PURE MATHEMATICS -3

9709

(March, June and November series 2020 – 2023 With marking scheme)

COMPLEX NUMBERS

EXERCISE -1

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1)SP-2020 _9709_3 Q6

The complex numbers $1 + 3\text{i}$ and $4 + 2\text{i}$ are denoted by u and v respectively.

(a) Find $\frac{u}{v}$ in the form $x + \text{i}y$, where x and y are real. [3]

(b) State the argument of $\frac{u}{v}$. [1]

In an Argand diagram, with origin O, the points A, B and C represent the complex numbers u, v and $u - v$ respectively.

(c) State fully the geometrical relationship between OC and BA. [2]

(d) Show that angle $AOB = \frac{1}{4}\pi$ radians. [2]

2)MARCH-2020 _9709_32 Q10

(a) The complex numbers v and w satisfy the equations

$$v + \text{i}w = 5 \qquad \text{and} \qquad (1 + 2\text{i})v - w = 3\text{i}.$$

Solve the equations for v and w, giving your answers in the form $x + \text{i}y$, where x and y are real. [6]

(b) **(i)** On an Argand diagram, sketch the locus of points representing complex numbers z satisfying $|z - 2 - 3\text{i}| = 1$. [2]

3)MARCH-2021 _9709_32 Q8

The complex numbers u and v are defined by $u = -4 + 2\text{i}$ and $v = 3 + \text{i}$.

(a) Find $\frac{u}{v}$ in the form $x + \text{i}y$, where x and y are real. [3]

(b) Hence express $\frac{u}{v}$ in the form $re^{\text{i}\theta}$, where r and θ are exact. [2]

In an Argand diagram, with origin O, the points A, B and C represent the complex numbers u, v and $2u + v$ respectively.

(c) State fully the geometrical relationship between OA and BC. [2]

(d) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

4)MARCH-2022 _9709_32 Q2

On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z + 2 - 3\text{i}| \leqslant 2$ and $\arg z \leqslant \frac{3}{4}\pi$. [4]

5)MARCH-2022 _9709_32 Q6

Find the complex numbers w which satisfy the equation $w^2 + 2\text{i}w^* = 1$ and are such that $\text{Re}\, w \leqslant 0$. Give your answers in the form $x + \text{i}y$, where x and y are real. [6]

6)MARCH-2023 _9709_32 Q2

(a) On an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $-\frac{1}{3}\pi \leqslant \arg(z - 1 - 2i) \leqslant \frac{1}{3}\pi$ and $\text{Re}\, z \leqslant 3$. [3]

(b) Calculate the least value of $\arg z$ for points in the region from **(a)**. Give your answer in radians correct to 3 decimal places. [2]

7)MARCH-2023 _9709_32 Q4

Solve the equation

$$\frac{5z}{1 + 2i} - zz^* + 30 + 10i = 0,$$

giving your answers in the form $x + iy$, where x and y are real. [5]

8)JUNE-2020 _9709_31 Q10

(a) The complex number u is defined by $u = \frac{3i}{a + 2i}$, where a is real.

(i) Express u in the Cartesian form $x + iy$, where x and y are in terms of a. [3]

(ii) Find the exact value of a for which $\arg u^* = \frac{1}{3}\pi$. [3]

(b) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 2i| \leqslant |z - 1 - i|$ and $|z - 2 - i| \leqslant 2$. [4]

(ii) Calculate the least value of $\arg z$ for points in this region. [2]

9)JUNE-2020 _9709_32 Q8

(a) Solve the equation $(1 + 2i)w + iw^* = 3 + 5i$. Give your answer in the form $x + iy$, where x and y are real. [4]

(b) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 2 - 2i| \leqslant 1$ and $\arg(z - 4i) \geqslant -\frac{1}{4}\pi$. [4]

(ii) Find the least value of $\text{Im}\, z$ for points in this region, giving your answer in an exact form. [2]

10)JUNE-2020 _9709_33 Q9

(a) The complex numbers u and w are such that

$$u - w = 2i \quad \text{and} \quad uw = 6.$$

Find u and w, giving your answers in the form $x + iy$, where x and y are real and exact. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities

$$|z - 2 - 2i| \leqslant 2, \quad 0 \leqslant \arg z \leqslant \frac{1}{4}\pi \quad \text{and} \quad \text{Re}\, z \leqslant 3.$$ [5]

11)JUNE-2021 _9709_31 Q5

(a) Solve the equation $z^2 - 2piz - q = 0$, where p and q are real constants. [2]

In an Argand diagram with origin O, the roots of this equation are represented by the distinct points A and B.

(b) Given that A and B lie on the imaginary axis, find a relation between p and q. [2]

(c) Given instead that triangle OAB is equilateral, express q in terms of p. [3]

12)JUNE-2021 _9709_32 Q2

On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z + 1 - \mathrm{i}| \leqslant 1$ and $\arg(z - 1) \leqslant \frac{3}{4}\pi$. [4]

13)JUNE-2021 _9709_32 Q5

The complex number u is given by $u = 10 - 4\sqrt{6}\mathrm{i}$.

Find the two square roots of u, giving your answers in the form $a + \mathrm{i}b$, where a and b are real and exact. [5]

14)JUNE-2021 _9709_33 Q10

(a) Verify that $-1 + \sqrt{2}\mathrm{i}$ is a root of the equation $z^4 + 3z^2 + 2z + 12 = 0$. [3]

(b) Find the other roots of this equation. [7]

15)JUNE-2022 _9709_31 Q7

The complex number u is defined by $u = \dfrac{\sqrt{2} - a\sqrt{2}\mathrm{i}}{1 + 2\mathrm{i}}$, where a is a positive integer.

(a) Express u in terms of a, in the form $x + \mathrm{i}y$, where x and y are real and exact. [3]

It is now given that $a = 3$.

(b) Express u in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ. [2]

(c) Using your answer to part **(b)**, find the two square roots of u. Give your answers in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ. [3]

16)JUNE-2022 _9709_32 Q10

The complex number $-1 + \sqrt{7}\mathrm{i}$ is denoted by u. It is given that u is a root of the equation

$$2x^3 + 3x^2 + 14x + k = 0,$$

where k is a real constant.

(a) Find the value of k. [3]

(b) Find the other two roots of the equation. [4]

(c) On an Argand diagram, sketch the locus of points representing complex numbers z satisfying the equation $|z - u| = 2$. [2]

(d) Determine the greatest value of $\arg z$ for points on this locus, giving your answer in radians. [2]

17)JUNE-2022 _9709_33 Q5

The complex number $3 - \mathrm{i}$ is denoted by u.

(a) Show, on an Argand diagram with origin O, the points A, B and C representing the complex numbers u, u^* and $u^* - u$ respectively.

State the type of quadrilateral formed by the points O, A, B and C. [3]

(b) Express $\frac{u^*}{u}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

(c) By considering the argument of $\frac{u^*}{u}$, or otherwise, prove that $\tan^{-1}\left(\frac{3}{4}\right) = 2\tan^{-1}\left(\frac{1}{3}\right)$. [2]

18)JUNE-2023 _9709_31 Q10

The polynomial $x^3 + 5x^2 + 31x + 75$ is denoted by $\mathrm{p}(x)$.

(a) Show that $(x + 3)$ is a factor of $\mathrm{p}(x)$. [2]

(b) Show that $z = -1 + 2\sqrt{6}\mathrm{i}$ is a root of $\mathrm{p}(z) = 0$. [3]

(c) Hence find the complex numbers z which are roots of $\mathrm{p}(z^2) = 0$. [7]

19)JUNE-2023 _9709_32 Q3

(a) On an Argand diagram, sketch the locus of points representing complex numbers z satisfying $|z + 3 - 2\mathrm{i}| = 2$. [2]

(b) Find the least value of $|z|$ for points on this locus, giving your answer in an exact form. [2]

20)JUNE-2023 _9709_32 Q5

The complex number $2 + y\mathrm{i}$ is denoted by a, where y is a real number and $y < 0$. It is given that $\mathrm{f}(a) = a^3 - a^2 - 2a$.

(a) Find a simplified expression for $\mathrm{f}(a)$ in terms of y. [3]

(b) Given that $\mathrm{Re}(\mathrm{f}(a)) = -20$, find $\arg a$. [3]

21)JUNE-2023 _9709_33 Q3

On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 3 - \mathrm{i}| \leqslant 3$ and $|z| \geqslant |z - 4\mathrm{i}|$. [4]

22)JUNE-2023 _9709_33 Q11

The complex number z is defined by $z = \frac{5a - 2i}{3 + ai}$, where a is an integer. It is given that $\arg z = -\frac{1}{4}\pi$.

(a) Find the value of a and hence express z in the form $x + iy$, where x and y are real. **[6]**

(b) Express z^3 in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the simplified exact values of r and θ. **[3]**

23)OCT-2020_9709_31 Q2

On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z| \geqslant 2$ and $|z - 1 + i| \leqslant 1$. **[4]**

24)OCT-2020_9709_31 Q7

(a) Verify that $-1 + \sqrt{5}i$ is a root of the equation $2x^3 + x^2 + 6x - 18 = 0$. **[3]**

(b) Find the other roots of this equation. **[4]**

25)OCT-2020_9709_32 Q6

The complex number u is defined by

$$u = \frac{7 + i}{1 - i}.$$

(a) Express u in the form $x + iy$, where x and y are real. **[3]**

(b) Show on a sketch of an Argand diagram the points A, B and C representing u, $7 + i$ and $1 - i$ respectively. **[2]**

(c) By considering the arguments of $7 + i$ and $1 - i$, show that

$$\tan^{-1}\left(\tfrac{4}{3}\right) = \tan^{-1}\left(\tfrac{1}{7}\right) + \tfrac{1}{4}\pi.$$ **[3]**

26)OCT-2021_9709_31 Q10

The complex number $1 + 2i$ is denoted by u. The polynomial $2x^3 + ax^2 + 4x + b$, where a and b are real constants, is denoted by $p(x)$. It is given that u is a root of the equation $p(x) = 0$.

(a) Find the values of a and b. **[4]**

(b) State a second complex root of this equation. **[1]**

(c) Find the real factors of $p(x)$. **[2]**

(d) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - u| \leqslant \sqrt{5}$ and $\arg z \leqslant \frac{1}{4}\pi$. **[4]**

(ii) Find the least value of $\operatorname{Im} z$ for points in the shaded region. Give your answer in an exact form. **[1]**

27) OCT-2021_9709_32 Q3

(a) Given the complex numbers $u = a + ib$ and $w = c + id$, where a, b, c and d are real, prove that $(u + w)^* = u^* + w^*$. [2]

(b) Solve the equation $(z + 2 + \mathrm{i})^* + (2 + \mathrm{i})z = 0$, giving your answer in the form $x + \mathrm{i}y$ where x and y are real. [4]

28) OCT-2021_9709_32 Q5

(a) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 3 - 2\mathrm{i}| \leqslant 1$ and $\operatorname{Im} z \geqslant 2$. [4]

(b) Find the greatest value of $\arg z$ for points in the shaded region, giving your answer in degrees. [3]

29) OCT-2021_9709_33 Q11

The complex number $-\sqrt{3} + \mathrm{i}$ is denoted by u.

(a) Express u in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ. [2]

(b) Hence show that u^6 is real and state its value. [2]

30) OCT-2022_9709_31 Q2

On a sketch of an Argand diagram shade the region whose points represent complex numbers z satisfying the inequalities $|z| \leqslant 3$, $\operatorname{Re} z \geqslant -2$ and $\frac{1}{4}\pi \leqslant \arg z \leqslant \pi$. [4]

31) OCT-2022_9709_31 Q5

The complex numbers u and w are defined by $u = 2\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}$ and $w = 3\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$.

(a) Find $\dfrac{u^2}{w}$, giving your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the exact values of r and θ. [3]

(b) State the least positive integer n such that both $\operatorname{Im} w^n = 0$ and $\operatorname{Re} w^n > 0$. [1]

32) OCT-2022_9709_32 Q5

(a) Solve the equation $z^2 - 6\mathrm{i}z - 12 = 0$, giving the answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [3]

(b) On a sketch of an Argand diagram with origin O, show points A and B representing the roots of the equation in part **(a)**. [1]

(c) Find the exact modulus and argument of each root. [3]

(d) Hence show that the triangle OAB is equilateral. [1]

33) OCT-2022_9709_33 Q5

(a) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z + 2| \leqslant 2$ and $\operatorname{Im} z \geqslant 1$. [4]

(b) Find the greatest value of $\arg z$ for points in the shaded region. [2]

34)OCT-2022_9709_33 Q6

Solve the quadratic equation $(1-3\mathrm{i})z^2-(2+\mathrm{i})z+\mathrm{i}=0$, giving your answers in the form $x+\mathrm{i}y$, where x and y are real. [6]

MARKING SCHEME

1)SP-2020 _9709_3 Q6

(a)	**EITHER** Solution 1 Multiply the numerator and denominator of $\frac{u}{v}$ by $4-2i$, or equivalent	1	**(M1**	
	Simplify the numerator to $10+10i$ or the denominator to 20	1	**A1)**	
	OR Solution 2 Obtain two equations in x and y, and solve for x or for y	1	**(M1**	
	Obtain $x=\frac{1}{2}$ or $y=\frac{1}{2}$, or equivalent	1	**A1)**	
	Obtain final answer $\frac{1}{2}+\frac{1}{2}i$	1	**A1**	Unsupported answer receives 0 marks
	Available marks	**3**		
(b)	State argument is $\frac{1}{4}\pi$ (or 0.785 radians or 45°)	**1**	**B1FT**	
(c)	State that OC and BA are equal (in length)	1	**B1**	
	State that OC and BA are parallel or have the same direction	1	**B1**	
		2		
(d)	**EITHER** Solution 1 Use angle $AOB = \arg u - \arg v = \arg\left(\frac{u}{v}\right)$	1	**(M1**	
	Obtain given answer (or 45°)	1	**A1)**	AG
	OR Solution 2 Obtain tan AOB from gradients of OA and OB and $\tan(A \pm B)$ formula	1	**(M1**	
	Obtain given answer (or 45°)	1	**A1)**	AG
	Available marks	**2**		

2)MARCH-2020 _9709_32 Q10

0(a)	Solve for v or w	**M1**
	Use $i^2=-1$	**M1**
	Obtain $v=-\dfrac{2i}{1+i}$ or $w=\dfrac{5+7i}{-1+i}$	**A1**
	Multiply numerator and denominator by the conjugate of the denominator	**M1**
	Obtain $v=-1-i$	**A1**
	Obtain $w=1-6i$	**A1**
		6
(b)(i)	Show a circle with centre $2+3i$	**B1**
	Show a circle with radius 1 and centre not at the origin	**B1**
		2
b)(ii)	Carry out a complete method for finding the least value of arg z	**M1**
	Obtain answer 40.2° or 0.702 radians	**A1**
		2

3)MARCH-2021 _9709_32 Q8

(a)	Multiply numerator and denominator by 3 – i	**M1**
	Obtain numerator – 10 + 10i or denominator 10	**A1**
	Obtain final answer – 1 + i	**A1**
		3
(b)	State or imply $r = \sqrt{2}$	**B1 FT**
	State or imply that $\theta = \frac{3}{4}\pi$	**B1 FT**
		2
(c)	State that OA and BC are parallel	**B1**
	State that $BC = 2OA$	**B1**
		2
(d)	Use angle $AOB = \arg u - \arg v = \arg\frac{u}{v}$	**M1**
	Obtain the given answer	**A1**
	Alternative method for question 8(d)	
	Obtain tan AOB from gradients of OA and OB and the $\tan(A \pm B)$ formula	**M1**
	Obtain the given answer	**A1**
	Alternative method for question 8(d)	
	Obtain $\cos AOB$ by using the cosine rule or a scalar product	**M1**
	Obtain the given answer	**A1**
		2

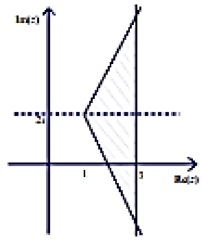
4)MARCH-2022 _9709_32 Q2

Show a circle with centre – 2 + 3i	**B1**	Must see (– 2, 3) or appropriate marks on axes
Show a circle of radius 2 and centre not at the origin.	**B1**	
Show correct **half line** from the origin	**B1**	$\frac{3\pi}{4}$ or $\frac{\pi}{4}$ seen, or half line that approximately bisects angle $\frac{\pi}{2}$.
Shade the correct region.	**B1**	
	4	N.B. Maximum 3 out of 4 if any errors seen.

5)MARCH-2022 _9709_32 Q6

Substitute and obtain a correct equation in x and y	B1	$(x+iy)^2+2i(x-iy)=1$
Use $i^2=-1$ at least once and equate real and imaginary parts	M1	
Obtain two correct equations, e.g. $x^2-y^2+2y=1$ and $2xy+2x=0$	A1	
Solve for x or for y	M1	
Using $y=-1$, obtain answer $w=-2-i$ only	A1	A0 if $w=2-i$ as well
Using $x=0$, obtain answer $w=i$	A1	
	6	

6)MARCH-2023 _9709_32 Q2

(a)	Show correct half-lines from $1+2i$, symmetrical about $y=2i$ (drawn between $\frac{\pi}{4}$ and $\frac{5\pi}{12}$).	B1	
	Show the line $x=3$ extending in both quadrants.	B1	
	Shade the correct region. Allow dashes on axes as scale. FT If only error is one of following: FULL lines or $x\neq 3$ or one sign error in $1+2i$ or angle outside tolerance or scale missing on one axis.	B1 FT	
			SC No scale on either axis allow B1 FT for otherwise correct figure in correct position.
		3	
(b)	Carry out a complete method for finding the least value of arg z	M1	e.g. $-\tan^{-1}\frac{(2\sqrt{3}-2)}{3}$ or $\tan^{-1}\frac{(-2\sqrt{3}+2)}{3}$.
	Obtain answer -0.454 3dp	A1	SC B1 0.454 .
		2	

7)MARCH-2023 _9709_32 Q4

Substitute $z=x+iy$ and $z^*=x-iy$ to obtain a correct equation, horizontal or with $(1-2i)/(1-2i)$ seen, in x and y	B1	$5(x+iy)-(x+iy)(x-iy)(1+2i)+(30+10i)(1+2i)=0$ $5(x+iy)(1-2i)/[(1+2i)(1-2i)]-(x+iy)(x-iy)+(30+10i)=0$ $x-2ix+iy+2y-x^2-y^2+30+10i=0$.
Use $i^2=-1$ at least once and equate real and imaginary parts to zero	*M1	OE For their horizontal equation.
Obtain two correct equations e.g. $x+2y-x^2-y^2+30=0$ and $-2x+y+10=0$	A1	$5x-(x^2+y^2)+10=0$ $5y-2(x^2+y^2)+70=0$ $5y-10x+50=0$ $x+2y-(x^2+y^2)+30=0$ Allow $-2ix+iy+10i=0$.
Solve quadratic equation for x or for y	DM1	$x^2-9x+18=(x-3)(x-6)=0$ $y^2+2y-8=(y+4)(y-2)=0$ DM0 If x or y imaginary.
Obtain answers $3-4i$ and $6+2i$	A1	
	5	

8)JUNE-2020 _9709_31 Q10

(a)(i)	Multiply numerator and denominator by $a-2\text{i}$, or equivalent	**M1**
	Use $\text{i}^2=-1$ at least once	**A1**
	Obtain answer $\dfrac{6}{a^2+4}+\dfrac{3ai}{a^2+4}$	**A1**
		3
(a)(ii)	Either state that $\arg u=-\frac{1}{3}\pi$ or express u^* in terms of a (FT on u)	**B1**
	Use correct method to form an equation in a	**M1**
	Obtain answer $a=-2\sqrt{3}$	**A1**
		3
(b)(i)	Show the perpendicular bisector of points representing 2i and $1+\text{i}$	**B1**
	Show the point representing $2+\text{i}$	**B1**
	Show a circle with radius 2 and centre $2+\text{i}$ (FT on the position of the point for $2+\text{i}$)	**B1FT**
	Shade the correct region	**B1**
		4
(b)(ii)	State or imply the critical point $2+3\text{i}$	**B1**
	Obtain answer 56.3° or 0.983 radians	**B1**
		2

9)JUNE-2020 _9709_32 Q8

(a)	Substitute and obtain a correct equation in x and y	**B1**
	Use $\text{i}^2=-1$ and equate real and imaginary parts	**M1**
	Obtain two correct equations in x and y, e.g. $x-y=3$ and $3x+y=5$	**A1**
	Solve and obtain answer $z=2-\text{i}$	**A1**
		4
(b)(i)	Show a point representing $2+2\text{i}$	**B1**
	Show a circle with radius 1 and centre not at the origin (FT is on the point representing the centre)	**B1 FT**
	Show the correct half line from 4i	**B1**
	Shade the correct region	**B1**
		4
(b)(ii)	Carry out a complete method for finding the least value of $\text{Im}\,z$	**M1**
	Obtain answer $2-\frac{1}{2}\sqrt{2}$, or exact equivalent	**A1**
		2

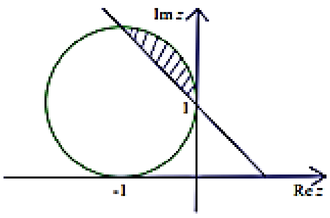
10) JUNE-2020 _9709_33 Q9

(a)	Eliminate u or w and obtain an equation w or u	**M1**
	Obtain a quadratic in u or w, e.g. $u^2-2iu-6=0$ or $w^2+2iw-6=0$	**A1**
	Solve a 3-term quadratic for u or for w	**M1**
	Obtain answer $u=\sqrt{5}+i$, $w=\sqrt{5}-i$	**A1**
	Obtain answer $u=-\sqrt{5}+i$, $w=-\sqrt{5}-i$	**A1**
		5
(b)	Show the point representing $2+2i$	**B1**
	Show a circle with centre $2+2i$ and radius 2 (**FT** is on the position of $2+2i$)	**B1 FT**
	Show half-line from origin at 45° to the positive x-axis	**B1**
	Show line for $\text{Re } z=3$	**B1**
	Shade the correct region	**B1**
		5

11) JUNE-2021 _9709_31 Q5

(a)	Use quadratic formula and $i^2=-1$	**M1**	
	Obtain answers $pi+\sqrt{q-p^2}$ and $pi-\sqrt{q-p^2}$	**A1**	Accept $\frac{2pi\pm\sqrt{-4p^2+4q}}{2}$ and ISW
		2	
(b)	State or imply that the discriminant must be negative	**M1**	
	State condition $q<p^2$	**A1**	
		2	
(c)	Carry out a correct method for finding a relation, e.g. use the fact that the argument of one of the roots is $(\pm)60°$	**M1**	
	State a correct relation in any form, e.g. $\frac{p}{\sqrt{q-p^2}}=(\pm)\sqrt{3}$	**A1**	
	Simplify to $q=\frac{4}{3}p^2$	**A1**	
	Alternative method for Question 5(c)		
	Carry out a correct method for finding a relation, e.g. use the fact that the sides have equal length	**M1**	
	State a correct relation in any form, e.g. $4\left(q-p^2\right)=p^2+q-p^2$	**A1**	
	Simplify to $q=\frac{4}{3}p^2$	**A1**	
		3	

12)JUNE-2021 _9709_32 Q2

Show a circle with centre – 1 + i.	B1	Need some indication of scale or a correct label. Could just be mark(s) on the axes
Show a circle with radius 1 and centre not at the origin (or relevant part thereof).	B1	
Show correct half line from 1(or relevant part thereof) .	B1	
Shade the correct region on a correct diagram.	B1	
	4	N.B. If they have very different scales on *their* 2 axes the diagram must match *their* scale - the 'circle' should be an ellipse. Allow freehand diagrams with clear correct intention.

13)JUNE-2021 _9709_32 Q5

Square $a+ib$, use $i^2=-1$ and equate real and imaginary parts to 10 and $-4\sqrt{6}$ respectively	M1	
Obtain $a^2-b^2=10$ and $2ab=-4\sqrt{6}$	A1	Allow $2abi=-4\sqrt{6}i$
Eliminate one unknown and find an equation in the other	M1	Must be sensible algebra e.g. use of $\sqrt{a^2-b^2}=a-b$ socres M0
Obtain $a^4-10a^2-24[=0]$, or $b^4+10b^2-24[=0]$, or 3-term equivalent	A1	Or equivalent horizontal equation from correct work
Obtain final answers $\pm\left(2\sqrt{3}-\sqrt{2}i\right)$, or exact equivalents	A1	e.g. $\pm\left(\sqrt{12}-\sqrt{2}i\right)$ from correct work
Alternative method for Question 5		
Use the correct method to find the modulus and argument of $\sqrt{u}$	M1	
Obtain modulus $\sqrt{14}$	A1	
Obtain argument $\tan^{-1}\frac{-1}{\sqrt{6}}$ using an exact method	A1	e.g. by using half angle formula which gives $2\sqrt{6}t^2-10t-2\sqrt{6}=0$
Convert to the required form	M1	$\pm\sqrt{14}\left(\frac{\sqrt{6}}{\sqrt{7}}-\frac{1}{\sqrt{7}}i\right)$ This mark is available if working in decimals
Obtain answers $\pm\left(2\sqrt{3}-\sqrt{2}i\right)$, or exact equivalents	A1	e.g. $\pm\left(\sqrt{12}-\sqrt{2}i\right)$
	5	

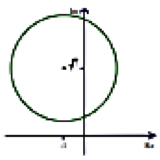
14) JUNE-2021 _9709_33 Q10

(a)	Substitute $-1+\sqrt{2}i$ and attempt expansions of the z^2 and z^4 terms	**M1**
	Use $i^2=-1$ at least once	**M1**
	Complete the verification correctly	**A1**
		3
(b)	State second root $-1-\sqrt{2}i$	**A1**
	Carry out a method to find a quadratic factor with zeros $-1\pm\sqrt{2}i$	**M1**
	Obtain z^2+2z+3	**A1**
	Commence division and reach partial quotient z^2+kz	**M1**
	Obtain second quadratic factor z^2-2z+4	**A1**
	Solve a 3-term quadratic and use $i^2=-1$	**M1**
	Obtain roots $1+\sqrt{3}i$ and $1-\sqrt{3}i$	**A1**
		7

15) JUNE-2022 _9709_31 Q7

(a)	Multiply numerator and denominator by $1-2i$, or equivalent	**M1**
	Obtain correct numerator $(1-2a)\sqrt{2}-(2+a)\sqrt{2}i$	**A1**
	Obtain final answer $\frac{1-2a}{5}\sqrt{2}-\frac{2+a}{5}\sqrt{2}i$	**A1**
	Alternative method for question 7(a)	
	Multiply $x+iy$ by $1+2i$ and compare real and imaginary parts	**M1**
	Obtain $x-2y=\sqrt{2}$ and $2x+y=a\sqrt{2}$	**A1**
	Obtain final answer $\frac{1-2a}{5}\sqrt{2}-\frac{2+a}{5}\sqrt{2}i$	**A1**
		3
(b)	Obtain $r=2$	**B1 FT**
	Obtain $\theta=-\frac{3}{4}\pi$	**B1**
		2
(c)	Use correct method to find r or θ	**M1**
	State answer $\sqrt{2}e^{-\frac{3}{8}\pi i}$	**A1 FT**
	State answer $\sqrt{2}e^{\frac{5}{8}\pi i}$	**A1 FT**
		3

16)JUNE-2022 _9709_32 Q10

a)	Substitute $x = -1+\sqrt{7}\mathrm{i}$ in the equation and attempt expansions of x^2 and x^3	*M1	
	Use $\mathrm{i}^2 = -1$ correctly at least once and solve for k	DM1	$2\left(20-4\sqrt{7}\mathrm{i}\right)+3\left(-6-2\sqrt{7}\mathrm{i}\right)+14\left(-1+\sqrt{7}\mathrm{i}\right)+k=0$
	Obtain answer $k = -8$	A1	
			SC B1 only for those who show no working for the cube and square and obtain answer $k = -8$.
	Alternative method for question 10(a)		
	Attempt division by $\left(x+1-\sqrt{7}\mathrm{i}\right)$ as far as $2x^2 + z_1x + ...$	*M1	See division on next page.
	Use $\mathrm{i}^2 = -1$ correctly at least once and obtain $2x^2 + z_1x + z_2 +$ remainder	DM1	
	Obtain answer $k = -8$	A1	
		3	
(b)	State answer $-1-\sqrt{7}\mathrm{i}$	B1	Can be seen simply stated on its own, or in a list of roots. Allow if stated clearly in part **10(a)**.
	Carry out a method for finding a quadratic factor with zeros $-1+\sqrt{7}\mathrm{i}$ and $-1-\sqrt{7}\mathrm{i}$	M1	Or state $\left(x-\left(-1+\sqrt{7}\mathrm{i}\right)\right)\left(x-\left(-1-\sqrt{7}\mathrm{i}\right)\right)(2x-p)$
	Obtain x^2+2x+8	A1	Or obtain $\left(-1+\sqrt{7}\mathrm{i}\right)\left(-1-\sqrt{7}\mathrm{i}\right)p=-8$ Or obtain $\left(-1+\sqrt{7}\mathrm{i}\right)+\left(-1-\sqrt{7}\mathrm{i}\right)+\frac{p}{2}=-\frac{3}{2}$
	Obtain root $x = \frac{1}{2}$, or equivalent, via division or inspection	A1	Needs to follow from the working.
		4	
(c)	Show a circle with centre $-1+\sqrt{7}\mathrm{i}$	B1	
	Show circle with radius 2 and centre not at the origin There needs to be some evidence of scale e.g. radius marked or a scale on the axes	B1	If the scales are very different from each other then B1 for centre in the correct position and B1 for an ellipse. If there is more than one circle the max score is B1.
		2	
(d)	Carry out a complete method for calculating the maximum value of arg z for correct circle	M1	e.g. $\frac{\pi}{2}+\tan^{-1}\frac{1}{\sqrt{7}}+\frac{\pi}{4}$ Can be implied by 155.7°.
	Obtain answer 2.72 radians	A1	CAO. The question requires radians.
		2	

17)JUNE-2022 _9709_33 Q5

a)	Show u and u^* in relatively correct positions. Must have sense of scale on axes	B1	$u = 3 - \mathrm{i}$, $u^* = 3 + \mathrm{i}$ Ignore labels.
	Show $u^* - u$ in a relatively correct position. Must have sense of scale on axes	B1	2i. Scale only on Imaginary axis is sufficient for this mark.
	State that $OABC$ is a parallelogram [independent of previous marks]	B1	Ignore 'quadrilateral'. Allow 'trapezium' from correct work.
		3	

b)	Multiply *their* numerator and the given denominator by 3 + i and attempt to evaluate either	M1	Can have missing term and arithmetic errors but need $i^2 = -1$ once, seen or implied.
	Obtain numerator 8 + 6i or denominator 10	A1	
	State final answer $\frac{4}{5}+\frac{3}{5}\text{i}$ or $\frac{8}{10}+\frac{6}{10}\text{i}$ or $0.8+0.6\text{i}$	A1	Correct answer with no working scores 0/3.
	Alternative method for question 5(b)		
	Obtain two equations in x and y, and attempt to solve for x or for y	M1	$3 = 3x + y$ and $1 = -x + 3y$
	Obtain $x = \frac{4}{5}$ or $\frac{8}{10}$ or 0.8 $y = \frac{3}{5}$ or $\frac{6}{10}$ or 0.6	A1	
	State final answer $\frac{4}{5}+\frac{3}{5}\text{i}$ or $\frac{8}{10}+\frac{6}{10}\text{i}$ or $0.8+0.6\text{i}$	A1	Correct answer with no working scores 0/3.
		3	
(c)	State or imply $\arg\frac{u^*}{u} = \arg u^* - \arg u$ or $2\arg u^*$	M1	
	Justify the given statement correctly	A1	AG $\arg\frac{u^*}{u} = \tan^{-1}\frac{3}{4}$, $\arg u^* = \tan^{-1}\frac{1}{3}$ and $\arg u = \tan^{-1} -\frac{1}{3}$ (or $\arg u = -\tan^{-1}\frac{1}{3}$), needed if use first expression in M1; or $\arg\frac{u^*}{u} = \tan^{-1}\frac{3}{4}$ and $\arg u^* = \tan^{-1}\frac{1}{3}$, needed if use second expression in M1.
	Alternative method for question 5(c)		
	Use tan $2A$ formula with $\tan A = \frac{1}{3}$	M1	$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$, $\tan A = \frac{1}{3}$, hence $\tan 2A = \frac{3}{4}$.
	Justify the given statement correctly	A1	AG So $2A = \tan^{-1}\frac{3}{4} = \arg\frac{u^*}{u}$ and $A = \tan^{-1}\frac{1}{3} = \arg u^*$ hence $\arg\frac{u^*}{u} = 2\arg u^*$.
		2	

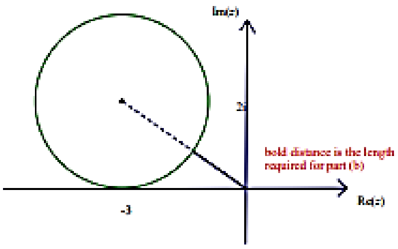
18) JUNE-2023 _9709_31 Q10

(a)	Substitute $x = -3$ to obtain value of $p(-3)$	M1	
	Obtain $p(-3) = 0$ and hence given result	A1	
	Alternative method for Question 10(a)		
	Divide $p(x)$ by $(x+3)$ to obtain quotient $x^2 \pm 2x + ...$	M1	
	Obtain quotient $x^2 + 2x + 25$, with zero remainder and hence given result	A1	
		2	

(b)	Substitute $z=-1+2\sqrt{6}\text{i}$ and attempt expansions of z^2 and z^3	M1	$z^2=-23-4\sqrt{6}\text{i},\quad z^3=-1+6\sqrt{6}\text{i}+72-48\sqrt{6}\text{i}$.
	Use $\text{i}^2=-1$	M1	Seen at least once.
	Obtain $\text{p}(z)=0$ and hence given result	A1	**SC B1** if there is no evidence of working for the square or the cube. Total 1/3.
	Alternative Method 1		
	Use roots $z=-1+2\sqrt{6}\text{i}$ to form quadratic factor	M1	$z^2+2z+25$.
	Divide $\text{p}(z)$ by *their* quadratic factor	M1	
	Obtain zero remainder and hence given result.	A1	
	Alternative Method 2		
	Set *their* quadratic factor from **(a)** equal to zero	M1	
	Solve for z	M1	Need to see method here as answer is given.
	Obtain $z=-1+2\sqrt{6}\text{i}$ (and $z=-1-2\sqrt{6}\text{i}$)	A1	
	Alternative Method 3		
	Substitute $z=-1+2\sqrt{6}\text{i}$ into *their* quadratic factor and attempt expansion of z^2	M1	
	Use $\text{i}^2=-1$	M1	
	Obtain 0 and hence given result	A1	
		3	

(c)	State $z_1=\sqrt{3}\text{i}$ and $z_2=-\sqrt{3}\text{i}$	B1	
	Expand $(x+\text{i}y)^2=-1+2\sqrt{6}\text{i}$ and compare real and imaginary parts	M1	Allow for use of $z^2=-1-2\sqrt{6}\text{i}$.
	Obtain $x^2-y^2=-1$ and $xy=\sqrt{6}$	A1	
	Solve to obtain x and y	M1	
	Obtain $z_3=\sqrt{2}+\sqrt{3}\text{i}$ and $z_4=-\sqrt{2}-\sqrt{3}\text{i}$	A1	
	Use $z^2=-1-2\sqrt{6}\text{i}$ to obtain z_5 and z_6	M1	Allow for use of $z^2=-1+2\sqrt{6}\text{i}$.
	Obtain $z_5=\sqrt{2}-\sqrt{3}\text{i}$ and $z_6=-\sqrt{2}+\sqrt{3}\text{i}$	A1	
		7	

19) JUNE-2023 _9709_32 Q3

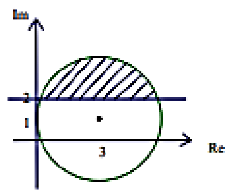
(a)		B1	Show a circle with centre –3 + 2i. Allow for a curved figure with 'centre' in roughly the correct position. Accept marks or numbers on axes, coordinates of centre shown. B0B1 available for axes the wrong way round (and M1 A1 in part **(b)**).
	Show a circle with radius 2	B1 FT	FT centre not at the origin. Allow 'near miss' on x axis. Different scales on axes require an ellipse for B1 B1. Scales on the axes and any label of the radius must be consistent for B1 B1. Correct circle shaded scores B1 B0.
		2	

(b)	Carry out a correct method for finding the least value of $\lvert z \rvert$	M1	e.g. distance of centre from origin – radius or find point of intersection of circle and $3y = -2x$ and use Pythagoras. If they subtract the wrong way round M0. If their diagram is a reflection or a rotation of the correct diagram, M1 A1 is available (requires equivalent work). Any other circle M0.
	Obtain answer $\sqrt{13} - 2$ or $\sqrt{17 - 4\sqrt{13}}$	A1	Or exact equivalent e.g. $\sqrt{17 - \frac{26}{3}\sqrt{\frac{36}{13}}}$. Correct solution only. Allow A1 if exact answer seen and then decimal given.
		2	

20)JUNE-2023 _9709_32 Q5

(a)	Substitute $2 + yi$ in $a^3 - a^2 - 2a$ and attempt expansions of a^2 and a^3	M1	$a^2 = 4 + 4yi - y^2$ $a^3 = 8 + 12yi - 6y^2 - y^3 i$. If using $a\left(a^2 - a - 2\right)$ must then expand fully. Must see working.
	Use $i^2 = -1$	M1	Seen at least once (e.g. in squaring).
	Obtain final answer $-5y^2 + (6y - y^3)i$	A1	Or simplified equivalent e.g. $6yi - 5y^2 - y^3 i$. Do not ISW.
		3	No evidence of working for the square or the cube can score SC B1 for the correct answer.
(b)	Equate *their* $-5y^2$ to -20 and solve for y	M1	Need to obtain a value for y. Available even if *their* y is not real.
	Obtain $y = -2$	A1	From correct work. Allow after incorrect f(a) if the real part was correct. Condone ± 2 with positive not rejected.
	Obtain final answer $\arg a = -\frac{\pi}{4}$	A1	Correct only (must have rejected y positive). OE e.g. $-\frac{\pi}{4} \pm 2n\pi$. Accept -0.785, 5.50. Allow after incorrect f(a) if the real part was correct. Accept degrees. Do not ISW.
		3	

21)JUNE-2023 _9709_33 Q3

Show a circle with centre $3 + i$	B1	Must be some evidence of scale on both axes or centre stated as 3 + i or (3, 1).
Show a circle with radius 3 and centre not at the origin	B1	Must be some evidence that radius = 3 or stated $r = 3$
Show the line $y = 2$	B1	Line $y = 2$ can be represented by 2 or correct dashes.
Shade the correct region	B1	Line and circle must be correct.
	4	Scales may be replaced by dashes on axes for all marks. Correct figure, with no scale on either axis then allow 1/3 and the B1 for correct shaded region Max 2/4. If B0 above for line but relatively correct position then B1 for correct shaded region Max 3/4. Re and Im axes interchanged but clearly labelled, allow **SCB1** for centre and radius of circle correct and **SCB1** for line and shading correct Max 2/4.

22) JUNE-2023 _9709_33 Q11

1(a)	Multiply numerator and denominator by $(3-ai)$	M1	Must perform complete multiplications but need not simplify i^2. Can have errors but no term duplicated or missing. $\frac{(5a-2i)(3-ai)}{9-a^2}=\frac{13a-i(5a^2+6)}{9-a^2}$ M0 M1 A0 No working so unsure if denominator multiplied by $3-ai$ M1 M1 A0
	Use $i^2=-1$ at least once and separate real and imaginary parts	M1	
	Obtain $\frac{13a-i(5a^2+6)}{9+a^2}$ or $\frac{13a-5a^2i-6i}{9+a^2}$	A1	OE If $15a-2a=13a$ seen later award this A1.
	Use $\arg z$ to form equation in a $-\frac{5a^2+6}{13a}=\pm\tan\left(\pm\frac{\pi}{4}\right)$ or $-\frac{13a}{5a^2+6}=\pm\tan\left(\pm\frac{\pi}{4}\right)$ or $\tan^{-1}\left(-\frac{5a^2+6}{13a}\right)=\pm\frac{\pi}{4}$ or $\tan^{-1}\left(-\frac{13a}{5a^2+6}\right)=\pm\frac{\pi}{4}$	M1	Allow expression given in answer column or $5a^2+6=\pm13a$ or use $-(x\pm xi)=(13a-i(5a^2+6))/(9+a^2)$ and eliminate x so $5a^2+6=\pm13a$ M1.
	Obtain $a=2$	A1	Need to reject $a=\frac{3}{5}$ or ignore it in future work. May not see second root, but if present, must be $\frac{3}{5}$.
	Obtain $z=2-2i$ only	A1	Allow $z=-2i+2$.
(a)	**Alternative Method 1 for the first four marks**		
	$\arg z=\arg(5a-2i)-\arg(3+ai)$	M1	
	$=\tan^{-1}\left(\frac{-2}{5a}\right)-\tan^{-1}\left(\frac{a}{3}\right)$ $=\tan^{-1}\left(\frac{-2}{5a}-\frac{a}{3}\right)/\left(1+\left(\frac{-2}{5a}\right)\left(\frac{a}{3}\right)\right)$	M1	Allow one sign error in second M1.
	$=\tan^{-1}\left(-\frac{5a^2+6}{13a}\right)$ or $\tan^{-1}\left(-\frac{13a}{5a^2+6}\right)$	A1	
	$\pm\frac{\pi}{4}=\tan^{-1}\left(-\frac{5a^2+6}{13a}\right)$ or $\tan^{-1}\left(-\frac{13a}{5a^2+6}\right)$	M1	Equate *their* $\tan^{-1}\left(-\frac{5a^2+6}{13a}\right)$ to $\pm\frac{\pi}{4}$. Then as original scheme for final 2 marks.
	Alternative Method 2 for the first four marks		
	$(x+iy)(3+ai)=5a-2i$ $3x-ay=5a$ and $ax+3y=-2$	M1 A1	
	$x=\pm y$ Find x or y in terms of a, e.g. $x=\frac{2}{3-a}$ or $x=\frac{5a}{3+a}$	M1	
	Substitute in other equation, for example $3\left(\frac{2}{3-a}\right)+a\left(\frac{2}{3-a}\right)=5a$	M1	Then as original scheme for final 2 marks.
		6	

(b)	State $\arg(z^3) = -\frac{3}{4}\pi$ or evaluate from $z = b - bi$ or from $-2b^3(1+i)$	B1	If 2 different values given award B0. Do not ISW.
	Complete method to obtain r from *their* z	M1	$\lvert z^3\rvert = \left(\sqrt{x^2+y^2}\right)^3$. If z correct, may see $\lvert z^3\rvert = \left(\sqrt{2^2+(-2)^2}\right)^3$ or $\lvert z^3\rvert = \sqrt{(-16)^2+(-16)^2}$.
	$r = 16\sqrt{2}$	A1	CAO A1 if $z = 2 - 2i$ obtained correctly. or $z =$ used with $a = 2$ found correctly, otherwise A0XP. May see arg and r given in a final answer i.e. $16\sqrt{2}e^{-\frac{3}{4}\pi i}$. Allow this form for arg and r to collect full marks, even if i missing. Ignore answers outside the given interval. If 2 different values given award A0.
		3	

23) OCT-2020_9709_31 Q2

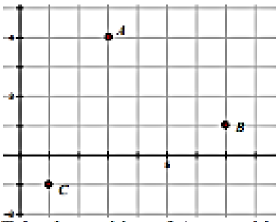
Show a circle with centre the origin and radius 2	B1	
Show the point representing $1 - i$	B1	
Show a circle with centre $1 - i$ and radius 1	B1 FT	The FT is on the position of $1 - i$.
Shade the appropriate region	B1 FT	The FT is on the position of $1 - i$. Shaded region outside circle with centre the origin and radius 2 and inside circle with centre $\pm 1 \pm i$ and radius 1
	4	

24) OCT-2020_9709_31 Q7

(a)	Substitute $-1+\sqrt{5}\,i$ in the equation and attempt expansions of x^2 and x^3	M1	All working must be seen. Allow M1 if small errors in $1-2\sqrt{5}i-5$ or $1-\sqrt{5}i-\sqrt{5}i-5$ and $4-2\sqrt{5}i+10$ or $4-4\sqrt{5}i+2\sqrt{5}i+10$
	Use $i^2 = -1$ correctly at least once	M1	$1-5$ or $4+10$ seen
	Complete the verification correctly	A1	$2\left(14-2\sqrt{5}i\right)+\left(-4-2\sqrt{5}i\right)+6\left(-1+\sqrt{5}i\right)-18=0$
		3	
(b)	State second root $-1-\sqrt{5}i$	B1	
	Carry out a complete method for finding a quadratic factor with zeros $-1+\sqrt{5}i$ and $-1-\sqrt{5}i$	M1	
	Obtain x^2+2x+6	A1	
	Obtain root $x = \frac{3}{2}$	A1	OE
	Alternative method for question 7(b)		
	State second root $-1-\sqrt{5}i$	B1	
	$(x+1-\sqrt{5}i)(x+1+\sqrt{5}i)(2x+a) = 2x^3+x^2+6x-18$	M1	
	$(1-\sqrt{5}\,i)(1+\sqrt{5}\,i)\,a = -18$	A1	
	$6a = -18$ $a = -3$ leading to $x = \frac{3}{2}$	A1	OE

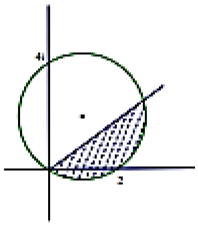
(b)	**Alternative method for question 7(b)**		
	State second root $-1-\sqrt{5}i$	B1	
	POR = 6 SOR = – 2	M1	
	Obtain x^2+2x+6	A1	
	Obtain root $x=\frac{3}{2}$	A1	OE
	Alternative method for question 7(b)		
	State second root $-1-\sqrt{5}i$	B1	
	POR $\left(-1-\sqrt{5}i\right)\left(-1+\sqrt{5}i\right)a=9$	M1 A1	
	Obtain root $x=\frac{3}{2}$	A1	OE
	Alternative method for question 7(b)		
	State second root $-1-\sqrt{5}i$	B1	
	SOR $\left(-1-\sqrt{5}i\right)+\left(-1+\sqrt{5}i\right)+a=-\frac{1}{2}$	M1 A1	
	Obtain root $x=\frac{3}{2}$	A1	OE
		4	

25)OCT-2020_9709_32 Q6

(a)	Multiply numerator and denominator by 1 + i, or equivalent	M1	Must multiply out
	Obtain numerator 6 + 8i or denominator 2	A1	
	Obtain final answer u = 3 + 4i	A1	
	Alternative method for question 6(a)		
	Multiply out $(1-i)(x+iy)=7+i$ and compare real and imaginary parts	M1	
	Obtain $x+y=7$ or $y-x=1$	A1	
	Obtain final answer u = 3 + 4i	A1	
		3	
(b)	Show the point A representing u in a relatively correct position	B1 FT	The FT is on $xy \neq 0$.
	Show the other two points B and C in relatively correct positions: approximately equal distance above / below real axis	B1	 Take the position of A as a guide to 'scale' if axes not marked
		2	

(c)	State or imply $\arg(1-i) = -\frac{1}{4}\pi$	B1	Arg$C$
	Substitute exact arguments in $\arg(7+i) - \arg(1-i) = \arg u$	M1	Must see a statement about the relationship between the Args e.g. Arg$A$ = Arg$B$ − Arg$C$ or equivalent exact method
	Obtain $\tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(\frac{1}{7}\right) + \frac{1}{4}\pi$ correctly	A1	Obtain given answer correctly from *their* $u = k(3+4i)$
		3	

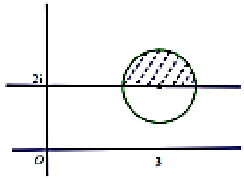
26)OCT-2021_9709_31 Q10

(a)	Substitute $1 + 2i$ in the polynomial and attempt expansions of x^2 and x^3	M1	$u^2 = -3+4i$, $u^3 = -11-2i$ Full substitution but need not simplify.
	Equate real and/or imaginary parts to zero	M1	$-18-3a+b=0$, $4+4a=0$
	Obtain $a = -1$	A1	
	Obtain $b = 15$	A1	
		4	
(b)	State second root $1 - 2i$	B1	
		1	
(c)	State the quadratic factor $x^2 - 2x + 5$	B1	
	State the linear factor $2x + 3$	B1	
		2	
d)(i)	Show a circle with centre $1 + 2i$	B1	
	Show circle passing through the origin	B1	
	Show the half line $y = x$ in the first quadrant (accept chord of circle)	B1	
	Shade the correct region on a correct diagram	B1	
		4	
d)(ii)	State answer $2-\sqrt{5}$	B1	
		1	

27)OCT-2021_9709_32 Q3

a)	Substitute for u and w and state correct conjugate of one side	B1	
	Express the other side without conjugates and confirm $(u+w)^* = u^* + w^*$	B1	**Given answer.** Needs explicit reference to conjugate of both sides.
		2	
b)	Substitute and remove conjugates to obtain a correct equation in x and y	B1	e.g. $x+2-(y+1)i+(2+i)(x+iy)=0$
	Use $i^2 = -1$ and equate real and imaginary parts to zero	M1	
	Obtain two correct equations in x and y	A1	e.g. $3x - y + 2 = 0$ and $x + y - 1 = 0$. Allow $xi + yi - i = 0$.
	Solve and obtain answer $z = -\frac{1}{4} + \frac{5}{4}i$	A1	Allow for real and imaginary parts stated separately.
		4	

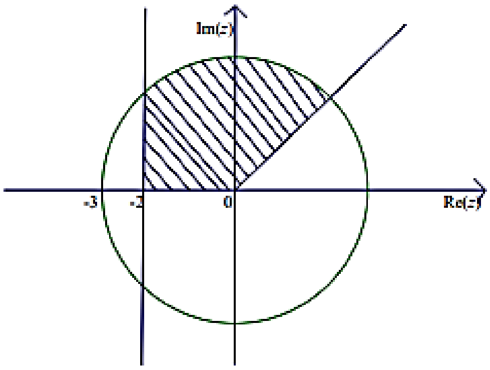
28)OCT-2021_9709_32 Q5

(a)	Show circle with centre 3 + 2i	B1	
	Show circle with radius 1. Must match *their* scales: if scales not identical should have an ellipse.	B1	
	Show line $y = 2$ in at least the diameter of a circle in the first quadrant	B1	
	Shade the correct region in a correct diagram	B1	
		4	
(b)	Identify the correct point	B1	
	Carry out a correct method for finding the argument	M1	e.g. $\arg z = \tan^{-1}\frac{2}{3} + \sin^{-1}\frac{1}{\sqrt{13}}$ Exact working required.
	Obtain answer 49.8°	A1	Or better. 0.869 radians scores **B1M1A0**.
		3	**Special Case 1: B1M0** for 45° if they have shaded the wrong half of the circle. **Special Case 2:** 3 out of 3 available if they identify the correct point on the correct circle and it is consistent with *their* shading.

29)OCT-2021_9709_33 Q11

(a)	State or imply $r = 2$	B1
	State or imply $\theta = \frac{5}{6}\pi$	B1
		2
(b)	Use a correct method for finding the modulus or argument of u^6	M1
	Show correctly that u^6 is real and has value -64	A1
		2
(c)(i)	Show half lines from the point representing $-\sqrt{3} + \mathrm{i}$	B1
	Show correct half lines	B1
	Show the line $x = 2$ in the first quadrant	B1
	Shade the correct region	B1
		4
(c)(ii)	Carry out a correct method to find the greatest value of $\|z\|$	M1
	Obtain answer 5.14	A1
		2

30)OCT-2022_9709_31 Q2

Show a circle with radius 3 and centre the origin	B1	 For the vertical line and the circle, allow the B1 marks if all you see is the relevant part.
Show the line $x = -2$	B1	
Show the correct half line for $\frac{\pi}{4}$	B1	
Shade the correct region	B1	
	4	

31)OCT-2022_9709_31 Q5

5(a)	State or imply $u^2 = 4e^{\frac{1}{3}\pi i}$	B1	
	Obtain answer $v = \frac{4}{3}e^{\frac{1}{6}\pi i}$	B1 + B1	For the modulus and the argument.
		3	
5(b)	State $n = 6$	B1	
		1	

32)OCT-2022_9709_32 Q5

5(a)	Use quadratic formula, or completing the square $\left((z-3i)^2 - 3 = 0\right)$ and use $i^2 = -1$ to find a root	M1	
	Obtain a root, e.g. $\sqrt{3} + 3i$	A1	Or exact 2 term equivalent e.g. $\frac{6i}{2} + \frac{\sqrt{12}}{2}$ ISW.
	Obtain the other root, e.g. $-\sqrt{3} + 3i$	A1	Or exact 2 term equivalent ISW.
		3	
5(b)	Show points representing the roots correctly	B1 FT	2 roots consistent with *their* (a) and with no errors seen on the diagram. B0 if they only have one root or more than 2 roots Must match their scale and $1 < \sqrt{3} < 2$ Linear scales seen or implied. Need some indication of scale (numbers or dashes). Scales along an axis must be approximately consistent but scales may be different on the 2 axes.
		1	

(c)	State modulus of either root is $2\sqrt{3}$, or simplified exact equivalent	B1 FT	ISW if converted to decimal. Ignore modulus of second root if seen. Follow their root(s) not on either axis (from (a) or (b)).
	Find the argument of one of their roots – get as far as $\tan^{-1}(...)$	M1	SOI but must be correct for their root.
	Obtain correct arguments $\frac{1}{3}\pi$ and $\frac{2}{3}\pi$, or simplified exact equivalents	A1	Must obtain values. Allow degrees.
		3	
(d)	Give a **complete justification** that the **correct triangle** is equilateral	B1	Check *their* diagram in (b). Possible justifications: 3 equal sides, or all angles equal to $\frac{\pi}{3}$, or isosceles and an angle of $\frac{\pi}{3}$.
		1	

33)OCT-2022_9709_33 Q5

(a)	Show a circle with centre -2	B1	
	Show a circle with radius 2 and centre not the origin	B1	
	Show the line $y = 1$	B1	
	Shade the correct region	B1	
		4	
(b)	Identify the correct point and carry out a correct method to find the argument	M1	
	Obtain answer $\frac{11}{12}\pi$	A1	2.88 radians or 165°.
		2	

34)OCT-2022_9709_33 Q6

Use quadratic formula to solve for z	M1	**SC M1**: For substitution of $x + iy$ and multiplying out.
Use $i^2 = -1$ throughout	M1	**SC M1**: Use $i^2 = -1$ throughout.
Obtain correct answer in any form	A1	**SC A1**: For two correct equations $x^2 - y^2 + 6xy - 2x + y = 0$ and $-3(x^2 - y^2) + 2xy - x - 2y + 1 = 0$.
Multiply numerator and denominator by $(1 + 3i)$, or equivalent	M1	
Obtain final answer, e.g. $-\frac{1}{2} + \frac{1}{2}i$	A1	
Obtain second final answer, e.g. $\frac{2}{5} + \frac{1}{5}i$	A1	
	6	