Date 11.12.23

P-3

Pure Maths - 3.

Complex Numbers.

Ex. 1. Solution (Revision)

SP-20	M-20	M-22	S-20	S-22	W-20
W-22	M-21	M-23	S-21	S-23	W-21

Suresh Goel
(Former Director)
Alliance World School.
Noida. Delhi. NCR
INDIA.

(+91 9810444804)

1 The complex numbers $1+3i$ and $4+2i$ are denoted by u and v respectively.

(a) Find $\frac{u}{v}$ in $x+iy$ where x and y are real. ---[3]

(b) State the argument of $\frac{u}{v}$ --[1]

In an argand diagram, with origin O, the points A, B and C represent the complex numbers u, v and $u-v$ respectively.

(c) State fully the geometrical relationship between OC and BA. ---[2]

(d) Show that angle $AOB = \frac{1}{4}\pi$ radians $\boxed{SP\text{-}20/03/Q6}$ ---[2]

Solution (a) $\frac{u}{v} = \frac{1+3i}{4+2i} = \frac{(1+3i)}{(4+2i)} \times \frac{(4-2i)}{(4-2i)}$

$$= \frac{(4+6)+i(-2+12)}{4^2+2^2} \qquad [(a+ib)(a-ib) = a^2+b^2]$$

$$= \frac{10+10i}{20} = \underline{\left(\tfrac{1}{2}+\tfrac{1}{2}i\right)} \checkmark$$

(b) $\arg\left(\frac{1}{2}+\frac{1}{2}i\right) = \tan^{-1}\left(\frac{1}{2}/\frac{1}{2}\right) = \tan^{-1}1 = \underline{\frac{\pi}{4}} \checkmark$

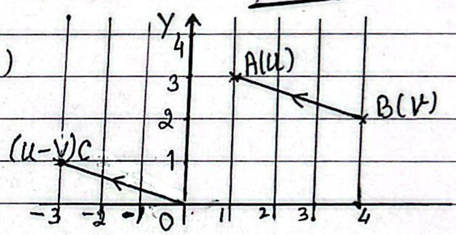
(c) $u-v = (1+3i)-(4+2i)$

or $\vec{OC} = (-3+i)$

$\vec{BA} = u-v = (-3+i)$

$\therefore \vec{OC} = \vec{BA}$

$\therefore$ OC and BA are equal and parallel.



(d) angle $AOB = \arg u - \arg v = \arg\frac{u}{v} = \underline{\frac{\pi}{4}} \checkmark$ (from Part (b))

Complex Numbers_ Revision

Example 2(a) The complex numbers v and w satisfy the equations:

$$v + iw = 5 \quad \text{and} \quad (1+2i)v - w = 3i$$

Solve the equations for v and w, giving your answers in the form $x+iy$, where x and y are real. ---[6]

(b) (i) On an argand diagram, sketch the locus of points representing complex numbers z satisfying:

$$|z-2-3i| = 1 \quad \text{---[2]}$$

(ii) Calculate the least value of $\arg z$ for the points on the locus. ---- [2]

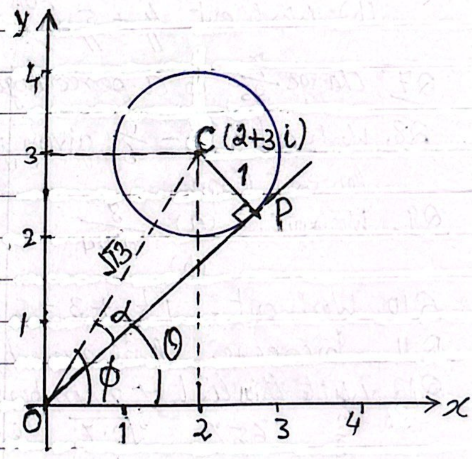
Solution (a) Solve for v and w (use $i^2=-1$) | M-20 | 32 | Q10 |

we get $v = \dfrac{-2i}{1+i}$ and $w = \dfrac{5+7i}{-1+i}$

multiply N^r and D^r by the conjugate of the D^r.

we get $v = -1-i$ ✓

and $w = 1-6i$ ✓



(b)(i) $|z-2-3i| = 1$

or $|z-(2+3i)| = 1$

represents a circle with centre at $(2+3i)$ and rad $= 1$.

Draw OP tangent to the circle

The arg. of $P = \angle POX = \theta$ is the least.

$\theta = \arg(2+3i) - \alpha$

$= \tan^{-1}\dfrac{3}{2} - \sin^{-1}\dfrac{1}{\sqrt{13}}$

$= 56{\cdot}3^\circ - 16{\cdot}1^\circ$

$= \underline{40{\cdot}2^\circ}$ ✓

Dis $OC = \sqrt{2^2+3^2} = \sqrt{13}$

$CP = r = 1$

3 The complex numbers u and v are defined by $u=-4+2i$ and $v=3+i$

(a) Find u/v in the form $x+iy$, where x and y are real. ...[3]

(b) Hence express u/v in the form $re^{i\theta}$, where θ and r are exact. ..[2]

In an argand diagram, with origin O, the points A, B and C represent the complex numbers u, v and $(2u+v)$ resp.

(c) State fully the geometrical relationship between OA and BC. --[2]

(d) Prove that angle $AOB = \frac{3}{4}\pi$.

M-21/32/Q8

Solution (a) $\frac{u}{v} = \frac{-4+2i}{3+i} = \frac{-4+2i}{3+i} \times \frac{(3-i)}{(3-i)} = \frac{(-12+2)+i(4+6)}{3^2+1^2} = \frac{-10+10i}{10}$

$= (-1+i)$ ✓

(b) $\frac{u}{v} = (-1+i) = r(\cos\theta + i\sin\theta) = re^{i\theta}$ --- ①

$\begin{cases} r\cos\theta = -1 \\ r\sin\theta = 1 \end{cases}$, Sq and add $r^2 = 1^2 + (-1)^2 = 2 \Rightarrow r = \sqrt{2}$ ✓

~~and $\tan\theta = \frac{-1}{1} = -1 = -\tan\frac{\pi}{4}$~~

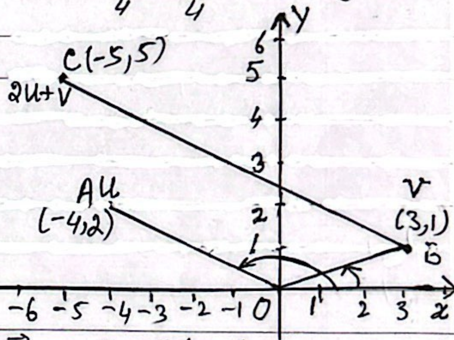
$\Rightarrow \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ ✓ --- ②

from ① $\frac{u}{v} = \sqrt{2}\,e^{i\cdot\left(\frac{3\pi}{4}\right)}$ ✓

(C) $u = -4+2i$, $v = 3+i$

$2u+v = 2(-4+2i)+(3+i) = (-5+5i)$

$\vec{BC} = \vec{OC} - \vec{OB} = (-5+5i)-(3+i)$
$= (-8+4i)$



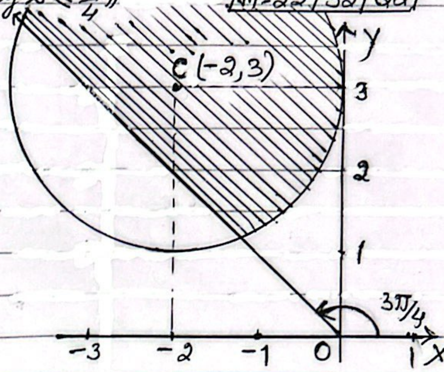
Now $\vec{BC} = (-8+4i) = 2(-4+2i) = 2\vec{OA}$

$\Rightarrow$ $\vec{BC} = 2\vec{OA} \Rightarrow$ $\vec{BC}$ and $\vec{OA}$ are Parallel and $BC = 2OA$.

(d) angle $AOB = \arg u - \arg v = \arg\frac{u}{v} = \arg(-1+i) = \frac{3\pi}{4}$ from ②

4. On the sketch of an Argan diagram, shade the region whose points represent complex numbers z satisfying the inequalities --- [4]

$|z+2-3i| \leq 2$ and $\arg z \leq \frac{3}{4}\pi$ [M-22/32/Q2]



Solution: $|z+2-3i| \leq 2$ —— ①

$\Rightarrow |z-(-2+3i)| \leq 2$

Represents a circle and its interior with centre $(-2, 3)$ and radius 2 units.

and $\arg z \leq \frac{3}{4}\pi$ represents a half line from origin inclined at angle $\frac{3\pi}{4}$ with positive x-axis, and between x-axis.

5. Find the complex numbers w which satisfy the equation: $w^2 + 2iw^* = 1$ and are such that $\text{Re } w \leq 0$.

Give your answer in the form $x+iy$, where x and y are real. --- [6]

[M-22/32/Q6]

Solution: $w^2 + 2iw^* = 1$ $\left\{ \begin{array}{l} \text{Let } w = x+iy \\ w^* = x-iy \end{array} \right.$

$\Rightarrow (x+iy)^2 + 2i(x-iy) = 1$

$\Rightarrow (x^2-y^2) + 2xyi + 2ix + 2y = 1$

Equating real and imaginary parts.

$x^2 - y^2 + 2y = 1$ —— ①

$2xy + 2x = 0$ —— ②

from ② $2x(y+1) = 0$

$\Rightarrow x = 0$ or $y = -1$

from ① $\left\{ \begin{array}{l} x = 0 \Rightarrow y^2 - 2y + 1 = 0 \Rightarrow (y-1)^2 = 0 \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad y = 1 \\ \therefore w = 0 + i \checkmark \end{array} \right.$

Again for $\left\{ \begin{array}{l} y = -1, \text{ in ①} \Rightarrow x^2 - 1 - 2 = 1 \\ x^2 = 4 \\ x = -2 \;\; \cancel{2} \;\; (\text{Re } w \leq 0) \end{array} \right.$

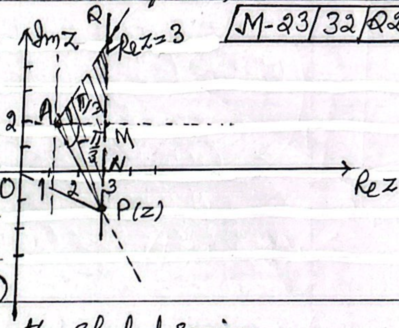
$\therefore w = (-2-i) \checkmark$

$\therefore w = i$ or $(-2-i) \checkmark$

6(a) On an argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $-\frac{1}{3}\pi \le \arg(z-1-2i) \le \frac{\pi}{3}$ and $\mathrm{Re}\, z \le 3$ ---[3]

(b) Calculate the least value of $\arg z$ for points in the region from (a). Give your answer in radians correct to 3 decimal places. ---[2]

M-23/32/Q2



Solution (a) Given $-\frac{\pi}{3} \le \arg(z-(1+2i)) \le \frac{\pi}{3}$

$A(1,2)$ and $\mathrm{Re}\, z \le 3$ is on the left of line PQ.

(b) In ΔAMP, $\frac{MP}{AM} = \tan\frac{\pi}{3}$

$\Rightarrow MP = 2\sqrt{3}$

$\Rightarrow PN = PM - MN = 2\sqrt{3} - 2$

In ΔPON, $\tan\theta = \frac{NP}{ON} = \frac{2\sqrt{3}-2}{3}$ --(1)

$\because$ for Point P has the least $\arg z$ in the shaded region.

for P: $\arg z = -\theta = -\tan^{-1}\left(\frac{2\sqrt{3}-2}{3}\right)$ ✓

7. Solve the equation: $\frac{5z}{1+2i} - z \cdot z^* + 30 + 10i = 0$ ---[5]

Give your answer in the form $(x+iy)$, where x and y are real.

M-23/32/Q4

Solution: Let $z = (x+iy)$, $z^* = (x-iy) \Rightarrow z \cdot z^* = x^2+y^2$

Now $\frac{5z}{(1+2i)} - zz^* + 30 + 10i = 0$

$\Rightarrow 5(x+iy) - (x^2+y^2)(1+2i) + (30+10i)(1+2i) = 0$

$\Rightarrow 5x + 5iy - (x^2+y^2) - 2(x^2+y^2)i + ((30-20) + i(60+10)) = 0$

$\Rightarrow 5x + 5iy - (x^2+y^2) - 2(x^2+y^2)i + 10 + 70i = 0$

$\Rightarrow (5x - (x^2+y^2) + 10) + i(5y - 2(x^2+y^2) + 70) = 0$

$\Rightarrow \begin{cases} 5x - (x^2+y^2) + 10 = 0 & \text{---(1)} \\ 5y - 2(x^2+y^2) + 70 = 0 & \text{---(2)} \end{cases}$

(2) $- 2\times$(1) $\Rightarrow 5y - 10x + 50 = 0 \Rightarrow y = (2x-10)$ ---(3)

for (1) and (3) $5x - x^2 - (2x-10)^2 + 10 = 0$

$5x - x^2 - (4x^2 - 40x + 100) + 10 = 0 \Rightarrow -5x^2 + 45x + 90 = 0$

$\Rightarrow x^2 - 9x + 18 = 0 \Rightarrow (x-3)(x-6) = 0 \Rightarrow x = 3, x = 6$

from (3) $\begin{cases} x = 3 \\ y = -4 \end{cases}$ and $\begin{cases} x = 6 \\ y = 2 \end{cases}$ $\therefore$ Solutions are $(3-4i)$ & $(6+2i)$ ✓

Example 8.: The complex number u is defined by, $u = \frac{3i}{a+2i}$, where a is real.

a(i) Express u in the cartesian form $(x+iy)$, where x and y are in terms of a. ---[3]

(ii) Find the exact value of a for which $\arg u^* = \frac{1}{3}\pi$ --[3]

(b)(i) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z-2i| \le |z-1-i|$ and $|z-2-i| \le 2$ --[4]

(ii) Calculate the least value of $\arg z$ for the points in this region. --[2]

S-20 | 31 | Q10

Solution (a)(i) Multiply N^r and D^r by $(a-2i)$ and use $i^2 = -1$

$\Rightarrow \quad u = \frac{6}{(a^2+4)} + \frac{3ai}{a^2+4}$

(a)(ii) $u^* = \frac{6}{(a^2+4)} - \frac{3ai}{a^2+4} \Rightarrow \arg u^* = \tan^{-1}\left(-\frac{3a}{6}\right) = \frac{\pi}{3}$ (given)

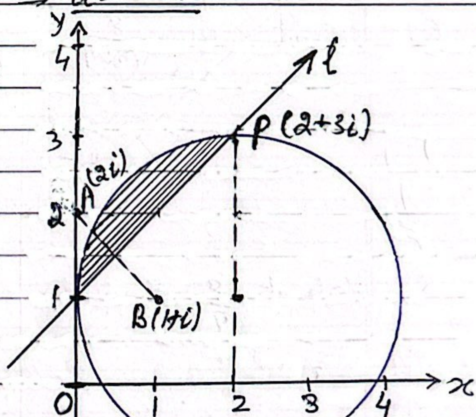
$\Rightarrow -\frac{3a}{6} = \sqrt{3}$

$\Rightarrow a = -2\sqrt{3}$ ✓

(b)(i) $|z-2i| = |z-(1+i)|$ represent the perp. bisector of seg. joining $A(2i)$, $B(1+i)$. ✓

and $|z-(2+i)| = 2$ represents a circle centre $(2+i)$ and $r = 2$.

Shaded region.



(ii) Least value of $\arg z$ in the shaded region at $P(2+3i)$

$\arg(2+3i) = \tan^{-1} 3/2$

$= 56.3^\circ$

Scanned with CamScanner

Example 9(a) Solve the equation $(1+2i)w + iw^* = 3+5i$. Give your answer in the form $x+iy$, where x and y are real. ---[4]

b(i) On a sketch of Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities; $|z-2-2i| \leq 1$ and $\arg(z-4i) \geq -\frac{1}{4}\pi$ --[4]

(ii) Find the least value of $\operatorname{Im} z$ for the points in this region, giving your answer in an exact form. --[2]

[S-20/32/Q8]

Solution (a) $(1+2i)(x+iy) + i(x-iy) = 3+5i$ [using $i^2 = -1$]

equating real and imaginary parts;

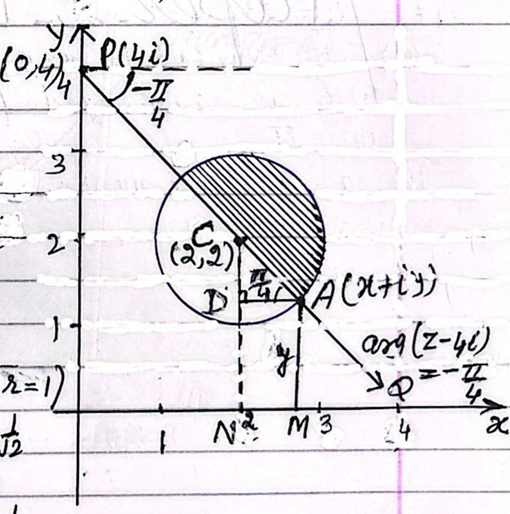
$x - y = 3$ and $3x + 5 = 5$

Solve for x and y, and get $w = (2-i)$ ✓

(b)(i) $|z-(2+2i)| \leq 1$; represents a circle centre $(2+2i)$, $r=1$

$(z-4i) \geq -\frac{\pi}{4}$ represents a half line from $4i$.

Shade the correct region.



(ii) A is the point with the least $\operatorname{Im} z$, in the shaded area,

Draw $AM \perp$ X-axis

$CN \perp$ X-axis

$AD \perp CN$

In rt $\triangle CAD$,

$\frac{CD}{CA} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $(CA = r = 1)$

$CD = CA \times \frac{1}{\sqrt{2}} = 1 \times \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

Now $AM = DN = CN - CD$

$= 2 - \frac{1}{\sqrt{2}}$

$\therefore \operatorname{Im} z = 2 - \frac{1}{\sqrt{2}} = \left(2 - \frac{1}{2}\sqrt{2}\right)$ ✓

Example 10 (a) Complex numbers u and w are such that.

$$u - w = 2i \quad \text{and} \quad uw = 6$$

Find u and w, giving your answers in the form $x+iy$, where x and y are real and exact. ---[5]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities:
$|z-2-2i| \leq 2$, $0 \leq \arg z \leq \frac{1}{4}\pi$ and $\operatorname{Re} z \leq 3$. $\boxed{S\text{-}20/33/Q9}$ ---[5]

Solution (a) $u - w = 2i$ and $uw = 6$

eliminating w, $\Rightarrow u^2 - 2iu - 6 = 0$

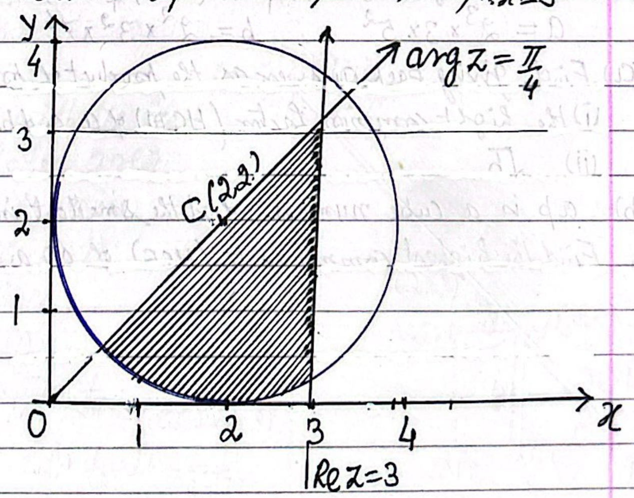
$\Rightarrow u = \sqrt{5} + i$, $w = \sqrt{5} - i$

or $u = -\sqrt{5} + i$, $w = -\sqrt{5} - i$

(b) $|z - (2+2i)| = 2$, represents a circle, centre $(2,2)$, rad $= 2$

$0 \leq \arg z \leq \frac{1}{4}\pi$, show half line from origin at 45° to the positive x-axis

$\operatorname{Re} z \leq 3$ represent a line parallel to y-axis, through $x=3$



11. (a) Solve the equation $z^2 - 2piz - q = 0$, where p and q are real constants. [2]

In an argand diagram with origin O, the roots of this equation are represented by the distinct points A and B.

(b) Given that A and B lie on the imaginary axis, find a relation between p and q. --[2]

(c) Given instead that triangle OAB is equilateral, express q in in terms of p.

S-21/31/Q5 --[3]

Solution (a) $z^2 - 2piz - q = 0$ --- ①

$$z = \frac{2pi \pm \sqrt{-4p^2 + 4q}}{2}$$

$$= pi \pm \sqrt{q - p^2} \checkmark$$

(b) For A and B lie on imaginary axis $\Rightarrow$ discriment of 1 be negative

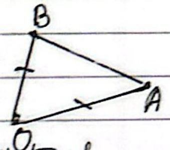
$\Rightarrow -4p^2 + 4q < 0$

$\Rightarrow q < p^2 \checkmark$ --- ②

(b)

$OA = pi + \sqrt{q - p^2}$

$OB = pi - \sqrt{q - p^2}$



for OAB to be equilateral

$OA = OB = AB$

$OA^2 = (p)^2 + q - p^2$ — ②

$= q$

$\vec{BA} = 2\sqrt{q - p^2}$

$AB^2 = 4(q - p^2)$ — ③

Now $AB^2 = OA^2$

$\Rightarrow 4(q - p^2) = q$

$\Rightarrow 3q = 4p^2 \Rightarrow q = \frac{4}{3}p^2 \checkmark$

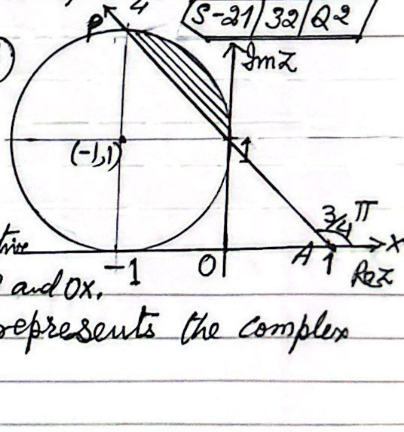
12. On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities:

$$|z + 1 - i| \leq 1 \text{ and } \arg(z - 1) \leq \frac{3}{4}\pi$$ --[4]

S-21/32/Q2

Solution: $|z + 1 - i| \leq 1 \Rightarrow |z - (-1 + i)| \leq 1$ — ①

represent a circle (and its interior) with centre at $(-1, 1)$ and rad $= 1$.



$\arg(z - 1) \leq \frac{3}{4}\pi$ ② represent a half line AP inclined at angle $\frac{3}{4}\pi$ with positive X-axis and the interior between AP and OX.

The common shaded area represents the complex numbers 'z' satisfying ① and ②.

13. The complex number u is given by, $u = 10 - 4\sqrt{6}i$
Find two square roots of u, giving your answers in the form $a + ib$, where a and b are real and exact. ---[5]
[S-21/32/Q5]

Solution: Let $(a+ib)$ is the square root of $(10-4\sqrt{6}i)$

$\Rightarrow (a+ib)^2 = 10 - 4\sqrt{6}i$ --- ①

$\Rightarrow (a^2-b^2) + 2abi = 10 - 4\sqrt{6}i$ -- ②

Equating real and imaginary parts of ② $\Rightarrow$ $a^2 - b^2 = 10$ --- ③

$2ab = -4\sqrt{6}$ --- ④

from ④ $b = -\frac{2\sqrt{6}}{a}$ Put in ③

$a^2 - \left(-\frac{2\sqrt{6}}{a}\right)^2 = 10$

$\Rightarrow a^4 - 10a^2 - 24 = 0$

$(a^2-12)(a^2+2) = 0 \Rightarrow$

$a^2 = 12$ or $\cancel{a^2 = -2}$

$a = \pm 2\sqrt{3}$

(i) $a = 2\sqrt{3}$, $b = \frac{-2\sqrt{6}}{2\sqrt{3}} = -\sqrt{2}$

(ii) $a = -2\sqrt{3}$, $b = \frac{-2\sqrt{6}}{-2\sqrt{3}} = \sqrt{2}$

$\therefore$ Required squared roots are

$(2\sqrt{3} - \sqrt{2}i)$, $(-2\sqrt{3} + \sqrt{2}i)$

or $\pm(2\sqrt{3} - \sqrt{2}i)$ ✓

---

14 (a) Verify that $-1+\sqrt{2}i$ is a root of the equation $z^4 + 3z^2 + 2z + 12 = 0$ --[3]
(b) Find the other roots of this equation. [S-21/33/Q10] --[7]

Solution: $z^4 + 3z^2 + 2z + 12 = 0$ --- ①

ⓐ Let $z = (-1+\sqrt{2}i) \Rightarrow z^2 = \left((-1)^2 - (\sqrt{2})^2\right) + 2\sqrt{2}i$

$\Rightarrow z^2 = (-1 - 2\sqrt{2}i)$ --- ②

$z^4 = (z^2)^2 = (1-8) + 4\sqrt{2}i = -7 + 4\sqrt{2}i$ --- ③

from ② and ③ in ①

$(-7+4\sqrt{2}i) + 3(-1-2\sqrt{2}i) + 2(-1+\sqrt{2}i) + 12 = 0$

$(-7-3-2+12) + i(4\sqrt{2} - 6\sqrt{2} + 2\sqrt{2}) + 12 = 0$

$\Rightarrow 0 = 0$ True ✓

$\therefore$ $(-1+\sqrt{2}i)$ is a root of equation ①

ⓑ If $(-1+\sqrt{2}i)$ is a root of ①

$\Rightarrow$ its conjugate $(-1-\sqrt{2}i)$ is also a root of ①

Sum $= (-1+\sqrt{2}i) + (-1-\sqrt{2}i) = -2$
Product $= (-1+\sqrt{2}i)(-1-\sqrt{2}i) = 1+2 = 3$

$\therefore$ Polynomial with these roots

$z^2 - (\text{Sum})z + \text{Product} = 0$

$(z^2 + 2z + 3)$ is a factor of ① ✓

To find the other factor of L.H.S of ①

$$
\begin{array}{r|l}
z^2+2z+3 & z^4 + 3z^2 + 2z + 12 \quad (z^2 - 2z + 4 \\
 & \underline{z^4 + 3z^2 \quad + 2z^3} \\
 & -2z^3 + 2z + 12 \\
 & \underline{-2z^3 - 6z \quad - 4z^2} \\
 & 4z^2 + 8z + 12 \\
 & \underline{4z^2 + 8z + 12} \\
 & \times
\end{array}
$$

$\therefore$ The second factor is $z^2 - 2z + 4$

$\therefore$ To find the roots of $z^2 - 2z + 4 = 0$

$z = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm \sqrt{3}i$ ✓

$= (1+\sqrt{3}i), (1-\sqrt{3}i)$

$\therefore$ other roots of equation ①

$(-1-\sqrt{2}i)$; $(1+\sqrt{3}i)$, $(1-\sqrt{3}i)$ ✓

15. The complex number u is defined by $u = \frac{\sqrt{2} - a\sqrt{2}\,i}{1+2i}$, where a is a positive integer.

(a) Express u in terms of a, in the form $x+iy$, where x and y are real and exact. ---[3]

It is now given that $a=3$

(b) Express u in the form $re^{i\theta}$, where $r>0$ and $-\pi<\theta\le\pi$, giving exact values of θ and r. ---[2]

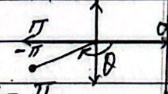
(c) Using your answer to part (b), find the two square roots of u. Give your answers in the form $re^{i\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. ...[3]

$\boxed{\text{S-22/31/Q7}}$

Solution

(a)
$$u = \frac{\sqrt{2} - a\sqrt{2}\,i}{(1+2i)} \times \frac{(1-2i)}{(1-2i)}$$
$$= \frac{(\sqrt{2} - 2a\sqrt{2}) + i(-2\sqrt{2} - a\sqrt{2})}{(1^2+2^2)}$$
$$\therefore\ u = \frac{(1-2a)\sqrt{2}}{5} - \frac{(2+a)\sqrt{2}}{5}\,i \quad \text{---(1)}$$

(b) for $a=3$
$$u = \frac{-5\sqrt{2}}{5} - \frac{5\sqrt{2}\,i}{5}$$
$$\text{or}\quad u = -\sqrt{2} - \sqrt{2}\,i \quad \text{---(2)}$$
from (2)
$$= r(\cos\theta + i\sin\theta)$$
$$r\cos\theta = -\sqrt{2} \quad \text{---(3)}$$
$$r\sin\theta = -\sqrt{2} \quad \text{---(4)}$$
Sq and add (3) & (4) $r^2 = 4 \Rightarrow r = 2$ ✓
$$\tan\theta = \left(\frac{-\sqrt{2}}{-\sqrt{2}}\right) = 1$$



basic angle $\alpha = \tan^{-1} 1 = \frac{\pi}{4}$
$$\therefore\ \theta = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{4}\right)$$
$$\theta = -\frac{3\pi}{4} ✓$$
$$\therefore\ u = re^{i\theta}$$
$$= 2 \cdot e^{-\frac{3\pi}{4}i} ✓ \quad \text{---(5)}$$

(c) $u = re^{i\theta}$

Then the square roots of z is w
$$w = (re^{i\theta})^{1/2} = \sqrt{r}\,e^{i\theta/2} \text{ and}$$
$$\begin{cases} \sqrt{r}\,e^{(\theta/2 - \pi)i} & \text{if } \theta \ge 0 \\ \text{or } \sqrt{r}\,e^{(\theta/2+\pi)} & \text{if } \theta < 0 \end{cases}$$

Now from (5)
$$u = 2e^{-\frac{3\pi}{4}i}$$
$\therefore$ Sq roots of u are
$$\sqrt{2} \cdot e^{\frac{1}{2}\left(-\frac{3\pi}{4}i\right)} \text{ and } \sqrt{2}\,e^{\left(\frac{1}{2}\left(-\frac{3\pi}{4}\right)+\pi\right)i}$$
or
$$= \underline{\sqrt{2}\,e^{-3\pi/8\,i}} \text{ and } \underline{\sqrt{2}\,e^{\frac{5}{8}\pi i}} ✓$$

16. The complex number $-1+\sqrt{7}i$ is denoted by u. It is given that u is a root of the equation: $2x^3+3x^2+14x+k=0$ where k is a real constant.

(a) Find the value of k. ---[3]

(b) Find the other two roots of the equation. --[4]

(c) On the argand diagram, sketch the locus of the points representing complex numbers z satisfying the equation $|z-u|=2$ --[2]

(d) Determine the greatest value of $\arg z$ for the points on this locus, giving your answer in radians. --[2]

[S-22/32/Q10]

Solution Eqn: $2x^3+3x^2+14x+k=0$ --- ①

(a) Let $x=(-1+\sqrt{7}i)$ ---②

$x^2=(1^2-7)+2(-1)\sqrt{7}i=(-6-2\sqrt{7}i)$

$x^3=x^2\cdot x=(-6-2\sqrt{7}i)(-1+\sqrt{7}i)$

$=(6+14)+i(-6\sqrt{7}+2\sqrt{7})i$

$=(20-4\sqrt{7}i)$ ——— ③

from ② and ③ ① as $x=(-1+\sqrt{7}i)$ is a root

7 ① $\Rightarrow 2(20-4\sqrt{7}i)+3(-6-2\sqrt{7}i)+14(-1+\sqrt{7}i)+k=0$

$\Rightarrow (8+k)+0i=0 \Rightarrow \underline{k=-8}$ ✓

(b) If one root of ① $z=(-1+\sqrt{7}i)$ ✓

then the second root $\bar{z}=(-1-\sqrt{7}i)$ ✓

$\begin{cases}\text{Sum of roots } z+\bar{z}=-2 ✓\\ \text{Product of roots } z\cdot\bar{z}=(-1+\sqrt{7}i)(-1-\sqrt{7}i)\\ \qquad =1^2+(\sqrt{7})^2=8 ✓\end{cases}$

∴ Quad with two roots

$x^2-(\text{Sum})x+\text{Product}=0$

∴ $x^2-(-2)x+8=0$

∴ x^2+2x+8 is a factor of ①

$$x^2+2x+8\,\big)\,2x^3+3x^2+14x-8\,\big(\,2x-1$$
$$\underline{-2x^3\mp4x^2\mp16x}$$
$$-x^2-2x-8$$
$$\underline{+-x^2\mp2x\mp8}$$
$$\times$$

∴ The third factor $2x-1$.

$(2x-1)=0 \Rightarrow$ ∴ Third root $=\frac{1}{2}$ ✓

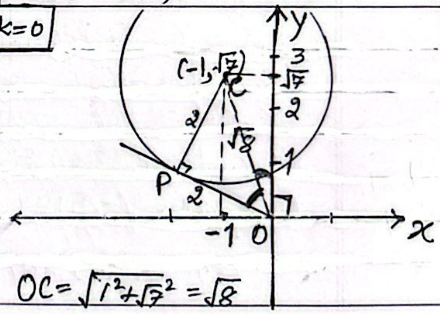
∴ The other two roots are $(-1-\sqrt{7}i)$ and $\frac{1}{2}$ ✓

(c)

$u=(-1+\sqrt{7}i)$

$|z-u|=2$

represent a circle with centre at $(-1,\sqrt{7})$; rad $=2$.



$OC=\sqrt{1^2+\sqrt{7}^2}=\sqrt{8}$

$CP=r=2$

In rt Δ OCP, $OP^2+PC^2=OC^2$

$OP^2+2^2=8$

$OP^2=4$

$OP=2=PC$

∴ ΔOPC is iso. rt triangle

angle $POC=\frac{\pi}{4}$

angle $COP=\tan^{-1}\frac{1}{\sqrt{7}}$

Arg z is max. at P ——— angle POX

$=\frac{\pi}{2}+\tan^{-1}\frac{1}{\sqrt{7}}+\frac{\pi}{4}=\underline{2.72\text{ radians}}$

17. The complex number $3-i$ is denoted by u.

(a) Show, on an argand diagram with origin O, the points A, B, and C representing the complex numbers u, u^* and u^*-u respectively. State the type of quadrilateral formed by the points O, A, B and C. --[3]

(b) Express $\frac{u^*}{u}$ in the form $x+iy$, where x and y are real. ---[3]

(c) By considering the argument of $\frac{u^*}{u}$, or otherwise, prove that

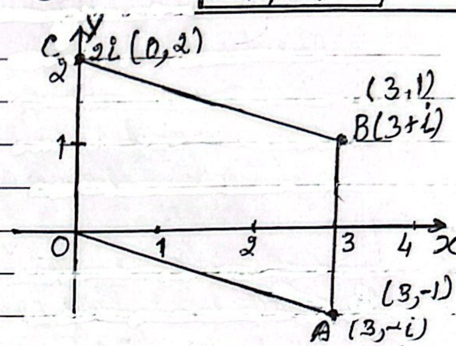
$$\tan^{-1}\left(\tfrac{3}{4}\right) = 2\tan^{-1}\left(\tfrac{1}{3}\right)$$ --[2]

[S-22/33/Q5]

Solution: $u=(3-i)$, $u^* = 3+i$

(a) $u^* - u = (3+i)-(3-i) = 2i$

The quadrilateral OABC is a Parallelgram. ✓



(b) $$\frac{u^*}{u} = \frac{(3+i)}{(3-i)} \times \frac{(3+i)}{(3+i)}$$

$$= \frac{(3^2-1^2)+2\times3\times1\times i}{3^2+1^2}$$

$$= \frac{8+6i}{10} = \left(\frac{4}{5}+\frac{3}{5}i\right) ✓$$

(c) $$\frac{u^*}{u} = \left(\frac{4}{5}+\frac{3}{5}i\right)$$

$$\Rightarrow \arg\frac{u^*}{u} = \tan^{-1}\frac{3/5}{4/5} = \tan^{-1}3/4 \quad \text{--(1)}$$

Also $$\arg\frac{u^*}{u} = \arg u^* - \arg u$$

$$= \arg(3+i) - \arg(3-i)$$

$$= \tan^{-1}1/3 - \tan^{-1}\left(-\frac{1}{3}\right) = \tan^{-1}\frac{1}{3} + \tan^{-1}1/3 = 2\tan^{-1}1/3 \quad \text{--(2)}$$

from (1) and (2) $\tan^{-1}3/4 = 2\tan^{-1}(1/3)$ ✓

Scanned with CamScanner

18. The polynomial $x^3+5x^2+31x+75$ is denoted by $p(x)$.

(a) Show that $(x+3)$ is a factor of $p(x)$. ...[2]

(b) Show that $z=-1+2\sqrt{6}i$ is a root of $p(z)=0$ ...[3]

(c) Hence find the complex numbers z which are roots of $p(z^2)=0$ ...[7]

[S-23 | 31 | Q10]

Solution: $p(x)=x^3+5x^2+31x+75$ ---- (1)

$\Rightarrow p(-3)=(-3)^3+5(-3)^2+31\times(-3)+75$

$=-27+45-93+75=0$ ✓

$\Rightarrow (x+3)$ is a factor of $p(x)$.

(using factor theorem: $(x+3)$ is a factor of $(p(x) \Leftrightarrow p(-3)=0)$

(b) from (1) $p(z)=z^3+5z^2+31z+75$ --- (2)

Let $z_1=(-1+2\sqrt{6}i)$, $z_2=\bar{z}_1=-1-2\sqrt{6}i$

Now $z_1+z_2=-2$ ✓ --- (3)

$z_1 . z_2=(-1+2\sqrt{6}i)(-1-2\sqrt{6}i)$

$=(1^2+(2\sqrt{6})^2)=25$ ✓ (4)

$\therefore$ quad. eqn with roots z_1, z_2.

$z^2-(z_1+z_2)z+z_1 . z_2=0$

$\Rightarrow z^2-(-2)z+25=0$ (frm (3) & (4))

$z^2+2z+25=0$ --- (5)

Now frm (2) & (5)

$$
\begin{array}{r|l}
z^2+2z+25 & z^3+5z^2+31z+75 \;\; (z+3) \\
 & \underline{-(z^3+2z^2+25z)} \\
 & 3z^2+6z+75 \\
 & \underline{-(3z^2+6z+75)} \\
 & \times
\end{array}
$$

$\therefore (z^2+2z+25)$ is a factor of $p(z)$

as $z_1=(-1+2\sqrt{6}i)$ is a factor of $(z^2+2z+25)$. hence

$z_1=(-1+2\sqrt{6}i)$ is a factor of $p(z)$

(C) The roots $p(z^2)=0$ will be the squre roots of the roots of the roots of the equation $p(z)=0$

Roots of the equation $p(z)=0$ are (i) -3 (ii) $(-1+2\sqrt{6}i)$ and (iii) $(-1-2\sqrt{6}i)$.

Case I when $z^2=-3 \Rightarrow z=\pm\sqrt{-3}$

$z=\pm\sqrt{3}i$ ✓ --- (6)

Case II when $z^2=-1+2\sqrt{6}i$

$z=x+iy=\sqrt{-1+2\sqrt{6}i}$

$\Rightarrow (x+iy)^2=-1+2\sqrt{6}i$ --- (7)

$\Rightarrow (x^2-y^2)+2xyi=-1+2\sqrt{6}i$

$\Rightarrow \begin{cases} x^2-y^2=-1 & \text{--- (7)} \\ 2xy=2\sqrt{6} \Rightarrow xy=\sqrt{6} & \text{--- (8)} \end{cases}$

Modulus of (7) on both sides

$|(x+iy)^2|^2=|-1+2\sqrt{6}i|$

$\Rightarrow x^2+y^2=5$ --- (9)

frm (7) & (8) $2x^2=4 \Rightarrow x=\pm\sqrt{2}$

frm (8) $y=\pm\sqrt{3}$ (10)

$\therefore$ Roots of $(-1+2\sqrt{6}i)$ are $\pm(\sqrt{2}+\sqrt{3}i)$ ✓

Case III when $z^2=(-1-2\sqrt{6}i)$

Same way as in Case II

$z=\pm(\sqrt{2}-\sqrt{3}i)$ ✓ --- (11)

hence the required roots of the equation $p(z^2)=0$ are:

$\pm\sqrt{3}i$; $\pm(\sqrt{2}+\sqrt{3}i)$; $\pm(\sqrt{2}-\sqrt{3}i)$

19 (a) On an Argand diagram, sketch the locus of points representing complex numbers z satisfying $|z+3-2i|=2$ --[2]

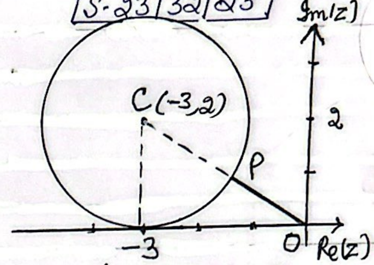
(b) Find the least value of $|z|$ for the points on this locus, giving your answer in an exact form. --[2]

S-23/32/Q3

Solution (a) $|z+3-2i| = 2$

$\Rightarrow |z-(-3+2i)| = 2$

Represents a circle with centre at $(-3+2i)$ and radius $= 2$



(b) Least value of $z =$ OP $=$ -

$= OC - CP = \sqrt{3^2+2^2} - 2 = (\sqrt{13} - 2)$ ✓

---

20. The complex number $2+yi$ is denoted by a, where y is a real number and $y<0$. It is given that $f(a) = a^3 - a^2 - 2a$

(a) Find a simplified expression for $f(a)$ in terms of y. ---[3]

(b) Given that $\text{Re}(f(a)) = -20$, find $\arg a$. ---[3]

S-23/32/Q5

Solution (a). $f(a) = a^3 - a^2 - 2a$ ---①

$a = (2+yi)$

$\Rightarrow a^2 = (2+yi)^2 = (4-y^2) + 4yi$ --②

and $a^3 = (2+yi)^3 = 8 + 12yi - 6y^2 - y^3 i$ --③

from ①, ② & ③

$f(a) = 8 + 12yi - 6y^2 - y^3 i - (4 - y^2 + 4yi) - 4 - 2yi$

$f(a) = -5y^2 + (6y - y^3)i$ —④ ✓

(b) Given $\text{Re}(f(a)) = -20 \Rightarrow -5y^2 = -20$ from ④

$\Rightarrow y^2 = 4 \Rightarrow y = -2$ (as $y<0$)

$\therefore a = 2-2i$

$\Rightarrow \arg a = \tan^{-1}\left(\frac{-2}{2}\right) = \tan^{-1}(-1)$

$= -\tan^{-1} 1$

$= -\frac{\pi}{4}$ ✓

($-\pi < \theta \leq \pi$ arg.)

21. On a sketch of an Argand diagram, shade the region whose points represent complex numbers z, satisfying: $|z-3-i| \leq 3$ and $|z| \geq |z-4i|$

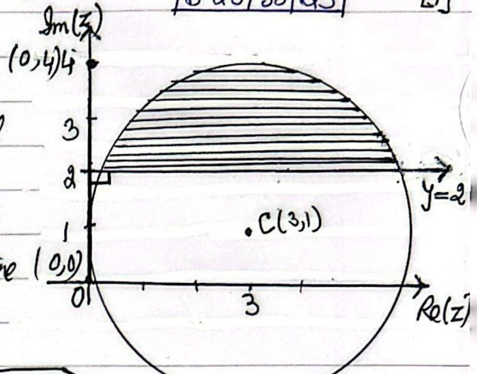
[S-23/33/Q3] ---[4]

Solution: $|z-(3+i)| \leq 3$ --- ①

Represent the point on a circle with centre $C(3+i)$ and radius 3, and its interior.

and $|z| \geq |z-4i| \Rightarrow |z-0| \geq |z-4i|$

represent the perp. bisector of the line joining $(0,0)$ and $(0,4) \Rightarrow$ line $y=2$



22. The complex number z is defined by $z = \dfrac{5a-2i}{3+ai}$, where a is an integer. It is given that $\arg z = -\frac{1}{4}\pi$. ---[6]

(a) Find the value of a and hence express z in the form $(x+iy)$, $x, y \in R$

(b) Express z^3 in the form $re^{i\theta}$, where $r>0$ and $-\pi < \theta \leq \pi$. Give simplify in exact form values of r and θ. ---[3]

[S-23/33/Q11]

Solution (a) $z = \dfrac{5a-2i}{3+ai}$ and $\arg z = -\dfrac{\pi}{4}$

$z = \dfrac{5a-2i}{(3+ai)} \times \dfrac{(3-ai)}{(3-ai)} = \dfrac{13a - i(5a^2+6)}{(9+a^2)}$ — ①

$\Rightarrow \arg z = -\dfrac{(5a^2+6)}{13a} = \tan(-\frac{\pi}{4})$

$\Rightarrow -\dfrac{(5a^2+6)}{13a} = -1 \Rightarrow 5a^2 - 13a + 6 = 0$

$\Rightarrow (a-2)(5a-3)=0$

$\Rightarrow a = 2$ ✓ or $a = \frac{3}{5}$ ✗ (Given a is an integer)

from ① for $a=2$

$z = \dfrac{26-26i}{13}$

$z = (2-2i)$ ✓

(b) $z = (2-2i)$

$z = r(\cos\theta + i\sin\theta)$

$z^3 = r_1^3(\cos\theta + i\sin\theta)^3$

$= (2\sqrt{2})^3\left[\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right]^3$

$\begin{cases} r_1\cos\theta = 2 \\ r_1\sin\theta = -2 \\ r_1 = \sqrt{2^2+2^2} = 2\sqrt{2} \\ \tan\theta_1 = -\frac{2}{2} = -1 \end{cases}$

$\theta_1 = \tan^{-1}(-1) = -\frac{\pi}{4}$

$z^3 = 16\sqrt{2}\left[\cos(-\frac{3\pi}{4}) + i\sin(-\frac{3\pi}{4})\right]$

for $z^3 \rightarrow r = 16\sqrt{2}$, $\theta = -\dfrac{3\pi}{4}$

$z^3 = r^3 e^{i\theta}$

$= 16\sqrt{2} \cdot e^{-\frac{3}{4}\pi i}$ ✓

[as: $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$]

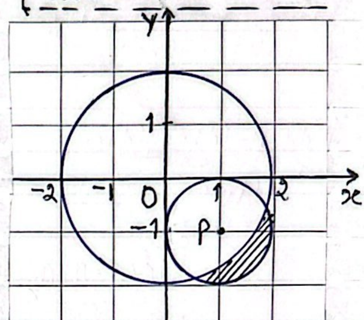
Scanned with CamScanner

23. On a sketch of argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z| \geq 2$ and $|z-1+i| \leq 1$. [W-20/31/Q2] ---[4]

Solution: $|z| \geq 2$ represents a circle and and outside with centre at origin (0,0) and radius = 2. ——①

$|z-z_0| = r$ Represent a circle with Centre at z_0 and rad = r

$|z-(1-i)| < 1$ represents a circle and its interior with centre at P(1-i) and rad = 1.



24.(a) Verify that $-1+\sqrt{5}i$ is root of the equation $2x^3+x^2+6x-18=0$ --[3]
(b) Find the other roots of this equation. [W-20/31/Q7] ---[4]

Solution (a) Let $x = -1+\sqrt{5}i \Rightarrow (x+1) = \sqrt{5}i \Rightarrow (x+1)^2 = (\sqrt{5}i)^2$

$\Rightarrow x^2+2x+1 = -5$

$\Rightarrow x^2 = (-2x-6)$ ——①

Now consider $2x^3+x^2+6x-18$

$= 2x \cdot x^2 + x^2 + 6x - 18$

$= 2x(-2x-6) + x^2+6x-18$

$= -4x^2 - 12x + x^2 - 18 + 6x$

$= -3x^2 - 6x - 18$

$= -3(-2x-6) - 18 - 6x$

$= 6x + 18 - 18 - 6x$

$= 0 = R.H.S$ ✓

(b) $(-1+\sqrt{5}i)$ is root of

$2x^3+x^2+6x-18=0$ ——①

$\therefore (-1-\sqrt{5}i)$ is also a root of ① (Second root) ✓

$\therefore (x+1-\sqrt{5}i) \& (x+1+\sqrt{5}i)$

$\Rightarrow (x+1)^2 + (\sqrt{5})^2 = (x^2+2x+6)$ is a factor of ①

$$
\begin{array}{r|l}
x^2+2x+6 & 2x^3+x^2+6x-18 \quad (2x-3 \\
 & -2x^3 \mp 4x^2 \mp 12x \\
\hline
 & -3x^2-6x-18 \\
 & -3x^2-6x-18 \\
\hline
\end{array}
$$

$\therefore (2x-3)$ is the third factor

$2x-3=0 \Rightarrow \frac{3}{2}$ is the third root. ✓

25. The complex number u is defined by, $u = \frac{7+i}{1-i}$

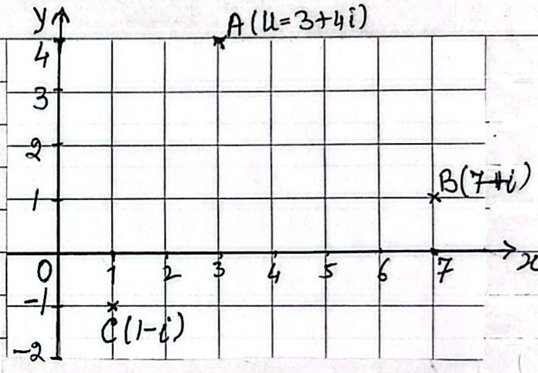
(a) Express u in the form $x+iy$, where x and y are real. ---[3]
(b) Show on a sketch of an Argand diagram the points A, B and C representing u, $7+i$ and $1-i$. --[2]
(c) By considering the arguments of $7+i$ and $1-i$, show that
$$\tan^{-1}\frac{4}{3} = \tan^{-1}\frac{1}{7} + \frac{1}{4}\pi$$ ---[3]

W-20/32/Q6

Solution (a) $u = \frac{7+i}{1-i} = \frac{(7+i)}{(1-i)} \times \frac{(1+i)}{(1+i)}$ $[(a+ib)(a-ib) = a^2+b^2]$

or $u = \frac{(7-1)+i(7+1)}{1^2+1^2} = \frac{6+8i}{2} = \underline{(3+4i)}$ ✓

(b) $u = (3+4i)$ A
$(7+i)$ B
$(1-i)$ C



(c) $\arg(7+i) = \tan^{-1}\frac{1}{7}$; $\arg(1-i) = \tan^{-1}\frac{-1}{1} = \tan^{-1}(-1)$
$= -\frac{\pi}{4}$

Now $u = \frac{7+i}{1-i}$

or $4+3i = \frac{(7+i)}{(1-i)}$ from Part (a)

$\Rightarrow \arg(4+3i) = \arg(7+i) - \arg(1-i)$

$\Rightarrow \tan^{-1}\frac{4}{3} = \tan^{-1}\frac{1}{7} - \left(-\frac{\pi}{4}\right)$

or $\tan^{-1}\frac{4}{3} = \tan^{-1}\frac{1}{7} + \frac{\pi}{4}$ ✓

26. The complex number $1+2i$ is denoted by u. The polynomial, $2x^3+ax^2+4x+b$, where a and b are real constants, is denoted by $p(x)$. It is given that u is a root of the equation $p(x)=0$

(a) Find the values of a and b. --[4]

(b) State a second complex root of this equation. --[1]

(c) Find the real factors of $p(x)$. --[2]

(d)(i) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities:

$$|z-u| \le \sqrt{5} \quad \text{and} \quad \arg z \le \tfrac{1}{4}\pi$$ --[4]

(ii) Find the least value of $\operatorname{Im} z$ for points in the shaded region. Give your answer in exact form. [W-21/31/Q10] --[1]

Solution: $p(x) = 2x^3+ax^2+4x+b$ --- ①

(a) $u=(1+2i) \Rightarrow u^2=(1+2i)^2=(1^2-2^2)+4i$
$=(-3+4i)$

$u^3 = u\cdot u^2 = (1+2i)(-3+4i) = (-3-8)+(4-6)i$
$=(-11-2i)$.

Put u, u^2, u^3 in $p(x)=0$ from ①

$2(-11-2i)+a(-3+4i)+4(1+2i)+b=0$

$(-22-3a+4+b)+i(-4+4a+8)=0$

Equating real and imaginary parts

$-3a+b=18$ --②

$4a=-4 \Rightarrow a=-1$ ✓

Put $a=-1$ in ② $\Rightarrow b=15$ ✓

(b) If $(1+2i)$ is a root of $p(x)=0$

Second complex conjugate root $= \underline{(1-2i)}$ ✓

(c) Sum of the complex roots $=(1+2i)+(1-2i)=2$

Product $=(1+2i)(1-2i)=1^2+2^2=5$

∴ The polynomial which has these factors

$x^2-(\text{Sum})x+\text{Product}=0$

$\Rightarrow (x^2-5x+5)$

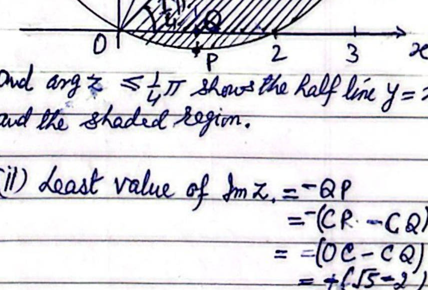
To find the linear factor

$$\begin{array}{r|l} x^2-2x+5 & 2x^3-x^2+4x+15 \;(2x+3 \\ & -2x^3 \mp 4x^2 \pm 10x \\ \hline & +3x^2-6x+15 \\ & 3x^2-6x+15 \\ \hline & \times \end{array}$$

The real factor $=(2x+3)$ ✓

d(i) $|z-u| \le \sqrt{5}$ and $\arg z \le \frac{1}{4}\pi$

rep. a circle centre $(1+2i)$ and rad $=\sqrt{5}$



and $\arg z \le \frac{1}{4}\pi$ shows the half line $y=x$ and the shaded region.

(ii) Least value of $\operatorname{Im} z = -QP$
$= -(CP - CQ)$
$= -(OC - CQ)$
$= -(\sqrt{5}-2)$
$= \underline{(2-\sqrt{5})}$ ✓

27.(a) Given the complex numbers $u = a+ib$ and $w = c+id$, where $a, b, c,$ and d are real, prove that $(u+w)^* = u^* + w^*$ ---[2]

(b) Solve the equation $(z+2+i)^* + (2+i)z = 0$ giving your answer in the form $x+iy$ where x and y are real. [W-21/32/Q3] ---[4]

Solution: $(u+w)^* = [(a+ib)+(c+id)]^*$

$= [(a+c)+i(b+id)]^*$

$= (a+c) - i(b+id)$

$= (a-ib) + i(c-id)$

$= (a+ib)^* + (c+id)^* = u^* + w^* = R.H.S$

(b) $(z+2+i)^* + (2+i)z = 0$

$\Rightarrow (x+iy+2+i)^* + (2+i)(x+iy) = 0$

$[(x+2)+(y+1)i]^* + (2x-y) + i(2y+x) = 0$

$(x+2) - (y+1)i + (2x-y) + i(2y+x) = 0$

$\Rightarrow (3x-y+2) + i(x+y-1) = 0$

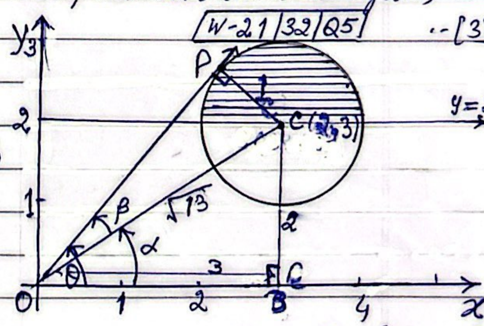
$\Rightarrow 3x-y+2=0$ --- ① and $x+y-1=0$

$\Rightarrow \begin{cases} 3x-y=-2 & \text{①} \\ x+y=1 & \text{②} \end{cases}$ Solve ① & ② $x = -\frac{1}{4}$, $y = \frac{5}{4}$

$\therefore z = \left(-\frac{1}{4} + \frac{5}{4}i\right)$

28.(a) On a sketch of an Argand diagram, shade the region whose point represent complex number z, satisfying the inequalities $|z-3-2i| \le 1$ and $\operatorname{Im} z \ge 2$ ---[4]

(b) Find the greatest value of $\arg z$ for the points in the shaded region, giving your answer in degrees. [W-21/32/Q5] ---[3]



Solution: (a) $|z-(3+2i)| \le 1$

Represents a circle (and its interior) with centre at $(3+2i)$ and radius 1.

$\operatorname{Im} z \ge 2$ reps a line $y=2$ and a half-plane above it.

(b) Let $\vec{OP}$ is tangent to the circle.

greatest value of $\arg z = \theta = \alpha + \beta = \tan^{-1} 2/3 + \sin^{-1} \frac{1}{\sqrt{13}}$

$= 33.69^\circ + 16.10$

$= 49.9^\circ$

ΔOBC, $\tan\alpha = \frac{2}{3}$

ΔOPC, $\sin\beta = \frac{1}{\sqrt{13}}$

Scanned with CamScanner

29. The complex number $-\sqrt{3}+i$ is denoted by u.

(a) Express u in the form $re^{i\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$, giving the exact values of r and θ. ---[2]

(b) Hence show that u^6 is real and state its value. ---[2]

(C)(i) On a sketch of an argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities:

$0\leq \arg(z-u)\leq \frac{1}{4}\pi$ and $\operatorname{Re} z\leq 2$ ---[4]

(ii) Find the greatest value of $|z|$ for the points in the shaded region. Give your answer correct to 3 significant figures. W-21/33/11 ---[2]

Solution (a) $u=-\sqrt{3}+i=r(\cos\theta+i\sin\theta)$

$r\cos\theta=-\sqrt{3}$, $r\sin\theta=1$ } Squ. and add $r^2=(-\sqrt{3})^2+1^2=4$

$\sin\theta=\frac{1}{2}=\sin\pi/6$ ✓, $\cos\theta=-\frac{\sqrt{3}}{2}$ | $r=2$ ✓

$\theta=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$.

$\therefore u=2\left[\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right]=\underline{2e^{i\cdot\frac{5\pi}{6}}}$ ✓

(b) $u^6=\left(2e^{i\cdot 5\pi/6}\right)^6=2^6\cdot e^{5\pi\cdot i}$

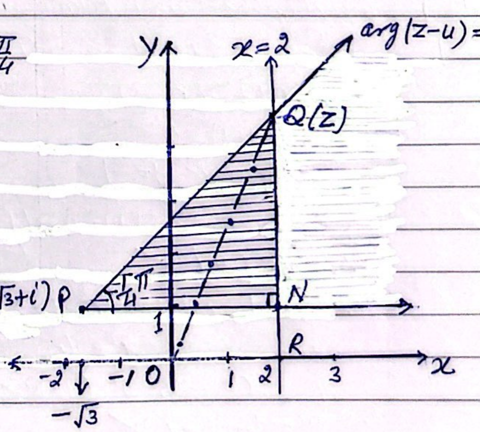
$=64[\cos 5\pi+i\sin 5\pi]$

$=64[-1+0]=\underline{-64}$ (Real)

c(i) $\leq\arg(z-u)\leq\frac{1}{4}\pi$, $u=-\sqrt{3}+i$

half line through $z=u$, at an angle $\frac{\pi}{4}$

and $\operatorname{Re} z\leq 2\Rightarrow x\leq 2$



(ii) Q is the at farthest distance of O.

Hence greatest value of $|z|=OQ$

$OQ=\sqrt{OR^2+QR^2}$ —— ①

Now $OR=2$ ✓

ΔQPN. $\frac{QN}{PN}=\tan\frac{\pi}{4}\Rightarrow QN=PN=(2+\sqrt{3})$

$QR=1+(2+\sqrt{3})=(3+\sqrt{3})$

or

ΔORQ: $OQ^2=2^2+(3+\sqrt{3})^2=4+22.39=26.39$

$OQ=\sqrt{26.39}=5.137$

$\therefore |z|=5.14$ ✓

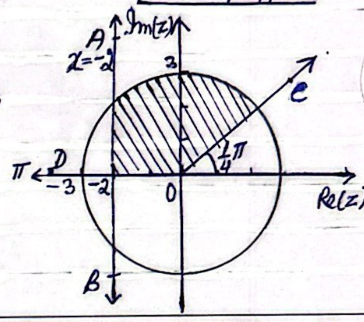
30. On a sketch of an Argand diagram shade the region whose points represents complex numbers z satisfying the inequalities $|z| \leq 3$, $\operatorname{Re} z \geq -2$ and $\frac{1}{4}\pi \leq \arg z \leq \pi$ ---[4]

[W-22/31/Q2]

Solution: $|z| \leq 3$ represent a circle, with centre at $(0,0)$ and radius 3, and its interior points.

$\operatorname{Re} z \geq -2 \Rightarrow x \geq -2$ is a vertical line AB through $(-2,0)$ and the points on the O-sides of it.

$\frac{1}{4}\pi \leq \arg z \leq \pi$ represent the points on $\arg z = \frac{\pi}{4}$ $(\vec{OD})$ half line $\vec{OC}$ and $\arg z = \pi$ represent a half line $\vec{OD}$ and the point in the interior of $\vec{OC}$ & $\vec{OD}$.



31. The complex numbers u and w are defined by $u = 2e^{\frac{1}{4}\pi i}$ and $w = 3e^{\frac{1}{3}\pi i}$

(a) Find $\frac{u^2}{w}$, giving your answer in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta < \pi$. Give the exact values of r and θ. ---[3]

(b) State the least positive integer n such that both $\operatorname{Im} w^n = 0$ and $\operatorname{Re} w^n = 0$ ---[1]

[W-22/31/Q5]

Solution: $u = 2e^{\frac{1}{4}i} \Rightarrow u^2 = (2e^{\frac{1}{4}\pi i})^2 = 4e^{\frac{1}{2}\pi i}$ ---(1)

(a) $w = 3e^{\frac{1}{3}\pi i} \Rightarrow \frac{u^2}{w} = \frac{4e^{\frac{1}{2}\pi i}}{3e^{\frac{1}{3}\pi i}} = \frac{4}{3}e^{(\frac{\pi}{2}-\frac{\pi}{3})i} = \underline{\frac{4}{3}e^{\frac{1}{6}\pi i}}$ ✓

(b) $w = 3e^{\frac{1}{3}\pi i} \Rightarrow w^n = (3e^{\frac{1}{3}\pi i})^n = 3^n \cdot e^{\frac{1}{3}\pi n i} = 3^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)$

Given $\operatorname{Im} w^n = 0$ and $\operatorname{Re} w^n > 0$

$\Rightarrow \sin\frac{n\pi}{3} = 0$; $\Rightarrow \cos\frac{n\pi}{3} > 0$

$\Rightarrow \frac{n\pi}{3} = 0, \pi, 2\pi, \ldots$; $0 \leq \frac{n\pi}{3} < \frac{\pi}{2}$ or $\frac{3\pi}{2} < \frac{n\pi}{3} \leq 2\pi$

$\Rightarrow n = \cancel{0}, 3, 6, \ldots$; $\Rightarrow 0 \leq n < \frac{3}{2}$ or $\frac{9}{2} < n \leq 6$

$n = 0, 1,$; $n = 6$

i.e. $\underline{n = 6}$ ✓

32. (a) Solve the equation $z^2 - 6iz - 12 = 0$, giving the answers in the form $x+iy$, where x and y are real and exact. --[3]

(b) On a sketch of an Argand diagram with origin O, show points A and B representing the roots of the equation in part (a) --[1]

(c) Find the exact modulus and argument of each root. --[3]

(d) Hence show that the triangle OAB is equilateral. --[1]

W-22/32/Q5

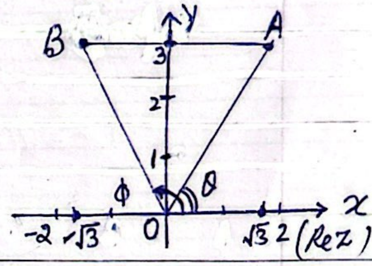
<u>Solution</u> (a) $z^2 - 6iz - 12 = 0$

$z = \dfrac{+6i \pm \sqrt{12}}{2}$ $\quad \left\{ \begin{aligned} b^2 - 4ac &= (-6i)^2 - 4\times1(-12) \\ &= -36 + 48 = 12 \end{aligned} \right.$

$= \dfrac{6i \pm 2\sqrt{3}}{2} = 3i \pm \sqrt{3}$

$\therefore$ Required roots are: $(\sqrt{3}+3i)$ & $(-\sqrt{3}+3i)$

(b) $A(\sqrt{3}+3i)$
$B(-\sqrt{3}+3i)$



(c) Now $|\sqrt{3}+3i| = \sqrt{(\sqrt{3})^2 + 3^2} = \sqrt{12} = 2\sqrt{3} = OA$

$|-\sqrt{3}+3i| = \sqrt{(-\sqrt{3})^2 + 3^2} = \sqrt{12} = 2\sqrt{3} = OB$

$\arg(\sqrt{3}+3i) = \theta \Rightarrow \tan\theta = \frac{3}{\sqrt{3}} = \sqrt{3} = \tan\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$

and $\arg(-\sqrt{3}+3i) = \phi \Rightarrow \tan\phi = \frac{3}{-\sqrt{3}} = -\sqrt{3} = -\tan\frac{\pi}{3} \Rightarrow \phi = \pi - \frac{\pi}{3} = 2\pi/3$

(d) In ΔOAB; $OA = OB = 2\sqrt{3}$ (Part C)

and angle $AOB = \frac{2\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3}$ (Part C)

$\therefore \angle OAB = \angle OBA = \frac{1}{2}\left(\pi - \frac{\pi}{3}\right) = \frac{\pi}{3}$ each.

$\therefore$ <u>ΔOAB is an equilateral triangle.</u> (each angle is $\frac{\pi}{3}$).

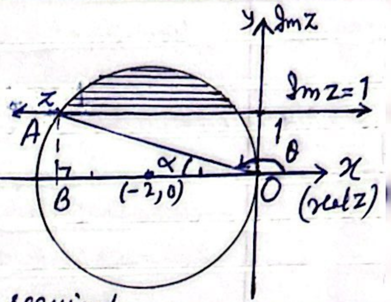
33 (a) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z, satisfying the inequalities: $|z+2| \leq 2$ and $\operatorname{Im} z \geq 1$ ---[4]

(b) Find the greatest value of $\arg z$ for the point in the shaded region. ---[2]

[W-22 | 33 | Q5]

Solution (a) $|z+2| \leq 2$ represents the points on a circle with its centre at $(-2,0)$ and radius 2, and its interior.

$\operatorname{Im} z \geq 1$, $(y \geq 1)$ represents a line $y=1$ (Parallel to x-axis) and the points above it, Shaded region is required.



b. The point A has the greatest $\arg z$.

A is the point of intersection of line $y=1$ & circle $(x+2)^2+y^2=2^2$

Solving $\begin{cases} (x+2)^2+1^2=4 \\ (x+2)^2=3 \Rightarrow x+2=\pm\sqrt{3} \\ x=-\sqrt{3}-2 \ \& \ \cancel{(\sqrt{3}-2)} \\ x=-(\sqrt{3}+2) \end{cases}$

at A,

Req. $\arg z = (\pi-\alpha)$ --- (1)

$\theta = \pi - 0.262$

$= 2.88$ rad. ✓

$\tan\alpha = \dfrac{AB}{OB} = \dfrac{1}{\sqrt{3}+2} = 0.2679$

$\alpha = \tan^{-1} 0.2679 = 0.262$ rad.

34. Solve the quadratic equation $(1-3i)z^2-(2+i)z+i=0$, giving your answer in the form $(x+iy)$, where x & y are real. ---[6]

[W-22 | 33 | Q6]

Solution: $(1-3i)z^2-(2+i)z+i=0$; $b^2-4ac = (2+i)^2 - 4(1-3i)\cdot i$

$= 4+\cancel{4i}+i^2-\cancel{4i}+12i^2 = 4-13=-9$

$z = \dfrac{(2+i)\pm\sqrt{-9}}{2(1-3i)}$

$= \dfrac{(2+i)\pm 3i}{2(1-3i)} = \dfrac{(2+4i)}{2(1-3i)}, \dfrac{(2-2i)}{2(1-3i)}$

$= \dfrac{(1+2i)}{1-3i}$ or $\dfrac{1-i}{1-3i}$

$= \dfrac{(1+2i)}{(1-3i)} \times \dfrac{(1+3i)}{(1+3i)}, \dfrac{(1-i)}{(1-3i)} \times \dfrac{(1+3i)}{(1+3i)}$

$= \dfrac{(1-6)+i(3+2)}{10}$ or $\dfrac{(1+3)+i(3-1)}{10}$

$= \dfrac{-5+5i}{10}$ or $\dfrac{4+2i}{10}$

$= \left(-\dfrac{1}{2}+\dfrac{1}{2}i\right)$ ✓ or $\left(\dfrac{2}{5}+\dfrac{1}{5}i\right)$ ✓