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Pure Maths - 3

Binomial theorem and Rational functions (Partial fractions).

Notes

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(P.3) Binomial theorem and Rational functions (Partial fractions)

Binomial theorem for positive integer n. (Done in P_1)

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n \quad \cdots (1)$$

or $$(1+x)^n = 1 + n\cdot x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n \quad \cdots (2)$$

$$\left\{\begin{aligned} \binom{n}{0} &= {}^nC_0 = 1 \\ \binom{n}{1} &= {}^nC_1 = n \\ \binom{n}{2} &= {}^nC_2 = \frac{n(n-1)}{2!} \\ &\vdots \\ \binom{n}{n} &= {}^nC_n = 1 \end{aligned}\right.$$

Now in general binomial theorem for any rational value 'n'

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!} + \cdots$$

Note 1. the series is infinite.

2. the expansion is only valid for $|x|<1$

Note: Symbols ${}^nC_0$, ${}^nC_2$ are not used. but their values like in (2)

Particular cases:

(i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 \cdots$ when $|x|<1$

(ii) $(1+x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots$

(iii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \cdots$

(iv) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$

(v) $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$

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$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

1. Given that $\sqrt[3]{(1+9x)} \approx 1+3x+ax^2+bx^3$ for small values of x, find the values of the coefficients a and b. ---[3]

[W-15/33/Q2]

Solution: $\sqrt[3]{(1+9x)} = 1+3x+ax^2+bx^3 \quad \cdots\cdots (1)$

Consider $\sqrt[3]{(1+9x)} = (1+9x)^{1/3}$

$$= 1 + \frac{1}{3}\cdot 9x + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}\cdot(9x)^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!}\cdot(9x)^3$$

$$= 1+3x-9x^2+45x^3 \quad \cdots (2)$$

from (1) and (2) $\Rightarrow$ $a=-9$ and $b=45$

2. Expand $(3+2x)^{-3}$ in ascending powers of x upto and including the terms in x^2, simplifying the coefficients. ---[4]

[S-17/33/Q2]

Solution: $(3+2x)^{-3} = [3(1+\frac{2}{3}x)]^{-3} = 3^{-3}(1+\frac{2}{3}x)^{-3}$

$$= \frac{1}{27}\left(1+(-3)\cdot\frac{2}{3}x + \frac{(-3)(-4)}{2!}\cdot\left(\frac{2}{3}x\right)^2 + \cdots\right)$$

$$= \frac{1}{27}\left(1-2x+\frac{8}{3}x^2+\cdots\right) = \frac{1}{27} - \frac{2}{27}x + \frac{8}{81}x^2 + \cdots$$

3. Expand $(2-x)(1+2x)^{-3/2}$ in ascending powers of x, upto and including the terms in x^2, simplifying the coefficients of x. ---[4]

[W-16/31/Q2]

Solution: $(2-x)(1+2x)^{-3/2} = (2-x)\left[1+(-\frac{3}{2})\cdot 2x + \frac{(-\frac{3}{2})(-\frac{3}{2}-1)}{2!}\cdot(2x)^2 + \cdots\right]$

$$= (2-x)\left(1-3x+\frac{15}{2}x^2+\cdots\right)$$

$$= 2-6x+15x^2-x+3x^2+\cdots$$

$$= 2-7x+18x^2+\cdots$$

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$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$

4. (i) Find the first three terms, ascending powers of x, in the expansion of: $(\sqrt{3-x^2})^5$ ---[3]

(ii) State the range of values of x for which the expansion is valid. ---[1]

Solution: (i) $(\sqrt{3-x^2})^5 = [(3-x^2)^{1/2}]^5 = (3-x^2)^{5/2} = [3(1-\frac{x^2}{3})]^{5/2} = 3^{5/2}(1-\frac{x^2}{3})^{5/2}$

$= 3^{2\frac{1}{2}} \cdot \{1 + \frac{5}{2}\cdot(-\frac{x^2}{3}) + \frac{5/2\,(5/2-1)}{2!}\cdot(-\frac{x^2}{3})^2 + \dots\}$

$= 9\sqrt{3}(1 - \frac{5}{6}x^2 + \frac{5}{24}x^4 + \dots)$

$= \underline{9\sqrt{3} - \frac{15\sqrt{3}}{2}x^2 + \frac{15\sqrt{3}}{8}x^4 + \dots}$ ✓

(ii) Expansion is valid for $|-\frac{x^2}{3}| < 1 \Rightarrow |x^2| < 3 \Rightarrow \underline{|x| < 3}$ ✓

5. When $(1+ax)^{-2}$, where a is a positive constant, is expanded in ascended powers of x, the coefficient of x and x^3 are equal.

(i) Find the exact value of a. ---[4]

(ii) When a has this value, obtain the expansion upto and including the term in x^2, simplifying the coefficients. ---[3]

W-12/31/Q4

Solution: $(1+ax)^{-2} = 1 + (-2)ax + \frac{(-2)(-3)}{2!}(ax)^2 + \frac{(-2)(-3)(-4)}{3!}(ax)^3 + \dots$

(i) $= 1 - 2ax + 3a^2x^2 - 4a^3x^3 + \dots$ --- (1)

from (1)

Coeff of $x = -2a$

Coeff of $x^3 = -4a^3$

Given coefficient of x = coefficient of x^3

$\Rightarrow -2a = -4a^3 \Rightarrow a^2 = \frac{1}{2} \Rightarrow \underline{a = \frac{1}{\sqrt{2}}}$ ✓ $(a>0)$

(ii) For $a = \frac{1}{\sqrt{2}}$

$(1+\frac{1}{\sqrt{2}}x)^{-2} = 1 + (-2)\cdot\frac{1}{\sqrt{2}}x + \frac{(-2)(-3)}{2!}(\frac{1}{\sqrt{2}})^2 + \dots$

$= \underline{1 - \sqrt{2}x + \frac{3}{2}x^2 + \dots}$ ✓

Rational functions and Partial fractions
(Binomial expansion)

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§ <u>Improper algebric fraction:</u>

Given $P(x)$ and $S(x)$ are polynomial in x.
Then $\frac{P(x)}{S(x)}$ is said to be improper fraction,
if degree of $P(x) \geq$ degree of $S(x)$

In numerical fraction; $\frac{12}{7} = 1 + \frac{5}{12}$

⊗ Then; $\frac{P(x)}{S(x)} = Q(x) + \frac{R(x)}{S(x)}$: $Q(x)$ is quotient when $P(x)$ is divided by $S(x)$ and $R(x)$ is remainder; $\deg R(x) < \deg S(x)$.

§ <u>Partial fraction:</u>

If the denominator of a proper algebric fraction has two (or more) factors, then we can split into as many fractions with each factor of the denominator.

<u>Case I</u>; The denominator has distinct linear factors.

Case II; The denominator has a repeated linear factor.

Case III: The denominator has a quadratic factor (which can not be factorised).

⊗
$$\frac{4x^2+9x-8}{(x+2)(2x-1)} = \frac{4x^2+9x-8}{2x^2+3x-2}$$

$\deg N^r = \deg D^r$ → {using long division}

$$= 2 + \frac{3x-4}{2x^2+3x-2}$$

$$= 2 + \frac{3x-4}{(x+2)(2x-1)}$$

$$2x^2+3x-2 \overline{)\, 4x^2+9x-8 \,(} \, 2$$
$$\underline{-4x^2 \pm 6x \mp 4}$$
$$3x - 4$$

Quotient = 2
Remainder = $3x-4$

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§ **Case I: Denominator has non-repeated linear factors.**

1. Let $f(x) = \dfrac{12x^2+4x-1}{(x-1)(3x+2)}$

(i) Express $f(x)$ in partial fractions. ---[5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, upto and including the terms in x^2. ---[5]

[S-18/31/Q9]

Solution: $f(x) = \dfrac{12x^2+4x-1}{(x-1)(3x+2)} = \dfrac{12x^2+4x-1}{3x^2-x-2} \rightarrow \text{Deg } N^r = \text{Deg } D^r$

Hence: $\dfrac{12x^2+4x-1}{(x-1)(3x+2)} = 4 + \dfrac{8x+7}{(x-1)(3x+2)}$ ---①

$$3x^2-x-2\ \big)\ 12x^2+4x-1\ \big(\ 4$$
$$\underline{-12x^2 \mp 4x \mp 8}$$
$$8x+7$$

Now consider (from ①)

$\dfrac{8x+7}{(x-1)(3x+2)} = \dfrac{a}{(x-1)} + \dfrac{b}{(3x+2)}$ ---②

multiply by $(x-1)(3x+2)$ on both sides of ② {Multiply by D^r

$8x+7 = a(3x+2) + b(x-1)$ ----③

Put $(x-1)=0 \rightarrow x=1$ in ③ $\Rightarrow 8\times1+7 = a(3\times1+2)+0 \Rightarrow a=3$ ✓

Again put $(3x+2)=0 \rightarrow x=-2/3$ in ③ $\Rightarrow 8\times(-\frac{2}{3})+7 = 0 + b(-\frac{2}{3}-1) \Rightarrow b=-1$

Put $a=3$ and $b=-1$ in ②

$\dfrac{8x+7}{(x-1)(3x+2)} = \dfrac{3}{x-1} + \dfrac{-1}{(3x+2)}$ ✓

from ① Required Partial fraction: $f(x) = 4 + \dfrac{3}{(x-1)} + \dfrac{-1}{(3x+2)}$ ---④

(ii) $f(x) = 4 + \dfrac{3}{(x-1)} + \dfrac{-1}{(3x+2)}$ $\quad\left\{\begin{array}{l}(3x+2) = 2(1+\frac{3}{2}x)\\ \text{and } x-1 = -1(1-x)\end{array}\right.$

$= 4 + \dfrac{3}{-(1-x)} + \dfrac{-1}{2(1+\frac{3}{2}x)}$

$= 4 - 3(1-x)^{-1} - \frac{1}{2}(1+\frac{3}{2}x)^{-1}$

$= 4 - 3\left(1 + (-1)(-x) + \dfrac{(-1)(-2)}{2!}(-x)^2 + \cdots\right) - \frac{1}{2}\left(1 + (-1)\cdot\frac{3}{2}x + \dfrac{(-1)(-2)}{2!}\left(\frac{3}{2}x\right)^2\right.$

$= 4 - 3(1 + x + x^2 + \cdots) - \frac{1}{2}(1 - \frac{3}{2}x + \frac{9}{4}x^2$

$= (4 - 3 - \frac{1}{2}) + (-3 + \frac{3}{4})x + (-3 - \frac{9}{8}x^2) + \cdots$

$= \frac{1}{2} - \frac{9}{4}x - \frac{33}{8}x^2 + \cdots$ ✓

§ Case II : Denominator has a repeated linear factor.

2. Let $f(x) = \dfrac{7x^2 - 15x + 8}{(1-2x)(2-x)^2}$

(i) Express $f(x)$ in partial fractions. ---[5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, upto and including x^2. ---[5]

[W-18/32/Q8]

Solution (i) $f(x) = \dfrac{7x^2 - 15x + 8}{(1-2x)(2-x)^2} = \dfrac{a}{(1-2x)} + \dfrac{b}{(2-x)^2} + \dfrac{c}{(2-x)}$ --- ①

multiply on both sides of ① by Dr. $(1-2x)(2-x)^2$.

$\Rightarrow \quad 7x^2 - 15x + 8 = a(2-x)^2 + b(1-2x) + c(1-2x)(2-x)$ --- ②

Put $1-2x=0 \Rightarrow x = \frac{1}{2}$ in ② $\Rightarrow 7\left(\frac{1}{2}\right)^2 - 15\times\frac{1}{2} + 8 = a\left(2-\frac{1}{2}\right)^2 + 0 + 0$

$\Rightarrow \frac{9}{4} = \frac{9}{4}a = \underline{a=1}$ --- ✓

Again put $2-x=0 \Rightarrow x=2$ in ②

$7\times2^2 - 15\times2 + 8 = 0 + b(1-2\times2) + 0 \Rightarrow \underline{b=-2}$ ✓

Put the values $a=1$ and $b=-2$ in ②

$7x^2 - 15x + 8 = 1\cdot(2-x)^2 - 2(1-2x) + c(1-2x)(2-x)$ --- ③

Put $x=0$ in ③ $\quad 8 = 4 - 2 + 2c \Rightarrow c = 3$ ✓

Hence the required partion fractions of $f(x) = \dfrac{1}{(1-2x)} + \dfrac{3}{(2-x)} + \dfrac{-2}{(2-x)^2}$ ✓

(ii) $f(x) = \dfrac{1}{(1-2x)} + \dfrac{3}{(2-x)} + \dfrac{-2}{(2-x)^2} \longrightarrow \left\{ \begin{array}{l} 2-x = 2(1-x/2) \\ (2-x)^2 = [2(1-x/2)]^2 = 4\left(1-\frac{x}{2}\right)^2 \end{array} \right.$

$= \dfrac{1}{(1-2x)} + \dfrac{3}{2(1-x/2)} + \dfrac{-2}{4(1-x/2)^2}$

$= (1-2x)^{-1} + \frac{3}{2}\left(1-\frac{x}{2}\right)^{-1} - \frac{1}{2}(1-x/2)^{-2}$

$= \left(1 + (-1)(-2x) + \frac{(-1)(-2)}{2!}(-2x)^2 + \cdots\right) + \frac{3}{2}\left(1 + (-1)\left(-\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{x}{2}\right)^2 \cdots\right)$
$- \frac{1}{2}\left(1 + (-2)\left(-\frac{x}{2}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{x}{2}\right)^2 \cdots\right)$

$= (1 + 2x + 4x^2 + \cdots) + \frac{3}{2}\left(1 + \frac{x}{2} + \frac{x^2}{4} \cdots\right) - \frac{1}{2}\left(1 + x + \frac{3}{4}x^2 + \cdots\right)$

$= \left(1 + \frac{3}{2} - \frac{1}{2}\right) + \left(2 + \frac{3}{4} - \frac{1}{2}\right)x + \left(4 + 3/8 - 3/8\right)x^2 + \cdots$

$= \underline{2 + \frac{9}{4}x + 4x^2 + \cdots}$

Case III: The denominator has a quadratic factor (which cannot be factorised)

3. Let $f(x) = \dfrac{x(6-x)}{(2+x)(4+x^2)}$

(i) Express $f(x)$ in partial fractions. ---[5]

(ii) Hence obtain the expansion of $f(x)$ in the ascending powers of x, upto and including the terms in x^2. ---[5]

M-17/32/Q9

Solution:
(i) $f(x) = \dfrac{x(6-x)}{(2+x)(4+x^2)} = \dfrac{a}{(2+x)} + \dfrac{bx+c}{(4+x^2)}$ ----- ①

multipleing both sides of ① by the D^r - $(2+x)(4+x^2)$.

$\Rightarrow \quad x(6-x) = a(4+x^2) + (bx+c)(2+x)$ --- ②

Put $2+x=0 \rightarrow x=-2$ in ② $\Rightarrow -2(6+2) = a(4+(-2)^2) \Rightarrow 8a = -16$

$\Rightarrow \underline{a=-2}$

Put $a=-2$ in ② $\Rightarrow$

$x(6-x) = -2(4+x^2) + (bx+c)(2+x)$ --- ③

Comparing the constant term on both side of ③ $0 = -8+2c \Rightarrow c = \underline{4}$

Comparing the coefficient of x^2 on both sides of ③ $\Rightarrow -1 = -2+b \Rightarrow b = \underline{1}$

Put the values of a, b and c is ① The required partial fracti

$f(x) = \dfrac{-2}{2+x} + \dfrac{x+4}{4+x^2}$ --- ④

(ii) $f(x) = \dfrac{-2}{2+x} + \dfrac{x+4}{4+x^2}$

$= \dfrac{-2}{2(1+x/2)} + \dfrac{x+4}{4(1+\frac{x^2}{4})}$

$= -(1+x/2)^{-1} + \frac{1}{4}(x+4)(1+\frac{x^2}{4})^{-1}$

$= -1\left(1+(-1)\frac{x}{2} + \frac{(-1)(-2)}{2!}\cdot\left(\frac{x}{2}\right)^2 + \ldots\right) + \frac{1}{4}(x+4)\left(1+(-1)\cdot\frac{x^2}{4} + \ldots\right)$

$= (-1+1) + \left(\frac{1}{2}+\frac{1}{4}\right)x + \left(\frac{-1}{4}-\frac{1}{4}\right)x^2 + \ldots$

$= \underline{\frac{3}{4}x - \frac{1}{2}x^2 + \ldots}$