underline{10.8}$

23 The sum S_n, of the first n terms of a arithmetic progression is given by: $S_n = n^2 + 4n$
The k_{th} term in the progression is greater than 200, Find the smallest possible value of k. --[5]

[W-20/12/Q4]

Solution: K_{th} term $= S_K - S_{K-1}$

$$= (k^2+4k) - [(k-1)^2 + 4(k-1)] \quad [\because S_n = n^2+4n]$$

$$= (k^2+4k) - (k^2 - 2k + 1 + 4k - 4$$

$$= k^2 + 4k - k^2 - 2k + 3$$

$$= 2k + 3 \quad \text{—— (1)}$$

Given K_{th} term > 200 —— (2)

from (1) & (2) $\quad 2k+3 > 200$

$$2k > 197$$

$$k > 98.5$$

$\therefore$ Minimum value of $k = 99$ ✓

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24 The first and second terms of an arithmetic progression are $\frac{1}{\cos^2\theta}$ and $-\frac{\tan^2\theta}{\cos^2\theta}$, where $0 < \theta < \frac{\pi}{2}$

(a) Show that the common difference is $\frac{-1}{\cos^4\theta}$ --[4]

(b) Find the exact value of 13_{th} term when $\theta = \frac{\pi}{6}$ --[3]

[W-20/13/Q7]

Solution (a) $d = -\frac{\tan^2\theta}{\cos^2\theta} - \frac{1}{\cos^2\theta}$

$$= -\frac{\sin^2\theta}{\cos^4\theta} - \frac{1}{\cos^2\theta}$$

$$= \frac{-\sin^2\theta - \cos^2\theta}{\cos^4\theta}$$

$$= \frac{-[\sin^2\theta + \cos^2\theta]}{\cos^4\theta}$$

Common difference

$$d = \frac{-1}{\cos^4\theta} \checkmark$$

(b) $a = \frac{1}{\cos^2\theta} = \frac{1}{\cos^2\frac{\pi}{6}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$ ✓ $\quad [\because \theta = \frac{\pi}{6}]$

$$d = -\frac{1}{\left(\frac{\sqrt{3}}{2}\right)^4} = \frac{-16}{9} \checkmark$$

$\therefore 13_{th}$ term $= a + 12d$

$$= \frac{4}{3} + 12 \times \frac{-16}{9}$$

$$= \frac{4}{3} - \frac{64}{3} = -\frac{60}{3}$$

$\therefore 13_{th}$ term $= -20$ ✓

25 The first term of an A.P is a and the common difference is −4. The first term of a G.P is 5a and the common ratio is $-\frac{1}{4}$. The sum to infinity of the G.P is equal to the sum of the first eight terms of the A.P.

(a) Find the value of a. ---[4]

The K_{th} term of A.P is zero.

(b) Find the value of K. --[2]

[W-21 / 11 / Q4]

Solution: (a) AP: first term = a; d = −4

G.P: first term = 5a; $r = -\frac{1}{4}$

G.P. $S_\infty = \frac{a}{1-r} = \frac{5a}{1-(-\frac{1}{4})} = 4a$ --①

AP. $S_8 = \frac{8}{2}[2a + 7\cdot(-4)] = 8a - 112$ --②

Given from ① & ②

$4a = 8a - 112 \Rightarrow a = 28$ ✓

(b) K_{th} term of AP = 0

$\Rightarrow 28 + (K-1)(-4) = 0$ [$T_n = a + (n-1)d$]

$-4K + 4 + 28 = 0$

$K = 8$ ✓

26. The first, third and fifth terms of an A.P are $2\cos x$, $-6\sqrt{3}\sin x$ and $10\cos x$ respectively, where $\frac{1}{2}\pi < x < \pi$.

(a) Find the exact value of x. ---[3]

(b) Hence find the exact sum of the first 25 terms of the progression. --[3]

[W-21 / 12 / 25]

Solution: (a)

$\left.\begin{aligned} &\text{First term} = a \\ &\text{third term} = a + 2d \\ &\text{fifth term} = a + 4d \end{aligned}\right\}$ for A.P

$\Rightarrow$ {third term − first term = 2d = fifth term − third term

$\Rightarrow (-6\sqrt{3}\sin x - 2\cos x) = 10\cos x + 6\sqrt{3}\sin x$

$\Rightarrow -12\sqrt{3}\sin x = 12\cos x$

$\Rightarrow \tan x = -\frac{1}{\sqrt{3}} = -\tan\frac{\pi}{6}$

$x = \pi - \frac{\pi}{6}$ ($\because \frac{1}{2}\pi < x < \pi$ Second quad.)

$x = \frac{5\pi}{6}$ ✓

(b) First term $= 2\cos x = 2\cos\frac{5\pi}{6} = -\sqrt{3}$ ✓

$2d = 10\cos x + 6\sqrt{3}\sin x$

$= 10\cos\frac{5\pi}{6} + 6\sqrt{3}\sin\frac{5\pi}{6} = -2\sqrt{3}$

$\Rightarrow d = -\sqrt{3}$ ✓

Sum of 25_{th} terms: $S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{25} = \frac{25}{2}[2\times(-\sqrt{3}) + 24\cdot(-\sqrt{3})]$

$= \frac{25}{2}\times(-26\sqrt{3})$

$= -325\sqrt{3}$ ✓

27. The second term of a G.P is 54 and the sum to infinity of the progression is 243. The common ratio is greater than $\frac{1}{2}$.
Find the tenth term, giving your answer in exact form. ---[5]

[W-21/12/Q6]

Solution: Let G.P is $a, ar, ar^2, \ldots$

Second term $ar = 54 \Rightarrow a = \frac{54}{r}$ --- (1)

and $S_\infty = \frac{a}{1-r} = 243$

$\Rightarrow a = (1-r) \times 243$

from (1) $\rightarrow \frac{54}{r} = 243(1-r)$

$\Rightarrow 243r^2 - 243r + 54 = 0$

$\Rightarrow 9r^2 - 9r + 2 = 0$

$(3r-2)(3r-1) = 0$

$\Rightarrow r = \frac{2}{3}$ ✓ or ($r = \frac{1}{3}$ ✗) ($\because r > \frac{1}{2}$)

from (1) $a = \frac{54}{r} = \frac{54}{2/3} = 81$ ✓

10th term: [nth term $= ar^{n-1}$]

$= 81 \times \left(\frac{2}{3}\right)^{10-1}$

$= 81 \times \left(\frac{2}{3}\right)^9 = \frac{512}{243}$ ✓

28. The first term of an A.P is 84 and the common difference is -3.
(a) Find the smallest value of n for which the nth term is negative. ---[2]
It is given that the sum of first $2k$ terms of this progression is equal to the sum of the first k terms.
(b) Find the value of k. ---[3]

[W-21/13/Q4]

Solution: A.P: $a = 84$ and $d = -3$

(a) nth term $= a + (n-1)d$

$= 84 + (n-1)(-3) < 0$ (Given)

$84 - 3n + 3 < 0$

$3n > 87$

$n > 29$

$\therefore$ Minimum value of $n = 30$ ✓

(b) Sum of $2k$ terms $= \frac{2k}{2}[2 \times 84 + (2k-1) \times -3]$

$S_{2k} = k[168 - 6k + 3]$ --- (1)

$S_k = \frac{k}{2}[2 \times 84 + (k-1) \times (-3)]$

$= \frac{k}{2}[168 - 3k + 3]$ --- (2)

Given $S_{2k} = S_k$

$\Rightarrow k[171 - 6k] = \frac{k}{2}[171 - 3k]$

$\Rightarrow 2(171 - 6k) = 171 - 3k$

$\Rightarrow 9k = 171$

$k = 19$ ✓

29 A tool for putting fence posts into the ground is called a 'post-rammer'. The distance in millimetres that the post sinks into the ground on each impact of the post-rammer follow a geometric progression. The first three impacts cause the post to sink into the ground by 50 mm, 40 mm and 32 mm respectively.

(a) Verify that the 9_{th} impact is the first in which the post sinks less than 10 mm into the ground. ---[3]

(b) Find to the nearest millimetres, the total depth of the post in the ground after 20 impacts. --[2]

(c) Find the greatest depth in the ground which could theoretically be achieved. [2]

W-22 | 11 | Q 7

Solution: Terms in G.P: 50, 40, 32, --

(a) $r = \frac{40}{50} = 0.8$, 8_{th} term $= 50 \times (.8)^7 = 10.5$

9_{th} term $= ar^{n-1} = 50 \times (.8)^9 = 8.39$ mm

$\therefore$ 9_{th} impact is required.

(b) $S_n = \frac{a[1-r^n]}{1-r} = \frac{50(1-(0.8)^{20})}{1-0.8} = \underline{247 \text{ mm}}$

(c) $S_\infty = \frac{a}{1-r} = \frac{50}{1-0.8} = \frac{50}{0.2}$

$= \underline{250 \text{ mm}}.$

30 The first, second and third terms of an arithmetic progression are a, $2a$ and a^2 respectively, where a is a positive constant. Find the sum of the first 50 terms of the progression. ---[5]

W-22/12/Q2

Solution: a, $2a$ and a^2 are in A.P $\left\{ \because a, b, \text{ and } c \text{ are in A.P} \Rightarrow b-a=c-b \right.$

$\Rightarrow 2a-a=a^2-2a$

$\Rightarrow a^2-3a=0 \Rightarrow a(a-3)=0 \Rightarrow a=3$ ✓, $a=0$ ✗ $(a>0$, given$)$

and common difference $d=2a-a=a=3 \Rightarrow d=3$ ✓

$\therefore$ First term of A.P. $a=3$ ✓, and the common difference $d=3$ ✓

Now $n=50$

$\therefore S_{50}=\frac{50}{2}[2\times 3+(50-1)\cdot 3]$ $\quad \left[S_n=\frac{n}{2}[2a+(n-1)d]\right]$

$=25\times 153$

$=\underline{3825}$ ✓

31. A geometric progression is such that third term is 1764 and the sum of the second and third terms is 3444. Find 50_{th} term. --[4]

[W-22 | 12 | Q4]

Solution: n_{th} term of a G.P $= ar^{n-1}$ $\Rightarrow$ 3rd term $ar^2 = 1764$ --- ①

and $ar + ar^2 = 3444$ — ②

from ① & ② $ar = 1680$ ---- ③

from ① & ③ $\frac{ar^2}{ar} = \frac{1764}{1680} \Rightarrow r = \frac{1764}{1680} = \frac{21}{20}$; $a \times \frac{21}{20} = 1680$ from ③

$\Rightarrow a = 1600$

$\therefore$ 50_{th} term $= a \cdot r^{49} = 1600 \times \left(\frac{21}{20}\right)^{49} = 17474.1$

$= 17500$ (approx)

32. The first term of a geometric term is 216 and the fourth term is 64.

(a) Find the sum to infinity of the progression. ---[3]

The second term of the geometric progression is equal to the second term of an arithmetic progression.

The third term of the G.P is equal to the fifth term of the same A.P

(b) Find the sum of the first 21 terms of the A.P. ---[6]

[W-22 | 13 | Q9]

Solution: G.P: $a = 216$, n_{th} term $= ar^{n-1}$

(a) 4_{th} term $ar^3 = 64 \Rightarrow 216r^3 = 64$

$r^3 = \frac{64}{216} = \frac{8}{27}$

$\Rightarrow r = 2/3$

$\therefore S_\infty = \frac{a}{1-r} = \frac{216}{1-2/3} = 648$

(b) Second term of GP $= a_1 r_1 = 216 \times \frac{2}{3} = 144$

$=$ second term of AP $a + d = 144$ — ①

5_{th} term of AP $=$ third term of G.P

$a + 4d = a_1 r_1^2 = 216 \times \left(\frac{2}{3}\right)^2 = 96$

$a + 4d = 96$ —— ②

Solving ① & ② $d = -16$, $a = 160$

$S_{21} = \frac{n}{2}[2a + (n-1)d]$

$= \frac{21}{2}[2 \times 160 + 20 \cdot (-16)] = \frac{21}{2} \times 0 = 0$