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P-3

Pure Maths - 3

Complex Numbers Notes

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Complex Numbers - Notes

§ **Imaginary numbers:**

Given a quadratic equation $x^2+1=0$ has no solution in the set of real numbers, as $x^2+1=0 \Rightarrow x^2=-1$ and there is no real number whose square is -1.

Here we introduce a number (Symbol) $i=\sqrt{-1}$ ✓

or $i^2=-1$

we may deduce $i^3=-i$

$i^4=1$.

Now to solve: $x^2+1=0$

$\Rightarrow x^2=-1 \Rightarrow x=\pm\sqrt{-1}$

or $x=\pm i$ ✓

- **Example 1:** write the following numbers in simplest form:

(a) $\sqrt{-16} = \sqrt{16\times -1} = \sqrt{16}\times\sqrt{-1} = \underline{4i}$ ✓

(b) $\sqrt{-7} = \sqrt{7\times -1} = \sqrt{7}\times\sqrt{-1} = \underline{\sqrt{7}i}$ ✓

(c) $\sqrt{-50} = \sqrt{50\times -1} = \sqrt{2\times 25}\times\sqrt{-1} = \sqrt{25}.\sqrt{2}.\sqrt{-1} = \underline{5\sqrt{2}i}$ ✓

(d) $i^{50} = (i^2)^{25} = (-1)^{25} = -1$ ✓

(e) $i^{52} = (i^2)^{26} = (-1)^{26} = +1$ ✓

(f) $i^{53} = i^{52}.i = \underline{i}$ ✓ (from part (e))

(g) $\sqrt{-9}+\sqrt{-16} = \sqrt{9\times(-1)}+\sqrt{16(-1)}$

$= 3i + 4i = 7i$ ✓

- **Example 2:** Solve:

(a) $4x^2+9=0 \Rightarrow x^2 = -\frac{9}{4}$

$= \frac{9}{4}\times -1$

$x = \pm\sqrt{\frac{9}{4}\times -1}$

$= \pm\frac{3}{2}i$ ✓

(Note: i, $\sqrt{7}i$, $\pm\frac{3}{2}i$ are all imaginary numbers)

§ Complex Numbers: We define a complex number,
$z = (x+iy)$: $x, y \in R$
Example: $(4+3i)$, $\frac{5}{3}$, $7i$ and 0 are all complex numbers.

§ Real and imaginary parts of a complex number:
Given a complex number $z = (a+ib)$
Then real part of $z = a$ [or $Re\, z = a$]
and the imaginary part of $z = b$ [or $Im\, z = b$]

Example: (i) $z_1 = 4+3i$ is a complex number.
$\begin{cases} \text{real part of } (4+3i) = 4 \\ \text{imaginary part of } (4+3i) = 3 \end{cases}$
(ii) $z_2 = \frac{5}{3} = \frac{5}{3} + i0$

Real part $= 5/3$
and ima $z = 0$ } Such complex numbers are called pure real. (when $Im\, z = 0$)
(iii) $z_3 = 7i = 0 + 7i$
real part $= 0$
Ima $z = 7$ } Such complex numbers are. are called pure imaginary (when $Re\, z = 0$)

§ Equal Complex numbers:
Given two complex numbers $u = (a+ib)$ and $v = (c+id)$
Then $(a+ib) = (c+id) \Leftrightarrow \begin{cases} a = c \\ b = d \end{cases}$ real and imaginary parts are separately equal.

Note: Inequality is not defined in set of complex numbers.
$u > v$ or $u < v$ is not defined.

• Example 3: Given $(2a-3b) + i(b+1) = 4+3i$
Find the value of a and the value of b.
Solution: $(2a-3b) + i(b+1) = (4+3i) \Rightarrow \begin{cases} 2a - 3b = 4 \quad \text{---(1)} \\ b+1 = 3 \Rightarrow b = 2, \; a = 5 \end{cases}$ from (1)

§ Conjugate of a complex Number:

Given $z = (x+iy)$: $x, y \in R$

Then conjugate of z (denoted by $z^{\circledast}$) $= x - iy$

Example (i) $z = 4+3i \Rightarrow z^{\circledast} = 4-3i$

(ii) $z = 2-5i \Rightarrow z^{\circledast} = 2+5i$

(iii) $z = 5$ $(5+i0) \Rightarrow z^{\circledast} = 5 = z$ (If z is pure real)

(iv) $z = -3i$ $(0-3i) \Rightarrow z^{\circledast} = +3i = -z$ (If z is pure Im.)

(v) $z = 3i+2$ (or $2+3i$) $\Rightarrow z^{\circledast} = 2-3i$ ✓ (Not $3i-2$)

§ Properties of conjugate complex Numbers:

Given $z = (a+ib)$ and $w = (c+id)$

$\Rightarrow z^{\circledast} = (a-ib)$ and $w^{\circledast} = (c-id)$

(i) $(z^{\circledast})^{\circledast} = z$

(ii) $z + z^{\circledast} = 2\,Re(z)$

(iii) $z - z^{\circledast} = 2i\,Im(z)$

(iv) $z = z^{\circledast} \Leftrightarrow z$ is pure real.

(v) $z + z^{\circledast} = 0 \Leftrightarrow z$ is pure Im.

(vi) $z \cdot z^{\circledast} = (Re z)^2 + (Im z)^2 = a^2 + b^2$. {or $z \cdot z^{\circledast} = |z|^2$ where $|z| = \sqrt{a^2+b^2}$

(vii) $(z+w)^{\circledast} = z^{\circledast} + w^{\circledast}$

(viii) $(z-w)^{\circledast} = z^{\circledast} - w^{\circledast}$

(ix) $(z \cdot w)^{\circledast} = z^{\circledast} \cdot w^{\circledast}$

(x) $\left(\dfrac{z}{w}\right)^{\circledast} = \dfrac{z^{\circledast}}{w^{\circledast}}$; $w \neq 0$

(xi) A quadratic equation with real coefficients:

$ax^2 + bx + c = 0$, $a, b, c \in R$ and $b^2 - 4ac < 0$

has complex conjugate roots.

- Example: Solve: $z^2 + 4z + 13 = 0$

$z = \dfrac{-4 \pm \sqrt{16-52}}{2}$

$\Rightarrow z = \dfrac{-4 \pm \sqrt{-36}}{2}$

$= \dfrac{-4 \pm 6i}{2} = -2 \pm 3i$ ✓

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§ Algebra of Complex Numbers: $u = (a+ib)$, $v = (c+id)$; $a, b, c, d \in \mathbb{R}$

(a) Addition of complex numbers:

Given $u = (a+ib)$ and $v = (c+id) \Rightarrow u+v = (a+c) + i(b+d)$
$a, b, c, d \in \mathbb{R}$.

- Example: $u = (2+3i)$, $v = (4-7i)$
$\Rightarrow u+v = (2+4) + i(3-7) = (6-4i)$ ✓

(b) Subtraction of complex numbers:
$u - v = (a+ib) - (c+id) = (a-c) + i(b-d)$ ✓

- Example:
$(2+3i) - (4-7i) = (2-4) + i(3-(-7))$
$= (-2+10i)$ ✓

(c) Multiplication of two complex numbers:

$(a+ib)(c+id) = (ac-bd) + i(ad+bc)$

$(a+ib)\cdot(c+id)$
$= ac + iad + ibc + i^2bd$
$= (ac-bd) + i(ad+bc)$
(as $i^2 = -1$)

- Example: $(2+3i)(4-7i)$
$= (2\times4 - 3(-7)) + i(2(-7) + 3\times4)$
$= (8+21) + i(-14+12)$
$= (29-2i)$ ✓

- Note:

(i) $(a+ib)^2 = (a^2-b^2) + 2abi$ $\quad \left\{ \begin{array}{l} (2+3i)^2 = (2^2-3^2) + 2\times2\times3i \\ = (-5+12i) ✓ \end{array} \right.$

(ii) $(a-ib)^2 = (a^2-b^2) - 2abi$ $\quad \left\{ \begin{array}{l} (2-3i)^2 = (2^2-3^2) - 2\times2\times3i \\ = (-5-12i) ✓ \end{array} \right.$

(iii) $(a+ib)(a-ib) = a^2+b^2$

or $z \cdot \bar{z} = |z|^2$

where $|z| = \sqrt{a^2+b^2}$

$\left\{ \begin{array}{l} (2+3i)(2-3i) \\ = (2\times2 - 3(-3)) + (2\times(-3) + 3(2))i \\ = (2^2+3^2) + 0i \\ = 13 ✓ \end{array} \right.$

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(d) Division of two Complex numbers:

$$\frac{u}{v} = \frac{(a+ib)}{(c+id)} \times \frac{(c-id)}{(c-id)}$$ (multiplying N^r and D^r by the conjugate of the D^r)

$$= \frac{(ac+bd)+i(bc-ad)}{(c^2+d^2)}$$

- Example: $\frac{(2+3i)}{(4-7i)} = \frac{(2+3i)}{(4-7i)} \times \frac{(4+7i)}{(4+7i)}$

$$= \frac{(8-21)+i(14+12)}{(4^2+7^2)}$$

$$= \frac{-13+26i}{65} = \left(-\frac{13}{65}+i\frac{26}{65}\right)$$

$$= \left(-\frac{1}{5}+\frac{2}{5}i\right)$$

Example 5: Find the complex number z, satisfying the equation: $z^{*}+1 = 2iz$, where z^{*} denotes the complex conjugate of z. Give your answer in form $x+iy$, where x and y are real numbers. [M-16/32/Q10(a)] -- [5]

Solution: $z^{*}+1 = 2iz$ $\left\{ \begin{array}{l} z = x+iy \\ \Rightarrow z^{*} = x-iy \end{array} \right.$

$\Rightarrow (x-iy)+1 = 2i(x+iy)$

$\Rightarrow (x+1)-iy = 2xi+2i^2y$

$\Rightarrow ((x+1)-iy) = (-2y+2xi)$

Equating real and imaginary parts on both sides:

$x+1 = -2y \Rightarrow x+2y = -1$ —— ①

$-y = 2x \Rightarrow y = -2x$ —— ②

from ① & ② $x+2(-2x) = 1 \Rightarrow -3x = -1$

$\Rightarrow x = +\frac{1}{3}$

from ② $y = -2(+\frac{1}{3}) = -2/3$

$\therefore$ Required $z = x+iy$

$= \left(\frac{1}{3}-\frac{2}{3}i\right)$

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Example 6: The complex number w is defined by $w = \dfrac{22+4i}{(2-i)^2}$. --- [3]
Show that $w = 2+4i$

[S-15/31/Q8(i)]

Solution: $w = \dfrac{22+4i}{(2-i)^2} = \dfrac{22+4i}{3-4i}$

$\left\{ \begin{array}{l} (2-i)^2 = (2^2-1^2) - 2\times2\times1\, i \\ = (3-4i) \end{array}\right.$

$[\because (a-ib)^2 = (a^2-b^2) - 2abi]$

$= \dfrac{(22+4i)}{(3-4i)} \times \dfrac{(3+4i)}{(3+4i)}$

$= \dfrac{(22\times3 - 4\times4) + i(22\times4 + 4\cdot3)}{3^2+4^2} = \dfrac{50+100i}{25}$

$= (2+4i)$ ✓

Example 7: The complex number $\dfrac{3-5i}{1+4i}$ is denoted by u. Express u in the form $(x+iy)$

[S-14/3. /Q7(a)] --- [3]

Solution: $u = \dfrac{(3-5i)}{(1+4i)} \times \dfrac{(1-4i)}{(1-4i)}$

$\left(z\cdot z^{*} = |z|^2 \right)$

$= \dfrac{(3\times1 - (-5)(-4)) + i(3(-4) + (-5)\times1)}{1^2+4^2}$ $\left((a+ib)(a-ib) = a^2+b^2 \right)$

$= \dfrac{-17-17i}{17} = (-1-i)$ ✓

Example 8: The complex numbers v and w satisfy the equations:
$v + iw = 5$ and $(1+2i)v - w = 3i$, solve the equations for v and w, giving your answers in the form $x+iy$; where x and y are real.

[M-20/32/Q10(a)] -- [6]

Solution: $v + iw = 5$ --- ①

$(1+2i)v - w = 3i$ --- ②

multiply ② by i

$(-2+i)v - iw = -3$ --- ③

add ② and ③ $(-1+i)v = 2$

$\Rightarrow v = \dfrac{2}{(-1+i)} \times \dfrac{(-1-i)}{(-1-i)}$

$v = \dfrac{-2-2i}{1^2+1^2} = (-1-i)$ ✓

Put $v = (-1-i)$ in ①

$-1-i+iw = 5$

$\Rightarrow iw = 6+i$

$w = \dfrac{(6+i)}{i} \times \dfrac{i}{i} = \dfrac{-1+6i}{-1}$

or $w = (1-6i)$ ✓

$\therefore v = (-1-i)$
and $w = (1-6i)$ ✓

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To find the square roots of a given complex number $(a+ib)$:

Let $(x+iy)$ is the square root of $(a+ib)$

$\Rightarrow (x+iy)^2 = a+ib$

$\Rightarrow (x^2-y^2)+2xyi = (a+ib)$ —— (1)

equating real and imaginary parts

$$\begin{cases} x^2-y^2 = a & \text{—— (2)} \\ 2xy = b & \text{—— (3)} \end{cases}$$

solve (2) & (3)

Example 9: Find the square root of the complex number $(-1+4\sqrt{3}i)$

[S-15/ 32(Q7(i)] ---[5]

Solution Let $(a+ib)$ is a square root of $(-1+4\sqrt{3}i)$

$\Rightarrow (a+ib)^2 = (-1+4\sqrt{3}i)$

$\Rightarrow (a^2-b^2)+2abi = -1+4\sqrt{3}i$ --- (1)

equating real and imaginary part

$$\begin{cases} a^2-b^2 = -1 & \text{—— (2)} \\ 2ab = 4\sqrt{3} & \text{---- (3)} \end{cases}$$

from (3) $b = \frac{2\sqrt{3}}{a}$ --- (4)

$\Rightarrow a^2 - \left(\frac{2\sqrt{3}}{a}\right)^2 = -1$

$\Rightarrow a^2 - \frac{12}{a^2} = -1$

$\Rightarrow a^4 + a^2 - 12 = 0$

$(a^2+4)(a^2-3) = 0$

$\Rightarrow a^2 = 3$ or $a^2 = -4$ ✗

$\Rightarrow a = +\sqrt{3}$; $-\sqrt{3}$

from (4) $\left.\begin{matrix} a=\sqrt{3} \\ b=2 \end{matrix}\right\}$ and $\begin{matrix} a=-\sqrt{3} \\ b=-2 \end{matrix}$

$\therefore$ Sq. roots are $\pm(\sqrt{3}+2i)$ ✓

§ To find a quadratic equation, given α, β are its roots.

$$\underbrace{x^2 - (\alpha+\beta)}_{\text{Sum}} x + \underbrace{\alpha\beta}_{\text{Product}} = 0$$

$$\left\{\begin{array}{l} (x-\alpha)(x-\beta) = 0 \\ \Rightarrow x^2 - \alpha x - \beta x + \alpha\beta = 0 \\ \Rightarrow x^2 - (\alpha+\beta)x + \alpha\beta = 0 \end{array}\right.$$

- **Example 10:** $z = 5 + i\sqrt{3}$ is a root of a quadratic equation. Find this quadratic equation.

Solution: One root let $\alpha = 5 + i\sqrt{3}$ (complex number)

Then the other root (will be conjugate of α), $\beta = 5 - i\sqrt{3}$

Sum of the roots $\alpha + \beta = (5+i\sqrt{3}) + (5 - i\sqrt{3}) = 10$ ---(1)

Product of the roots $\alpha\beta = (5+i\sqrt{3})(5-i\sqrt{3})$

$= 5^2 + (\sqrt{3})^2$

$= 28$ ---(2)

$\left[\begin{array}{l} \because (a+ib)(a-ib) \\ = a^2 + b^2 \\ \text{or } z \cdot z^* = |z|^2 \end{array}\right.$

$\therefore$ The required quadratic equation:

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

$$\Rightarrow x^2 - 10x + 28 = 0 \quad \text{from (1) \& (2)}$$

- **Example 11:** Solve the equation: $(1+2i)w^2 + 4w - (1-2i) = 0$ giving your answers in the form $(x+iy)$, where x and y are real. ---[5]

[W-16/31/Q9(a)]

Solution: $(1+2i)w^2 + 4w - (1-2i) = 0$

$$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\left[\begin{array}{l} a = (1+2i),\ b = 4,\ c = -(1-2i) \\ b^2 - 4ac = 4^2 - 4(1+2i) \times -(1-2i) \\ = 16 + 4[1^2 + 2^2] \\ = 36 \quad [\because (a+ib)(a-ib) = a^2 + b^2 \end{array}\right.$

$$w = \frac{-4 \pm \sqrt{36}}{2(1+2i)}$$

$$= \frac{-4 \pm 6}{2(1+2i)}$$

$$= \frac{1}{1+2i} \ ; \ \frac{-5}{1+2i}$$

$\left[\begin{array}{l} \because \frac{1}{1+2i} = \frac{1}{(1+2i)} \times \frac{(1-2i)}{(1-2i)} \\ = \frac{(1-2i)}{1^2 + 2^2} \\ = \frac{1}{5}(1-2i) \end{array}\right.$

$$= \frac{1}{5}(1-2i) \ ; \ -5 \times \frac{1}{5}(1-2i)$$

$$= \left(\frac{1}{5} - \frac{2}{5}i\right) \ ; \ (-1+2i)$$

Example 12: Solve the equation: $(1+i)z^2-(4+3i)z+5+i=0$
Give your answer in the form $(x+iy)$, where x and y are real.
[M-19/32/Q7]... [6]

Solution: $(1+i)z^2-(4+3i)z+(5+i)=0$

$a=(1+i),\ b=-(4+3i),\ c=(5+i)$
$b^2-4ac=(4+3i)^2-4\times(1+i)(5+i)$
$=(7+24i)-(16+24i)$
$=-9$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{+(4+3i)\pm\sqrt{-9}}{2(1+i)}$$

$$=\frac{(4+3i)\pm 3i}{2(1+i)}=\frac{(4+6i)}{2(1+i)};\ \frac{(4)}{2(1+i)}$$

$$=\frac{\cancel{2}(2+3i)}{\cancel{2}(1+i)}\times\frac{(1-i)}{(1-i)};\ \frac{2}{(1+i)}\times\frac{(1-i)}{(1-i)}$$

$$=\frac{5+i}{1^2+1^2};\ \frac{2(1-i)}{1^2+1^2}$$

$$=\left(\frac{5}{2}+\frac{1}{2}i\right);\ (1-i)$$

Example 13: It is given that $-1+\sqrt{5}i$ is a root of the equation;
$z^3+2z+a=0$, where a is real. Find the value of a and write down the other complex roots of this equation. --[4]
[S-14/32/Q7(a)]

Solution: $z^3+2z+a=0$ --- ①

Let $\alpha=-1+\sqrt{5}i$ --- ②

⊗ $\alpha^2=(-1+\sqrt{5}i)^2=(1-5)-2\sqrt{5}i=-4-2\sqrt{5}i$

$\alpha^3=\alpha^2\cdot\alpha=(-4-2\sqrt{5}i)(-1+\sqrt{5}i)$
$=(4+10)+i(-4\sqrt{5}+2\sqrt{5})$
$=(14-2\sqrt{5}i)$ --- ③

Now α is a root of equ ① from ② & ③

$(14-2\sqrt{5}i)+2(-1+\sqrt{5}i)+a=0$
$(14-2+a)+i(-2\sqrt{5}+2\sqrt{5})=0$
$12+a=0$
$\Rightarrow a=-12$

⊗ $[(a+ib)^2=(a^2-b^2)+2abi]$

Now put $a=-12$ in ①
$z^3+2z-12=0$ --- ④
as the equ^n has real coefficients if one complex root is $(-1+\sqrt{5}i)$ the second will be $\alpha^*=(-1-\sqrt{5}i)$
$\therefore (z-(-1+\sqrt{5}i))(z-(-1-\sqrt{5}i))$ is a factor
$=(z+1-\sqrt{5}i)(z+1+\sqrt{5}i)$
$=(z+1)^2+5=z^2+2z+6$ is a factor of $z^3+2z-12$
$\Rightarrow (z^3+2z-12)=(z^2+2z+6)(z-2)=0$
$\therefore$ The third root $z-2=0$
$z=2$
$\therefore$ Other roots are $(-1-\sqrt{5}i);\ 2$

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Example 14. The complex number $1+\sqrt{2}i$ is denoted by u. The polynomial x^4+x^2+2x+6 is denoted by $p(x)$.

(i) Verify that u is a root of the equation $p(x)=0$ and write down a second complex root of the equation. --- [4]

(ii) Find the other two roots of the equation $p(x)=0$ -- [6]

[W-12/31/Q9]

Solution: Given equation $x^4+x^2+2x+6=0$ --- (1)

(i) $u=(1+\sqrt{2}i)$ ---- (2)

$u^2=(1^2-\sqrt{2}^2)+2\sqrt{2}i=(-1+2\sqrt{2}i)$ -- (3) $\quad [(a+ib)^2=(a^2-b^2)+2abi]$

$u^4=(u^2)^2=(-1+2\sqrt{2}i)^2=((-1)^2-(2\sqrt{2})^2)-4\sqrt{2}i=(-7-4\sqrt{2}i)$ -- (4)

Put the values of u, u^2 and u^4 in (1)

$(-7-4\sqrt{2}i)+(-1+2\sqrt{2}i)+2(1+\sqrt{2}i)+6=0.$

$\Rightarrow (-7-1+2+6)+(-4\sqrt{2}+2\sqrt{2}+2\sqrt{2})=0$

$\Rightarrow \quad 0+i0=0$ True

$\therefore$ u is a root of the equation $p(x)=0$

Now as the poly. eqn $p(x)=0$ has real coefficients and one root is $u=(1+\sqrt{2}i) \Rightarrow$ the other root is $u^* = (1-\sqrt{2}i)$

(ii) as $(1+\sqrt{2}i)$ is a root of $p(x)=0 \Rightarrow (x-(1+\sqrt{2}i))$ is factor of $p(x)$

also $(1-\sqrt{2}i)$ is a root of $p(x)=0 \Rightarrow (x-(1-\sqrt{2}i))$ is factor of $p(x)$

$\therefore$ product $(x-1-\sqrt{2}i)(x-1+\sqrt{2}i)$ is a factor of $p(x)$

$\Rightarrow (x-1)^2+(\sqrt{2})^2 = (x^2-2x+3)$ is a factor of $p(x)$.

$p(x)=(x^2-2x+3)(x^2+2x+2)=0.$

$$
\begin{array}{r}
x^2+2x+2 \\
x^2-2x+3 \,\big)\, x^4+x^2+2x+6 \\
-x^4 \mp 2x^3 + 3x^2 \\
\hline
2x^3-2x^2+2x+6 \\
-2x^3 \mp 4x^2 \pm 6x \\
\hline
2x^2-4x+6 \\
2x^2-4x+6 \\
\hline
\times
\end{array}
$$

for other two roots, solve,

or $x^2+2x+2=0$

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\dfrac{-2\pm\sqrt{-4}}{2} \quad \{b^2-4ac=2^2-4\times1\times2$
$=$

$=\dfrac{-2\pm 2i}{2} = (-1\pm i)$

$\therefore$ The other two roots are $(-1+i)$, $(-1-i)$

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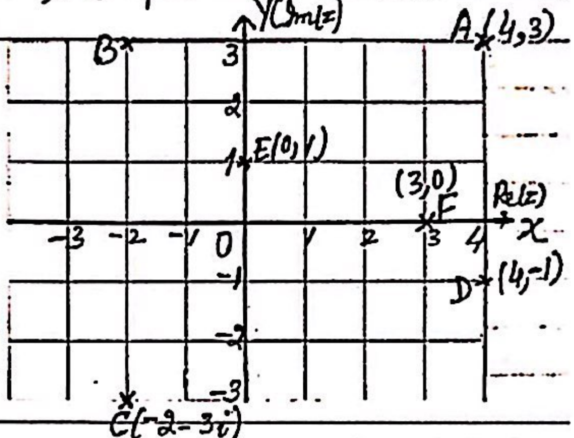
§ Geometric Representation of Complex Number (Argand Diagram)

To each complex number $z = (x+iy)$, there corresponds a unique ordered pair (x, y), which can be represent on a cartesian plane (Argand diagram), by a point $P(x,y)$

• Example: Represent the following complex numbers on the Argand plane:

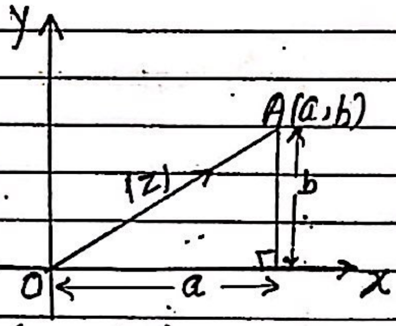
(i) $z = (4+3i) \rightarrow A(4,3)$
(ii) $w = (-2+3i) \rightarrow B(-2,3)$
(iii) $u = (-3-2i) \rightarrow C(-2,-3)$
(iv) $v = (4-i) \rightarrow D(4,-1)$
(v) $z_1 = i \rightarrow E(0,1)$
(vi) $z_2 = 3 \rightarrow F(3,0)$



$z = (4+3i)$ can also be represent by Position vector $\vec{OA} = \binom{4}{3}$

§ Modulus of a Complex Number

Given a complex number $z = a+ib$ which can be represented by a point $A(a,b)$ on the Argand plane.



Then modulus of z: $|z| = OA$

or $|z| = \sqrt{a^2+b^2}$ ✓ ; $|z| \geq 0$

• Example: $z = 4+3i$

$|z| = \sqrt{4^2+3^2} = 5$ ✓

or $|z| = |\vec{OA}|$ Magnitude of $\vec{OA}$; where $\vec{OA} = \binom{4}{3}$

§ Important Property: $z \cdot \bar{z} = |z|^2$

Proof: $(x+iy)(x-iy) = x^2+y^2 = \left(\sqrt{x^2+y^2}\right)^2 = |z|^2$ ✓

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§ Properties of Modulus of a Complex Number:

$z, z_1, z_2 \in C$

(i) $|z| = 0 \Leftrightarrow z = 0$ (or $\text{Re}\, z = 0$ and $\text{Im}\, z = 0$)

(ii) $|z| = |z^{\circledR}| = |-z|$

(iii) $-|z| \leq \text{Re}\, z \leq |z|$ and $-|z| \leq \text{Im}\, z \leq |z|$

(iv) $z \cdot z^{\circledR} = |z|^2$

(v) $|z_1 z_2| = |z_1||z_2|$

(vi) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$; $z_2 \neq 0$

(vii) $\dfrac{1}{z} = \dfrac{z^{\circledR}}{|z|^2}$

(viii) $|z^2| = |z|^2$

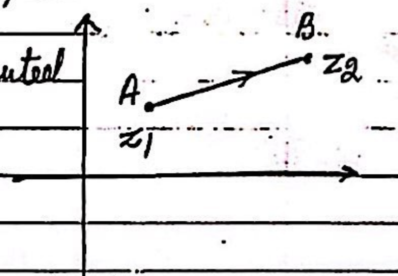
§ Important application of modulus of Complex numbers:

(i) Complex numbers z_1 and z_2 are represented by the points A and B respectively.

$\vec{AB} = z_2 - z_1$

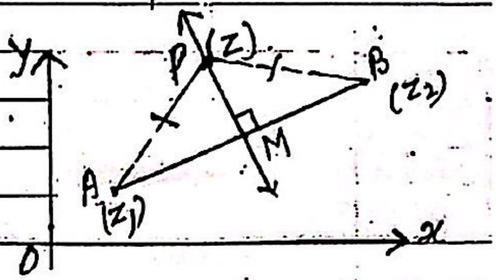
and distance $AB = |z_2 - z_1|$

and $OA = |z_1|$



(ii) Given $|z - z_1| = |z - z_2|$

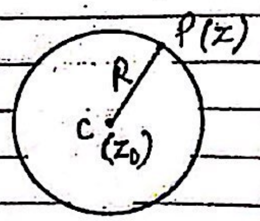
Represents the locus of $P(z)$ which is equidistant from A and B or P lies on the perpendicular bisector of AB.



(iii) Equation of Circle:

$|z - z_0| = R$

Represents locus of z, is a circle with centre at $C(z_0)$ and radius R.



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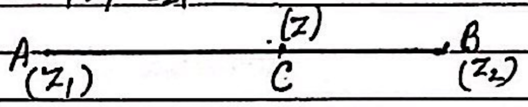
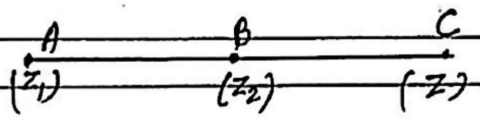
(iv) $|z-z_1|+|z-z_2| = |z_1-z_2|$



or $AC+CB=AB$

$\Rightarrow$ C lies on segment AB.

(v) $|z-z_1|-|z-z_2| = |z_1-z_2|$



$AC - BC = AB$

C lies on $\overrightarrow{BC}$.

Example 15: The complex number u is defined by $u = \dfrac{(1+2i)^2}{(2+i)}$

(i) Express u in the form $(x+iy)$ where x and y are real --- [4]

(ii) Sketch an argand diagram showing the locus of complex number z, such that $|z-u| = |u|$ [S-12/31/Q4] --- [3]

Solution: $u = \dfrac{(1+2i)^2}{2+i}$ $\quad [\because (a+ib)^2 = (a^2-b^2)+2abi]$

$= \dfrac{[(1^2-2^2)+2\times1\times2i]}{(2+i)} \times \dfrac{(2-i)}{(2-i)}$ $\quad [(a+ib)(a-ib) = a^2+b^2]$

$= \dfrac{(-3+4i)(2-i)}{2^2+1^2}$

$u = \dfrac{(-6+4)+i(3+8)}{5} = \left(-\dfrac{2}{5}+\dfrac{11}{5}i\right)$

(ii) $|z-u| = |u|$ $\quad \left[\because |u| = \sqrt{\left(\tfrac{2}{5}\right)^2+\left(\tfrac{11}{5}\right)^2} = \sqrt{5}\right]$

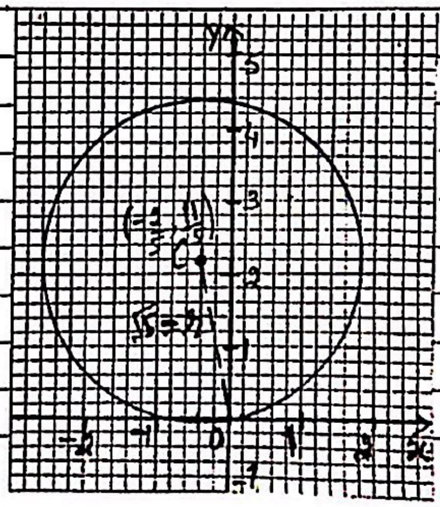
$\Rightarrow \left|z-\left(-\dfrac{2}{5}+\dfrac{11}{5}i\right)\right| = \sqrt{5}$

Locus of z:

Represents a circle with centre $C\left(-\dfrac{2}{5}, \dfrac{11}{5}\right)$ and radius $r=\sqrt{5}$

$OC = r = \sqrt{5}$

[$\because$ $|z-z_1| = r$ represents a circle with centre at z_1 and radius r.]



Example 16 (i) On a sketch of argand diagram, show the locus representing complex numbers satisfying the equation:

$|z| = |z-4-3i|$ --- [2]

(ii) Find the complex number represented by the point on the locus, where $|z|$ is least, Find the modulus of this complex number. [2]

[S-16 | 32 | Q10(b)]

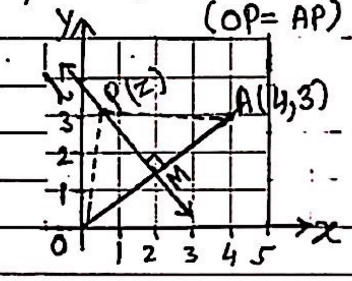
Solution: $|z| = |z-4-3i|$

(i) $\Rightarrow |z-0| = |z-(4+3i)|$

Locus the perpendicular bisector of segment joining the complex numbers represented by O (origin) and (4+3i). A.

{ $|z-z_1| = |z-z_2|$ represents the locus of z, which is the line, perpendicular bisector of the segment joining the points z_1 and z_2 }

(OP= AP)



Now let 'M' is the mid point of OA.

$M\left(2, \frac{3}{2}\right)$, $M\left(2+\frac{3}{2}i\right)$

$OM = \left|2+\frac{3}{2}i\right| = \sqrt{2^2+\left(\frac{3}{2}\right)^2} = \frac{5}{2}$

$\therefore$ $|z|$ is least at M and here $|z| = \frac{5}{2}$

Example 17: On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities:

$|z-2-i| \leq 1$ and $|z-i| \leq |z-2|$ --- [4]

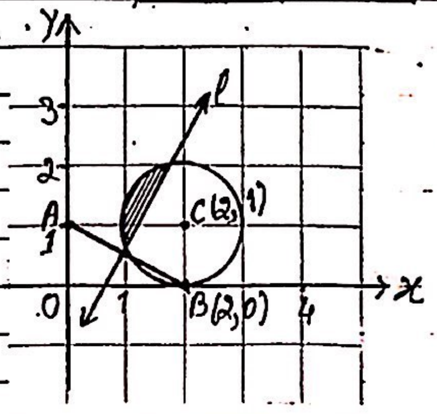
[S-14 | 33 | Q7(b)(i)]

Solution: $|z-(2+i)| \leq 1$

represents a circle with centre C(2,1) and radius r = 1 and its interior.

$|z-(0+i)| \leq |z-(2+0i)|$

represents the perpendicular bisector of the segment joining the point A(0,1) and B(2,0) and the half plane on the 0-side of it.



Required complex number are on the common shaded area.

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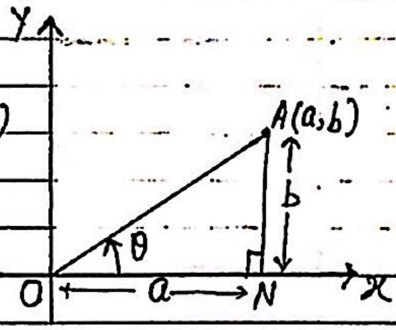
§ Argument of a Complex Number:

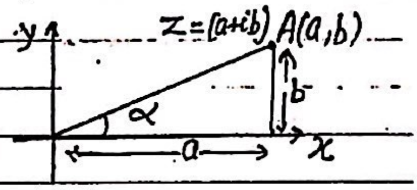


Given a complex number $z = (a+bi)$ represented by point A on the argand diagram. Then Argument of z:

$$\operatorname{Arg} z = \theta \;:\; \tan\theta = \frac{b}{a}$$

or $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ for $-\pi \leq \theta \leq \pi$

here θ is the principal argument of z.



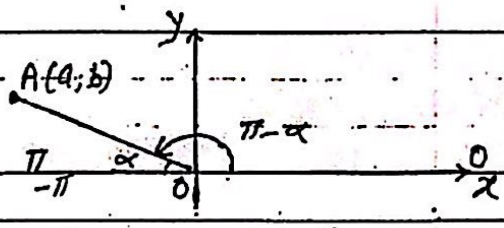
$\boxed{\text{Case I}}$: when $a>0$ and $b>0$

$$\theta = \tan^{-1}\frac{b}{a} = \alpha \checkmark$$

Example: $z = 1+\sqrt{3}\,i$

$\arg z = \theta = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3} \checkmark$

$\boxed{\text{Case II}}$ $a<0$ and $b>0$



basic angle $\alpha = \tan^{-1}\left|\frac{b}{a}\right|$

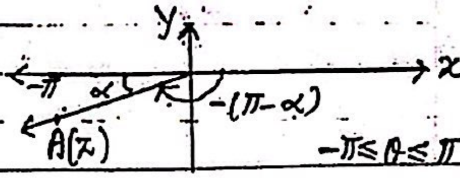
$\arg z = \theta = \pi - \alpha \checkmark$

Example: $z = -2\sqrt{3} + 2i$.

basic angle $\alpha = \tan^{-1}\left|\frac{2}{-2\sqrt{3}}\right| = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$\therefore\ \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \checkmark$

$\boxed{\text{Case III}}$: $a<0$ and $b<0$



basic angle $\alpha = \tan^{-1}\left|\frac{b}{a}\right|$

$\arg z = \theta = -(\pi-\alpha) \checkmark$

Example: $z = -3-3i$

basic angle $\alpha = \tan^{-1}\left|\frac{-3}{-3}\right| = \tan^{-1} 1 = \frac{\pi}{4}$

$\operatorname{Arg} z = -(\pi-\alpha) = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4} \checkmark$

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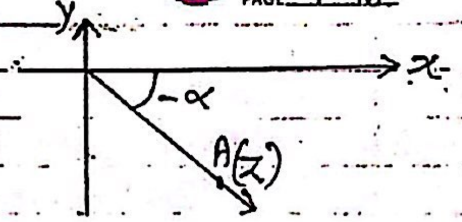
$\boxed{\text{Case (IV)}}$ $a > 0$ and $b < 0$

basic angle $\alpha = \tan^{-1}\left|\frac{b}{a}\right|$

$\text{Arg}\, z = \theta = -\alpha$ ✓



Example:

$z = \sqrt{3} - i$

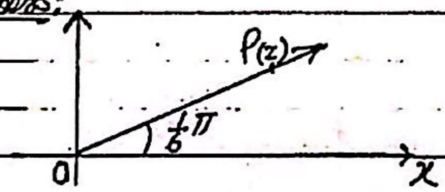
basic angle $\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\left|\frac{-1}{\sqrt{3}}\right| = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$\therefore \arg z = \theta = -\frac{\pi}{6}$ ✓

§ Application of argument of complex numbers:

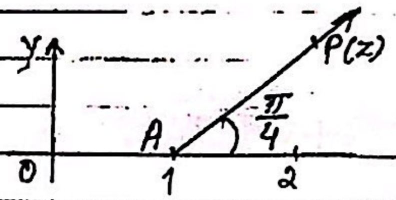
(i) $\arg z = \frac{\pi}{6}$

Locus of z is half line $\overrightarrow{OP}$



(ii) $\arg(z-1) = \frac{\pi}{4}$

locus of z is half line $\overrightarrow{AP}$, $A(1,0)$

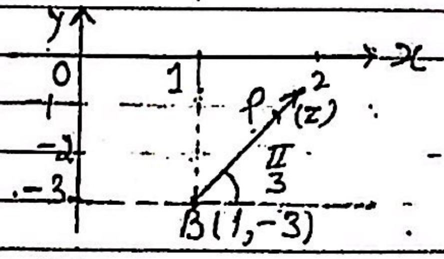


(iii) $\arg(z - 1 + 3i) = \frac{\pi}{3}$

or $\arg\left(z - (1-3i)\right) = \frac{\pi}{3}$

Locus of z is the half line $\overrightarrow{BP}$

$B(1,-3)$

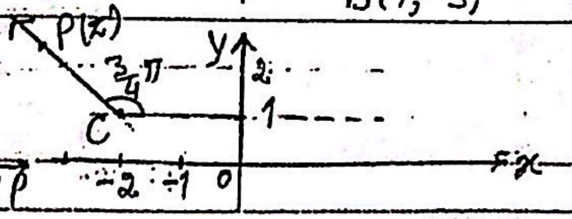


(iv) $\arg(z + 2 - i) = \frac{3\pi}{4}$

or $\arg\left(z - (-2+i)\right) = \frac{3}{4}\pi$

Locus of z is half line $\overrightarrow{CP}$

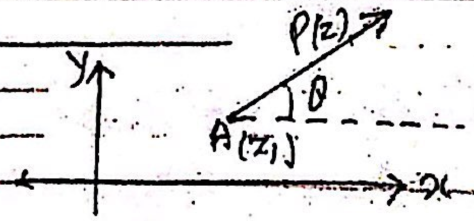
$C(-2,1)$



§ Locus of $\arg(z - z_1) = \theta$

is a half line $\overrightarrow{AP}$

$A(z_1)$



Example 18(i) On an Argand diagram, sketch the loci of points rep complex numbers w and z, such that,
$|W-1-2i| = 1$ and $\arg(z-1) = \frac{3}{4}\pi$ ---[4]

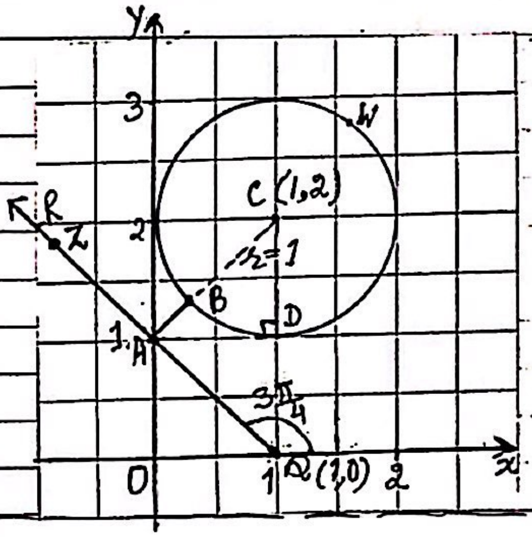
(ii) Calculate the least value of $|W-Z|$ for points on these loci. --[2]

[S-16/31/Q10(b)]

Solution (i) $|W-(1+2i)| = 1$ ----①
locus of W is a circle with centre $C(1,2)$ and radius $= 1$.
$\arg(z-1) = \frac{3}{4}\pi$ ----②
locus of z : is half line $\overrightarrow{QR}$, $Q(1,0)$.

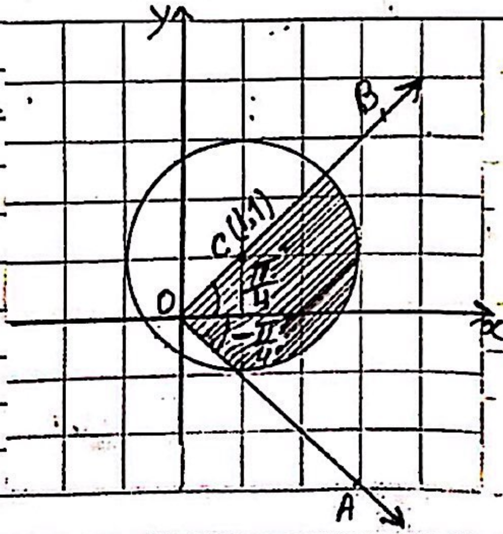
(ii) Min $|W-Z| = AB = AC - BC$
$= (\sqrt{2}-1)$ ✓

(In rt Δ ADC, $AC = \sqrt{1^2+1^2} = \sqrt{2}$
$BC = r = 1$)



Example 19: On the sketch of an argand diagram, shade the region whose points represent complex numbers satisfying the inequalities: --[5]
$|z-1-i| \le 2$ and $-\frac{1}{4}\pi \le \arg z \le \frac{1}{4}\pi$ [W-16/32/Q9(b)]

Solution: $|z-(1+i)| \le 2$ ----①
represents the circle with centre $C(1,1)$ and radius $r = 2$ and its "interior."
and $-\frac{1}{4}\pi \le \arg z \le \frac{1}{4}\pi$ --② represents the points on the points on the half lines $\overrightarrow{OA}$ and $\overrightarrow{OB}$, and the interior of the angle AOB.
Shaded region represents the points satisfying both ① and ②



Example 20: On a sketch of an Argand diagram, shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z-2i| \leq 2$ and $0 \leq \arg(z+2) \leq \frac{1}{4}\pi$. Calculate the greatest value of $|z|$ for the points in this region. [S-13/32/Q9] --[6]

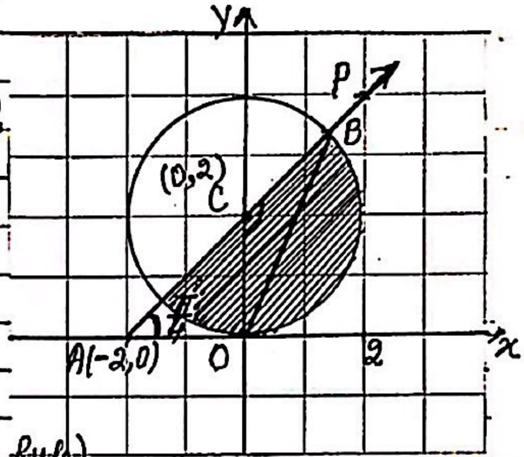
Solution: $|z-(0+2i)| \leq 2$ --- ①

represents the circle with centre $C(0,2)$, and radius $r=2$ and its interior,

and $0 \leq \arg(z-(-2+i0)) \leq \frac{1}{4}\pi$ is the half line $\overrightarrow{AP}$, $A(-2,0)$, $\arg = \frac{\pi}{4}$,

Let the circle and half line $\overrightarrow{AP}$ intersect at B.

∴ Greatest value of $|z|$ in the shaded region $= OB$ (using cosine Rule)



In $\triangle OCB$; $OB^2 = OC^2 + CB^2 - 2 \times CO \times CB \times \cos\frac{3\pi}{4} = 2^2 + 2^2 - 2\times2\times2\times(-\frac{1}{\sqrt{2}})$

$= 8 + 4\sqrt{2} = 13.656$

$\therefore OB = \sqrt{13.656} = 3.7$

Example 21 (i) On a sketch of Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities: --[4]

$|z+1-3i| \leq 1$ and $\text{Im}\, z \geq 3$, where $\text{Im}\, z$ denotes the imaginary part of z.

(ii) determine the difference between the greatest and least values of $\arg z$ for the points in this region. [M-16/32/Q10(b)] --[2]

Solution: $|z-(-1+3i)| \leq 1$ --- ①

represents the circle, Centre $(-1,3)C$, and radius $r=1$ and its interior.

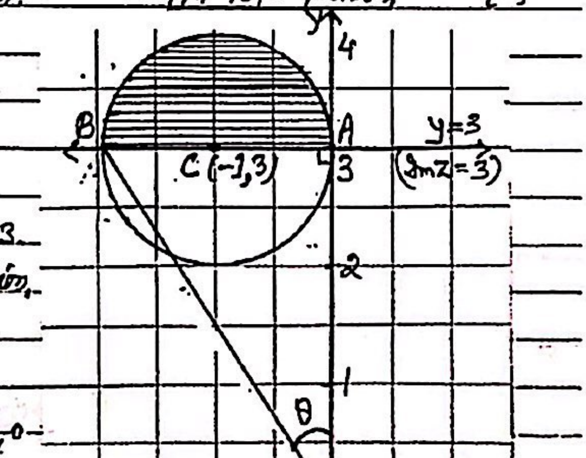
$\text{Im}\, z \geq 3$ denotes the region above the line $y=3$. ($y \geq 3$) and line $y=3$.

Common shaded area is required region.

Least and the greatest $\arg z$ are at points A and B of the region.

Their diff is θ.

$\tan\theta = \frac{AB}{OA} = \frac{2}{3} \Rightarrow \theta = \tan^{-1}\frac{2}{3} = 33.7°$



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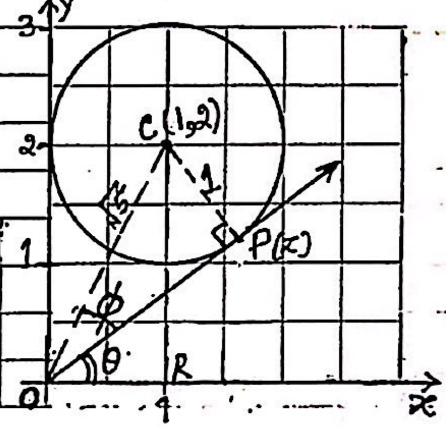
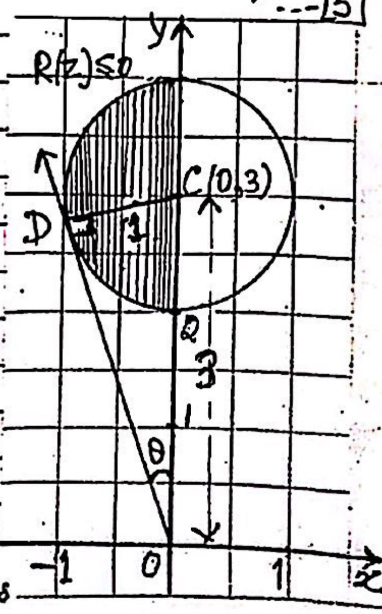
Example 22: The complex number $1+2i$ is denoted by u.
On the Argand diagram sketch the locus of points representing complex numbers z satisfying the equation $|z-u|=1$.
Determine the least value of $\arg z$ for the points on this locus. --[4]
Give your answer in radians correct to 2 decimal places [M-18/32/Q9(b)]



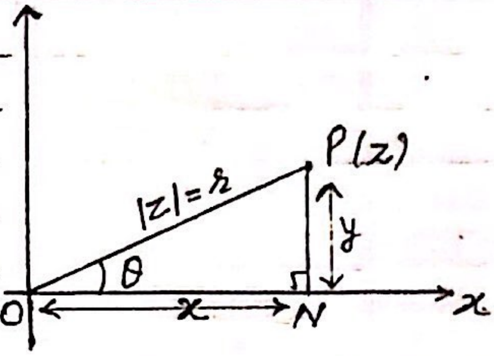
Solution: $|z-u|=1 \Rightarrow |z-(1+2i)|=1$ as $(u=1+2i)$
represents a circle with centre $C(1,2)$ and radius $r=1$.
Draw OP tangent to the circle.
Least value of $\arg z$ on circle $=\angle POX=\theta$
$\theta = \angle COR - \angle COP$ $[(\theta+\phi)-\phi]$
$= \cos^{-1}\frac{1}{\sqrt{5}} - \sin^{-1}\frac{1}{\sqrt{5}}$ $\left[OC=\sqrt{1^2+2^2} = \sqrt{5}\right.$
$= 1.107 - 0.463$
$\arg P(z) = 0.64$ radians.

Example 23: On a sketch of an Argand diagram, shade the region whose points represent complex numbers z, satisfying both the inequalities $|z-3i|\le 1$ and $\operatorname{Re} z \le 0$, where $\operatorname{Re} z$ denotes the real part of z. --[5]
Find the greatest value of $\arg z$, for the points in this region, giving your answer in radians. [W-18/32/Q9]



Solution: $|z-(0+3i)|\le 1$ ---(1)
denotes a circle with centre $C(0,3)$ and radius $r=1$ and its interior.
$\operatorname{Re}(z)\le 0$ ---(2) denote the half plane on the left of Y-axis.
Shaded region satisfy both (1) & (2)
Draw OD tangent to the circle. Then $D(z)$ has the greatest value of $\arg(z) = \frac{\pi}{2}+\theta$
$= \frac{\pi}{2} + \sin^{-1}\frac{1}{3}$ (In rt ΔODC)
$= 1.57 + 0.34 = 1.91$ radians

§ Polar form of a Complex Numbers:



Given a complex $z = x + iy$ ---(i)

$|z| = \sqrt{x^2+y^2} = r$ (let)

and $\arg z = \tan^{-1}\frac{y}{x} = \theta$ let

Now in right triangle PON

$x = r\cos\theta$ and $y = r\sin\theta$

$\therefore$ from (i) $z = r\cos\theta + ir\sin\theta$

or $\underline{z = r(\cos\theta + i\sin\theta)}$ ✓ is the polar form of a complex numbers

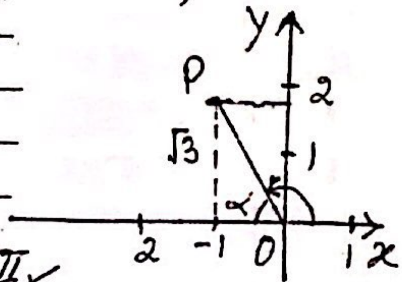
§ Exponential form a complex numbers:

Given a complex number $z = r(\cos\theta + i\sin\theta)$

Then exponential form: $\underline{z = re^{i\theta}}$

Example: Given a complex number $u = -1 + i\sqrt{3}$
Express u in (i) Polar form. (ii) Exponential form.

Solution:



$u = -1 + i\sqrt{3}$

(i) $r = |u| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$ ✓

$\arg u = \pi - \alpha \qquad \because \alpha = \tan^{-1}\left|\frac{\sqrt{3}}{-1}\right| = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$ ✓

$= \pi - \frac{\pi}{3}$

$\theta = \frac{2\pi}{3}$ ✓

$\therefore \quad u = r(\cos\theta + i\sin\theta)$

$u = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ ✓ Polar form of u.

(ii) Exponential form of $u = re^{i\theta}$

$= 2e^{i\frac{2\pi}{3}}$ ✓

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§ Multiplication of complex numbers in polar form:

Given complex numbers: $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ ---(i)
$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ ---(ii)

Then:

$$z_1 \cdot z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2[\cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \sin\theta_2] + i[\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2]$$
$$\therefore z_1 \cdot z_2 = \underline{r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]}$$

Note (i) $|z_1 z_2| = r_1 r_2 = |z_1||z_2|$ ✓

(ii) $\arg(z_1 z_2) = \theta_1 + \theta_2$

$$= \arg z_1 + \arg z_2 + k\cdot(2\pi) \quad \text{where } k = \begin{cases} 0 & \text{if } -\pi < \theta_1+\theta_2 \leq \pi \\ -1 & \text{if } (\theta_1+\theta_2) > \pi \\ +1 & \text{if } (\theta_1+\theta_2) < -\pi \end{cases}$$

§ Division of complex numbers in polar form:

Given complex numbers: $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\therefore \frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$
$$= \frac{r_1}{r_2} \cdot \frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$
$$= \frac{r_1}{r_2} \cdot \frac{[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]}{\cos^2\theta_2 + \sin^2\theta_2}$$
$$\therefore \frac{z_1}{z_2} = \frac{r_1}{r_2}\cdot[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)] \checkmark \quad [\cos^2\theta_2 + \sin^2\theta_2 = 1]$$

Note: (i) $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}$

(ii) $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$

$$= \arg z_1 - \arg z_2 + k\cdot 2\pi. \quad k = \begin{cases} 0 & \text{if } -\pi < \theta_1-\theta_2 \leq \pi \\ -1 & \text{if } (\theta_1-\theta_2) > \pi \\ +1 & \text{if } (\theta_1-\theta_2) < -\pi \end{cases}$$

§ To find the squre root of a complex number in polar form:

Given a complex number $z = r(\cos\theta + i\sin\theta)$

Let the square of z is $w = p(\cos\alpha + i\sin\alpha)$ ---(i)

$\therefore\ w^2 = z \Rightarrow [p(\cos\alpha + i\sin\alpha)]^2 = r(\cos\theta + i\sin\theta)$

$\Rightarrow p^2(\cos 2\alpha + i\sin 2\alpha) = r(\cos\theta + i\sin\theta)$

$\Rightarrow p^2 = r$ and $2\alpha = \theta$

$\Rightarrow p = \sqrt{r}$ and $\alpha = \theta/2$

here $\arg w = \alpha = \begin{cases} \frac{\theta}{2} \text{ and } \left(\frac{\theta}{2} - \pi\right) \text{ if } \theta \geq 0 \\ \text{or} \\ \frac{\theta}{2} \text{ and } \left(\frac{\theta}{2} + \pi\right) \text{ if } \theta < 0 \end{cases}$

$\therefore$ squre roots are:

$w = \sqrt{r}(\cos\theta/2 + i\sin\theta/2);\ \sqrt{r}(\cos\alpha + i\sin\alpha)$

§ In Exponential form (Square roots):

Given $z = re^{i\theta}$

Let w is the square root of z

$\Rightarrow w = (re^{i\theta})^{1/2} = \sqrt{r}\,e^{i\theta/2}$ and $\sqrt{r}\,e^{(\frac{\theta}{2} - \pi)i}$ if $\theta \geq 0$

(or $\sqrt{r}\,e^{(\theta/2 + \pi)i}$ if $\theta < 0$)

Example 24: The complex numbers w and z are defind by:

$w = (5+3i)$ and $z = 4+i$

Find wz and hence by considering arguments, show that:

$\tan^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\frac{1}{4} = \frac{\pi}{4}$ [W-14/33/Q5(ii)] ---[4]

Solution: $w = (5+3i)$ ---(i)

$z = (4+i)$ ---(ii)

$wz = (5+3i)(4+i)$

$= (20-3) + i(5+12)$

$= (17+17i)$ ---(iii)

from (i) $\arg w = \tan^{-1}\left(\frac{3}{5}\right)$ ---(iv)

from (ii) $\arg z = \tan^{-1}\left(\frac{1}{4}\right)$ ---(v)

and from (iii) $\arg(wz) = \tan^{-1}\frac{17}{17}$

$= \tan^{-1} 1 = \frac{\pi}{4}$ ---(vi)

We know

$\arg(wz) = \arg w + \arg z$

$\therefore\ \frac{\pi}{4} = \tan^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4}$

[from (iv), (v) & (vi)]

Example 25: The complex number z is defined by $z = \dfrac{9\sqrt{3}+9i}{(\sqrt{3}-i)}$

(i) Find an expression for z in the form $re^{i\theta}$, $-\pi < \theta \leq \pi$ --- [5]

(ii) Find two square roots of z, giving your answer in the form $r = e^{i\theta}$, where $r > 0$, $-\pi < \theta \leq \pi$. [S-14/31/Q5] --- [3]

Solution: $z = \dfrac{9\sqrt{3}+9i}{\sqrt{3}-i} = \dfrac{u}{v}$ (let) ---- ①

where $u = 9\sqrt{3}+9i$,

$$\left.\begin{array}{l} |u| = \sqrt{(9\sqrt{3})^2+9^2} = 18 \\ \text{and } \arg u = \tan^{-1}\dfrac{9}{9\sqrt{3}} = \tan^{-1}\dfrac{1}{\sqrt{3}} = \dfrac{\pi}{6} \end{array}\right\}$$

$\therefore u = 18\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ ---- ②

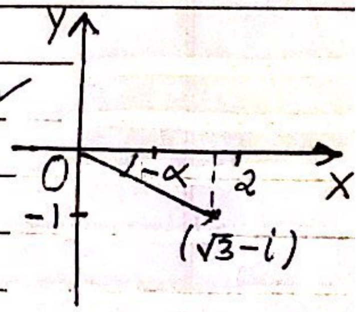
and $v = \sqrt{3} - i$,

$|v| = \sqrt{(\sqrt{3})^2+(-1)^2} = 2$ ✓

and $\arg v = -\tan^{-1}\dfrac{1}{\sqrt{3}}$

$= -\dfrac{\pi}{6}$ ✓

$\therefore v = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$ ---- ③



Now from ① $z = \dfrac{u}{v}$

$$= \frac{18\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}{2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)} \quad \left\{\begin{array}{l}\arg\dfrac{u}{v} \\ = \arg u - \arg v\end{array}\right.$$

$$= 9\left[\cos\left(\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right) + i\sin\left(\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right)\right]$$

$z = 9\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ (Polar form)

or $\underline{z = 9e^{i\frac{\pi}{3}}}$ ✓ ------------→ (Exponential form)

(ii) Let w is the square root of z,

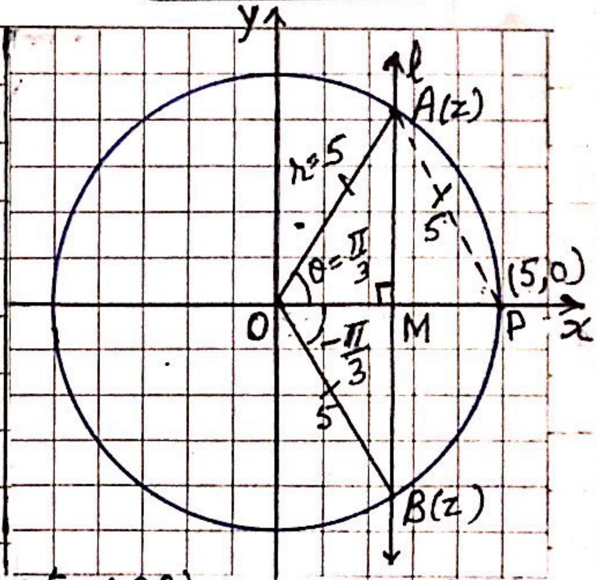
$w = \left(9e^{i\frac{\pi}{3}}\right)^{1/2}$

$= 9^{\frac{1}{2}} \cdot e^{i\frac{\pi}{6}}$ $\quad [\because \theta = \frac{\pi}{3} > 0]$

$\therefore w = 3e^{i\frac{\pi}{6}}$ and $3e^{\left(\frac{\pi}{6}-\pi\right)i}$

Two square roots are $\underline{3e^{\frac{\pi}{6}i}}$ and $\underline{3e^{-\frac{5\pi}{6}i}}$ ✓

Example 26: On a single Argand diagram sketch the loci $|z|=5$ and $|z-5|=|z|$. Hence determine the complex numbers represented by the points common to both loci, giving your answers in the form $re^{i\theta}$. [W-15/33/Q7(b)] --[4]



Solution: $|z|=5$ ---(i)

represent a circle with centre $C(0,0)$ and radius $r=5$.

$|z-5|=|z-0|$ represent the line which perp. bisector 'l' of the segment joining origin $(0,0)$ and $(5,0)$

The line 'l' intersects the circle at points A and B,

$OA = AP$ (as A lies on the perp. bisector of OP)

$\therefore OA = AP = OP = r = 5$, $\therefore \Delta OAP$ is an equilateral triangle.

$\therefore$ arg of point $A = \angle AOP = \frac{\pi}{3}$, $OA=5$

$\therefore A(z) = re^{i\theta} = 5e^{i\pi/3}$ ✓

and $B(z) = re^{-i\theta} = 5e^{-\frac{\pi}{3}i}$ ✓

Example 27: The complex numbers u and w are defined as:

$u = 4\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$ and $w = 2e^{i\pi}$. Find and simplify expressions for uw and $\frac{u}{w}$, giving your answer in the form $re^{i\theta}$, where $r>0$ and $-\pi < \theta \le \pi$.

Solution: $u = 4\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$ --①

$w = 2e^{i\pi} = 2(\cos\pi + i\sin\pi)$ --②

$\therefore uw = 4\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{4}\right) \times 2(\cos\pi + i\sin\pi)$

$uw = 8\left[\cos\left(\pi+\frac{5\pi}{12}\right) + i\sin\left(\pi+\frac{5\pi}{12}\right)\right]$

(Here $\theta = \frac{5\pi}{12}+\pi > \pi$)

$\therefore uv = 8\left[\cos\left(\pi+\frac{5\pi}{12}-2\pi\right) + i\sin\left(\pi+\frac{5\pi}{12}-2\pi\right)\right]$

$= 8\left[\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right] = 8e^{-\frac{7\pi}{12}}$ ✓

$\frac{u}{w} = \dfrac{4\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)}{2(\cos\pi + i\sin\pi)}$

$= 2\left[\cos\left(\frac{5\pi}{12}-\pi\right) + i\sin\left(\frac{5\pi}{12}-\pi\right)\right]$

$= 2\left[\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right]$

$\frac{u}{w} = 2e^{-\frac{7\pi}{12}}$ ✓